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MASS DEPENDENCES IN RG SOLUTIONS

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I. FORMULATION OF THE PROBLEM

Consider the problem of reconstruction of RG equation solutions in the asymptotic UV region from one logarithmic regime to another. Such a problem naturally arises if the considered physical system contains particles with masses of different scales. For instance in QCD masses of light quarks u, d and s equal 10-100 MeV; while those of heavy quarks c, b (and probably t), 1-10 GeV. We assume that there exists a small mass m of a light particle (or particles) and a large mass M of a heavy particle (particles) such that

$$\ln \frac{M^2}{m^2} \gg 1 \ . \tag{1}$$

Then there exists the region of energy (momentum transfer) $m^2 \ll Q^2 \sim M^2$ where the contributions of diagrams with light internal lines are in the logarithmic regime ($\sim \ln Q^2/m^2$), whereas the contributions of diagrams with heavy lines are small ($\lesssim 1$) and can be neglected. On the other hand, in the region $Q^2 \gg M^2$ the effects of heavy diagrams are logarithmic ($\sim \ln Q^2/M^2$) and contribute to the logarithmic asymptotics of invariant coupling, Green functions, and matrix elements.

For illustration consider the gluon propagator in QCD. Its dimensionless transverse coefficient in the one-loop approximation has the form

$$d(Q^2, m^2, M^2, \mu^2, \alpha_s) = 1 - \frac{\alpha_s}{4\pi} J(Q^2, \dots, \mu^2) \ , \tag{2}$$

$$J(Q^2, \dots) = 11 \ln \frac{Q^2}{\mu^2} - \frac{2}{3} \sum_{light} \left\{ I\left(\frac{Q^2}{m_\ell^2}\right) - I\left(\frac{\mu^2}{m_\ell^2}\right) \right\} - \frac{2}{3} \sum_{heavy} \left\{ I\left(\frac{Q^2}{M_h^2}\right) - I\left(\frac{\mu^2}{M_h^2}\right) \right\}, \tag{3}$$

where $I(z)$ is the well-known single fermion-loop contribution

$$I(z) = 6 \int_0^1 d\xi (1-\xi) \xi \ln\left[1 + 4z\xi(1-\xi)\right] \left(\to \ln z - 0.28 \ \text{as} \ z \to \infty\right).$$

The first term in the r.h.s. of (3) represents massless gluons and ghosts; second, light quarks; third, heavy quarks. In the "intermediate" asymptotic region I reaching the creation threshold of $c\bar{c}$ pair at $Q^2 = M_4^2 \simeq 10$ Gev , one has

$$J(Q^2, \mu^2) \simeq 9 \ln (Q^2/\mu^2) + c \, , \quad m_\ell^2 \ll Q^2 \sim M_h^2 \, , \tag{4}$$

while in the true asymptotic domain II

$$J(Q^2, \mu^2) \simeq 7 \ln (Q^2/\mu^2) \quad , \qquad Q^2, \mu^2 \gg M_h^2 \, . \tag{5}$$

The regular procedure for obtaining solutions of RG differential equations for the case of pure logarithmic asymptotic like (4), (5) is well known. Our goal is to generalize the procedure and obtain such solutions that could describe simultaneously both the domains I, II and the intermediate region (a neighbourhood of threhold M), i.e., depict the transition from one logarithmic regime to another.

2. MASS DEPENDENCES IN THE INVARIANT COUPLING

Let us start by analysing the invariant coupling. Proceeding from the standard perturbation expansion in the subtraction scheme (MOM renormalization)

$$\bar{g}_{p.th}(z,t,g) = g - g^2 U(z,t) + \\ + g^3 \left[U^2(z,t) - V(z,t) \right] + g^4 \left[-W(z,t) + \ldots \right] + \ldots \tag{6}$$

where

$U(z,t) = J_1(z) - J_1(t)$ is the sum of one-loop contributions like (3) ;

$V(z,t) = J_2(z) - J_2(z)$, the sum of the genuine 2-loop contributions;

$W(z,t) = J_3(z) - J_3(t)$, the sum of the genuine 3-loop contribution;

we get the differential RG equation

$$\frac{\partial \bar{g}(z,t,g)}{\partial \ln z} = -U'(z)\bar{g}^2 - V'(z)\bar{g}^3 - W'(z)\bar{g}^4 - \ldots \quad . \tag{7}$$

Let us rewrite it as follows

$$\frac{\partial}{\partial \ln z}\left(1/\bar{g}\right) = J_1'(z) + J_2'(z)\bar{g} + J_3'(z)\bar{g}^2 + \ldots$$

and solve by the successive approximation method

$$\frac{1}{\bar{g}_1(z)} - \frac{1}{g} = J_1(z) - J_1(t) = U(z,t) ,$$
$$\frac{1}{\bar{g}_2(z)} - \frac{1}{g} = U(z,t) + \int_t^z J_2'(\xi)\bar{g}_1(\xi)\, d\ln\xi ,$$
$$\frac{1}{\bar{g}_3(z)} - \frac{1}{g} = U(z,t) + \tag{8}$$
$$+ \int_t^z J_2'(\xi)\bar{g}_2(\xi)\, d\ln\xi + \int_t^z J_3'(\xi)\bar{g}_1^2(\xi)\, d\ln\xi .$$

The integrals in the r.h.sides have the similar structure and can be approximately evaluated by simple general trick exploiting the mentioned above peculiar mechanism of transition from one logarithmic regime to another. We demonstrate it in detail for the integral in the r.h.s. of the second Eq. (8), which will be conveniently rewritten in terms of the logarithmic variables $l = \ln z$, $\lambda = \ln\xi$, $U(\lambda) = U(\xi,t)$, $V(\lambda) = V(\xi,t)$:

$$\int^z J_1'(\xi)\bar{g}_1(\xi)\, d\ln\xi = \int^l \frac{g V'(\lambda)d\lambda}{1 + g U(\lambda)} .$$

Defining the ratio

$$R(l) = \frac{V(l)}{U(l)} , \tag{9}$$

we note that far away from the threshold of the pair creation at $z = z_h$ this ratio is equal to constant

$$R(l) = R_1 + O(1/l) \qquad z \ll z_h ,$$
$$R_2 + O(1/l) \qquad z \gg z_h .$$

In the vicinity of the threshold the function $R(l)$ changes rather quickly. Therefore, approximately one can put

$$\frac{\partial R(l)}{\partial l} = R'(l) \simeq (R_2 - R_1)\,\delta(l - l_h) + O(l^{-2}) .$$

Going in the integrand from V to R and U we get performing integration by parts

$$\int^{\ell} V'(\lambda)\bar{g}_1(\lambda)\,d\lambda = R(\ell)\ln\left[1+gU(\ell)\right] + \Delta_2(\ell)\ , \tag{10}$$

where

$$\Delta_2(\ell) = \int^{\ell} R'(\lambda) F_2\left[gU(\lambda)\right] d\lambda \quad , F_2(\gamma) = \frac{\gamma}{1+\gamma} - \ln(1+\gamma)\,. \tag{11}$$

As far as $R'(\lambda)$ is close to the delta-function, this integral can be estimated with the help of the mean value theorem:

$$\Delta_2(\ell) \simeq \left[R(\ell) - R_1\right] F_2(gU(\ell_h)) + O(g^2\ell)\,.$$

Here the first term is of order $O(g^2)$ and can be made rather small by an appropriate choice of the normalization point μ^2. Due to this the "correction" term Δ_2 in Eq. (10) is of order $g^2\ell$ and consequently can be disregarded at the 2-loop level. Hence

$$\bar{g}_2(z,t,g) = \\ = \frac{g}{1 + gU(z,t) + g\dfrac{V(z,t)}{U(z,t)}\ln\left[1+gU(z,t)\right]}\ . \tag{12}$$

The obtained explicit formula, on the one hand, in the pure logarithmic limit $U \to \beta_1 \ln z/t$, $V \to \beta_2 \ln z/t$ coincides with the well known 2-loop logarithmic expression. On the other hand, it precisely corresponds to the perturbation expansion with mass dependences. Note that it contains only ingredients U and V of the usual perturbation theory and does not depend on components of generalized beta-function

$$\beta(z,g) = -U'(z)g^2 - V'(z)g^3 - \ldots\ . \tag{13}$$

From here it follows, in particular, that the continuous parametrization /1/ of flavour numbers $f(Q^2, M_h^2)$ by the logarithmic derivative of the polarization operator

$$f(Q^2, M_h^2) = \sum_h I'\left(\frac{Q^2}{M_h^2}\right)$$

is not adequate and, e.g., in the one-loop contribution it must be replaced by the sum

$$f(Q^2, M, \mu^2) = \sum_h \frac{I(Q^2/M_h^2) - I(\mu^2/M_h^2)}{\ln Q^2/\mu^2} , \tag{14}$$

essentially dependent on the subtraction parameter μ^2.

We give without calculation also the 3-loop formula

$$\frac{1}{\bar{g}_3(z,t,g)} - \frac{1}{g} = U(z,t) + \frac{V(z,t)}{U(z,t)} \ln \frac{g}{\bar{g}_2(z,t,g)} + \frac{WU - V^2}{U(1+gU)} + \Delta_3 , \tag{15}$$

where

$$\Delta_3 = \int_{\ln t}^{\ln z} \{ R'(\lambda) F_{31}(g,U,V) + P'(\lambda) F_{32}(g,\bar{g}_1) \} d\lambda , \tag{16}$$

$$F_{31} = \bar{g}_2(\lambda,g) U(\lambda) + \ln \frac{\bar{g}_2(\lambda,g)}{g} + \bar{g}_1(\lambda,g) \left[2g V(\lambda) + \ln \frac{\bar{g}_2}{g} \right] ,$$

$$F_{32} = \frac{[\bar{g}_1(\lambda,g) - g]^3}{g^2 \bar{g}_1(\lambda,g)} ,$$

$$P(\lambda) = \frac{W(\lambda)}{U(\lambda)}$$

and $\bar{g}_2$ is defined by Eq. (12).

Here the "correction" Δ_3 is of the same order as Δ_2 in (10), (11), i.e., of an order of 3-loop terms. Due to this in contradistinction to the 2-loop case it cannot be neglected. It can be estimated by the mean value theorem.

The expression (15), except for the last correction term, precisely corresponds to the 3-loop logarithmic formula

$$\frac{g}{\bar{g}_3(\ell,g)} = 1 + g\beta_1 \ell + g \frac{\beta_2}{\beta_1} \ln \left[1 + g\beta_1 \ell + g \frac{\beta_2}{\beta_1} \ln (1 + g\beta_1 \ell) \right] + \frac{\beta_1\beta_3 - \beta_2^2}{\beta_1} \frac{g^3 \ell}{1 + g\beta_1 \ell} \tag{17}$$

—compare, e.g., Eq. (8) from /2/ .

3. THE MASS DEPENDENCES IN SINGLE-ARGUMENT RG FUNCTIONS

Let us perform the analogous analysis for the single-argument RG function s (e.g., the boson propagator, fermion mass operator, structure function moment). The differential RG Lie equation for s is of the form [3]

$$\frac{\partial \ln s(z,t,g)}{\partial \ln z} = \sigma_1(z)\,\bar{g}(z,t,g) + \sigma_2(z)\,\bar{g}^2 \ldots \tag{18}$$

where σ_ℓ is the logarithmic derivative of the ℓ-loop contribution to s :

$$\sigma_\ell(z) = S_\ell'(z) = \frac{\partial S_\ell(z)}{\partial \ln z} .$$

The quadrature of the one-loop approximation of Eq. (18)

$$\ln s_1(z,t,g) = \int_t^z \sigma_1(\xi)\,\bar{g}_1(\xi,t,g)\,d\ln\xi$$

can be calculated by using the trick from section 2. Introducing the ratio

$$R_1(z,t) = \frac{S_1(z,t)}{V(z,t)}$$

and integrating by parts we get

$$\int^z \frac{g S_1' \, d\ln\xi}{1+gU} = R_1 \ln(1+gU) + \int^z R_1' F_2(gU)\, d\ln\xi .$$

So

$$\ln s_1(z,t,g) = \frac{S_1(z,t)}{U(z,t)} \ln\left[1+gU(z,t)\right] + \delta_1 ,$$

where δ_1 is of an order of $g^2 \ln z/t$ and can be neglected. Hence, the final formula of the one-loop approximation is of the form

$$s_1(z,t,g) = \left(\frac{g}{\bar{g}_1(z,t,g)}\right)^{d(z,t)} , \quad d(z,t) = \frac{S_1(z,t)}{U(z,t)} . \tag{19}$$

Analogous calculations at the 2-loop level yield

$$s_2(z,t,g) = \left(\frac{g}{\bar{g}_2(z,t,g)}\right)^{d(z,t)} \exp\left\{(g-\bar{g}(z,t,g))\,\ell_2(z,t) + \delta_2\right\} , \tag{20}$$

where
$$C_2(z,t) = \frac{U(z,t)\,S_2(z,t) - V(z,t)\,S_1(z,t)}{U^2(z,t)} \tag{21}$$

and
$$\delta_2(z,t) = \int_{\ln t}^{\ln z} \left\{ R'(\lambda)\,\Phi(\lambda) + R_1'(\lambda)\,\Phi_1(\lambda) + R_2'(\lambda)\,\Phi_2(\lambda) \right\} d\lambda \, . \tag{22}$$

Here
$$R_i = \frac{S_i}{U} \quad , \quad \Phi(\ell) = R_1(\ell)\left[\, \bar{g}_1(\ell) \ln \frac{\bar{g}_1(\ell)}{g} + g - \bar{g}_1(\ell)\right] \sim O(g^3),$$

$$\Phi_1(\ell) = \left[g - \bar{g}_1(\ell)\right]\left\{ \frac{\bar{g}_2(\ell)}{g\bar{g}_1(\ell)} + R(\ell) \right\} + \ln \frac{\bar{g}_2(\ell)}{g} \sim -g^2 \frac{U^2(\ell)}{2} + O(g^3),$$

$$\Phi_2(\ell) = -\left[g - \bar{g}_1(\ell)\right]^2 / g \sim O(g^3) \, .$$

The correction term δ_2 is of the same order as the 2-loop contribution and cannot be disregarded. This situation is quite analogous to the 3-loop case for invariant coupling.

4. CONCLUSION

The obtained Eqs. (12), (15) for invariant coupling and Eqs. (19), (20) for the single-variable RG function give the solution of the formulated problem. The 2-loop Eq. (12) for $\bar{g}$ and one-loop Eq. (19) for s obey the remarkable property: their structure precisely follows to corresponding logarithmic formulae. In higher approximations this property vanishes and the characteristic dependence on "corrections" $\Delta_3, \dots, \delta_2, \dots$ appears. However, in a number of cases these corrections can be significantly reduced by a proper choice of the normalization μ^2. Neglecting them by this reasoning, we arrive at a simple heuristic recipe or obtaining formulae with mass dependences. It consists of four points:

(1) Perform the perturbative calculation in the standard R - operation renormalization (subtraction in the momentum representation also known as the MOM scheme) and obtain explicit expressions for mass-depending coefficients $S_1, \dots S_\ell(Q^2, m^2)$ of the function s under consideration, as well as $J_1, \dots J_\ell$ for invariant coupling $\bar{g}$;

(2) Proceed to the pure logarithmic region, assuming all masses to be small ($m^2 \to 0$):

$$S_\kappa(Q^2,m^2) \to \sigma_\kappa \ln\frac{Q^2}{m^2}\ ,\quad J_\kappa \to \beta_\kappa \ln\frac{Q^2}{m^2}\ ;$$

(3) Obtain logarithmic RG solutions in the considered ℓ-loop approximation

$$\bar{g}_\ell\left(\ln\frac{Q^2}{\mu^2};\ldots\beta_\kappa\ldots\right)\ ,\quad s_\ell\left(\ln\frac{Q^2}{\mu^2},\sigma_\kappa,\beta_\kappa\right)\ ;$$

(4) By substitution

$$\beta_\kappa \ln\frac{Q^2}{\mu^2} \to J_\kappa\left(\frac{Q^2}{m^2}\right) - J_\kappa\left(\frac{\mu^2}{m^2}\right)\ ,$$

$$\sigma_\kappa \ln\frac{Q^2}{\mu^2} \to S_\kappa\left(\frac{Q^2}{m^2}\right) - S_\kappa\left(\frac{\mu^2}{m^2}\right)$$

restore the mass dependence

$$\bar{g}_\ell\left(\ln\frac{Q^2}{\mu^2},\beta_\kappa\right) \to \bar{g}_\ell(Q^2,m^2,\mu^2,g)\ ,$$

$$s_\ell\left(\ln\frac{Q^2}{\mu^2},\beta_\kappa,\sigma_\kappa\right) \to s_\ell(Q^2,m^2,\mu^2,g)\ .$$

Besides the description of the transition region from one logarithmic regime to another, such formulas also correspond to the transition from the low energy $Q^2 \lesssim m^2$ to high energy logarithmic region $Q^2 \gg m^2$.

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