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NONLINEAR SCHRÖDINGER EQUATION WITH U(p,q) ISOGROUP.

Part 1. General Analysis

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INTRODUCTION

There are by now a great amount of papers of both physical and mathematical nature devoted to the nonlinear Schrödinger equation (NLS):

$$i\Psi_t = -\Psi_{xx} \quad 2\kappa\,|\Psi|^2\,\Psi\,. \tag{0}$$

For the simplest version U(1) symmetry NLS has been studied in detail on the classical level [1,2] as well as on the quantum one [3]. In the quantum case it describes Bose gas with δ - function pair interaction. The properties of such a gas were considered in refs. [4,5], where the ground state and excitation spectrum have been found. Remind also that in the continuum limit (0) describes a great body of various physical phenomena: from waves on water and spin waves in ferromagnets up to vertices in superfluids and laser beams in glass fibres [17].

More recent investigations dealt with the two component NLS with U(2) isogroup on both the classical [6] and the quantum [7] levels. In the first case we have elliptically polarized wave in non-linear media with dispersion relation $\omega = k^2 - 2\kappa|\vec{E}|^2$; in the latter, a gas of Bose particles possessing an internal degree of freedom.

Both U(1) and U(2) versions of equation (0) have been shown to be integrable Hamiltonian systems with all appropriate consequences [8]. In the first case the complete integrability of the system was in addition set up [9].

In the present paper we will convince that in the more complex case of noncompact U(p,q) isosymmetry group the integrability of the system is preserved. In particular U(1,1) variant the equation describes, for example, one-dimensional Hubbard model in long-wave approximation [10], as well as the system of two interacting Bose gases, "gravitating" and "antigravitating". The properties of such systems are considerably richer even in this simplest variant [11].

The present paper will have three parts. In Part 1 we give a general analysis of equation (0) with U(p,q) isosymmetry group. In section 1 of this part we present two variants of the associated linear problem: in the Lax form (L-A pair) and in the form of the flatness condition (Riemann problem). In section 2 we study the internal symmetry of the problem. We find n^2 local conservation laws corresponding to this symmetry. In section 3 the scattering data are obtained for the vanishing boundary conditions. In this case the infinite set of local conservation laws exists and all of them are in involution. In Appendix 1 we construct the Hermitian generators of the pseudounitary group $U(p,q)$. In Appendix 2 we derive the infinite series of conserving quantity in more detail.

In Part 2 of this paper we will study the properties of $U(1,1)$ version. The soliton solutions, their dispersion relations and spectra will be described in detail. The simplest physical interpretation will be given on the quasi-classical level.

In Part 3 the exact solution of the $U(1,1)$ variant will be considered.

1. STATEMENT OF THE PROBLEM

Let us consider a set of n coupled equations with cubic nonlinearity for the complex functions $\Psi^{(a)}(x,t)$ on the axis $-\infty < x < +\infty$:

$$i\Psi_t^{(a)} + \Psi_{xx}^{(a)} + 2\kappa\Big(\sum_{b=1}^{p} |\Psi^{(b)}|^2 - \sum_{b=p+1}^{p+q} |\Psi^{(b)}|^2\Big)\Psi^{(a)} = 0 \tag{1}$$

$a = 1,\dots,n\,;\quad p + q = n.$

For further discussion it is convenient to rewrite them in the matrix form. Introduce the column vector Ψ of n complex functions $(\Psi)_a = \Psi^{(a)}(x,t)$ and the Dirac conjugate row vector $\bar{\Psi} = \Psi^+\gamma_0$, where $\gamma_0 = \mathrm{diag}(\underbrace{+1,\dots,+1}_{p}, \underbrace{-1,\dots,-1}_{q})$ and the cross + means the Hermitian conjugate operation.

Defining inner product

$$(\bar{\Psi}\Psi) = \sum_{b=1}^{p} |\Psi^{(b)}|^2 - \sum_{b=p+1}^{p+q} |\Psi^{(b)}|^2$$

we can write set (1) in the conventional form of NLS

$$i\Psi_t + \Psi_{xx} + 2\kappa(\bar{\Psi}\Psi)\Psi = 0 \tag{2}$$

then conjugate one is

$$-i\bar{\Psi}_t + \bar{\Psi}_{xx} + 2\kappa(\bar{\Psi}\Psi)\bar{\Psi} = 0. \tag{3}$$

These equations correspond to the Lagrangian density

$$\mathcal{L} = \frac{i}{2}[(\bar{\Psi}\Psi_t) - (\bar{\Psi}_t \Psi)] - (\bar{\Psi}_x \Psi_x) + \kappa (\bar{\Psi}\Psi)^2 . \tag{4}$$

In terms of canonically conjugate variables $\Psi^{(a)}$ and $\bar{\Psi}^{(a)} (a=1,..n)$ with the Poisson brackets

$$\{\Psi^{(a)}(x), \bar{\Psi}^{(b)}(y)\} = i\delta^{ab}\delta(x-y)$$

the Hamiltonian of system (2) assumes the form:

$$H = \int_{-\infty}^{\infty} dx [(\bar{\Psi}_x \Psi_x) - \kappa (\bar{\Psi}\Psi)^2]. \tag{5}$$

It is easy to see that the Hamiltonian equations

$$i\Psi_t^{(a)} = i\{H, \Psi^{(a)}\} = \frac{\delta H}{\delta \bar{\Psi}^{(a)}},$$
$$i\bar{\Psi}_t^{(a)} = i\{H, \bar{\Psi}^{(a)}\} = -\frac{\delta H}{\delta \Psi^{(a)}}$$

coincide with systems (2) and (3) if the Poisson brackets for two functionals A and B are defined in the conventional manner

$$\{A, B\} = i \sum_{a=1}^{n} \int_{-\infty}^{\infty} dx \left(\frac{\delta A}{\delta \Psi^{(a)}}\frac{\delta B}{\delta \bar{\Psi}^{(a)}} - \frac{\delta B}{\delta \Psi^{(a)}}\frac{\delta A}{\delta \bar{\Psi}^{(a)}}\right).$$

The Lax pair for equation (2) consists of $(n+1)\times(n+1)$ matrix differential operators

$$\hat{L} = \left[\begin{array}{c|c} 1+s & \\ \hline & (1-s)I_n \end{array}\right] i\frac{\partial}{\partial x} + \left[\begin{array}{c|c} 0 & \bar{\Psi} \\ \hline \Psi & 0 \cdot I_n \end{array}\right] \tag{6}$$

$$\hat{A} = -sI_{n+1}\frac{\partial^2}{\partial x^2} + \left[\begin{array}{c|c} \frac{(\bar{\Psi}\Psi)}{1+s} & i\bar{\Psi}_x \\ \hline -i\Psi_x & -\frac{\Psi \otimes \bar{\Psi}}{1-s} \end{array}\right] \tag{7}$$

with $\Psi \otimes \bar{\Psi}$ being the $n\times n$ matrix of the direct (Kronecker) product of the column Ψ and row $\bar{\Psi}$, I_n is the unit $(n\times n)$ matrix and $\kappa = (1-s^2)^{-1}$ *. Then the Lax equation $\hat{L}_t = i[\hat{L},\hat{A}]$ is equivalent to equation (2).

The system (2) may be also presented as the flatness condition for certain connection [8]. Then the corresponding linear problem is the couple of $(n+1)$-component equations

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* Linear problem in the Lax form has been constructed earlier by the authors and Makhaldiani for the U(1,1) system in [12], for U(p,q) system in [13].

$$f_x = \hat{U} f\,,$$
$$f_t = \hat{V} f \qquad (8)$$

with their compatibility condition $\hat{U}_t - \hat{V}_x + [\hat{U},\hat{V}] = 0$ being equivalent to system (2), where

$$\hat{U}(x,t;\lambda) = -i\lambda \left[\begin{array}{c|c} \frac{1}{1+s} & 0 \\ \hline 0 & \frac{1}{1-s} I_n \end{array}\right] + \left[\begin{array}{c|c} 0 & \frac{i\bar{\Psi}}{1+s} \\ \hline \frac{i\Psi}{1-s} & 0\cdot I_n \end{array}\right], \qquad (9)$$

$$\hat{V}(x,t;\lambda) = -i\lambda^2 \left[\begin{array}{c|c} \frac{s}{(1+s)^2} & 0 \\ \hline 0 & \frac{s}{(1-s)^2} I_n \end{array}\right] + 2i\lambda \left[\begin{array}{c|c} 0 & \frac{s}{(1-s^2)(1+s)}\bar{\Psi} \\ \hline \frac{s\Psi}{(1-s^2)(1-s)} & 0\cdot I_n \end{array}\right] \qquad (10)$$

$$+ \left[\begin{array}{c|c} -\frac{i(\bar{\Psi}\Psi)}{1-s^2} & \frac{1}{1+s}\bar{\Psi}_x \\ \hline -\frac{1}{1-s}\Psi_x & \frac{i}{1-s^2}\Psi\otimes\bar{\Psi} \end{array}\right]$$

We may now use the method of the Riemann problem /8/ searching for solutions to system (8). But we shall follow the traditional scheme suggested in the early papers by Zakharov and Shabat /1,2/.

Substituting

$$f(x,t) = \exp\left(-i\frac{\alpha\lambda x}{1-s^2}\right)\hat{T}\phi(x,t),$$
$$\qquad (11)$$
$$\Psi^{(b)}(x,t) = (1-s^2)^{1/2} q^{(b)}(x,t)$$

and

$$T = \left[\begin{array}{c|c} (1-s)^{1/2} & 0 \\ \hline 0 & (1+s)^{1/2} I_n \end{array}\right], \quad b = 1,\dots,n$$

in system (8) we come to the couple of linear equations

$$\phi_x = \hat{U}_1\phi\,, \quad \phi_t = \hat{V}_1\phi \qquad (12)$$

with

$$\hat{U}_1(x,t;\lambda) = \left[\begin{array}{c|c} \frac{i\lambda(\alpha-1+s)}{1-s^2} & i\bar{q} \\ \hline iq & \frac{i\lambda(\alpha-1-s)}{1-s^2} I_n \end{array}\right], \qquad (13)$$

$$\hat{V}_1(x,t;\lambda) = \left[\begin{array}{c|c} -\frac{i\lambda^2 s}{(1+s)^2} - i(\bar{q}q) & \frac{2i\lambda s}{1-s^2}\bar{q} + \bar{q}_x \\ \hline \frac{2i\lambda s}{1-s^2} q - q_x & -\frac{i\lambda^2 s}{(1-s)^2} I_n + iq \otimes \bar{q} \end{array}\right]. \qquad (14)$$

It is convenient to have the Wronskian of the first equation of (12) space independent, the condition for it is $Sp\hat{U}=0$, which yields $\alpha = 1 + \frac{n-1}{n+1}s$ in eq. (11). Introducing $\xi = -\frac{2n}{n+1}\left(\frac{\lambda s}{1-s^2}\right)$ and omitting the subscript 1 in equations (13) and (14) we obtain

$$U(x,t;\xi) = \left[\begin{array}{c|c} -i\xi & i\bar{q} \\ \hline iq & i\frac{\xi}{n} I_n \end{array}\right], \quad V(x,t;\xi) = \left[\begin{array}{c|c} \left(\frac{n+1}{n}\right)^2 i\xi^2 - i(\bar{q}q) & -\frac{n+1}{n} i\xi\bar{q} + \bar{q}_x \\ \hline -\frac{n+1}{n} i\xi q - q_x & iq \otimes \bar{q} \end{array}\right] \qquad (15)$$

here we have used the freedom in defining matrix $\hat{V}$ under transformation $\hat{V} \to \hat{V} + c\hat{I}$ (with arbitrary c). The $(n+1)\times(n+1)$ linear problem governed by the operator $\hat{U}$ is the n-fold degenerated one (n identical eigenvalues). In the next section we show that this fact is related to nontrivial isotopic properties of system (2).

2. ISOTOPICAL SYMMETRY

The form of Lagrangian (4) implies that space-time independent linear transformations $\Psi' = R\Psi$ when conserve the inner product

$$(\bar{\Psi}\Psi) = |\Psi_1|^2 + \dots + |\Psi_p|^2 - |\Psi_{p+1}|^2 - \dots - |\Psi_{p+q}|^2 \qquad (16)$$

are the transformations of the system symmetry. As is known[14], the Hermitian form (16) is conserved under the transformations of pseudounitary matrix group $U(p,q)$. Matrix $R \in U(p,q)$ satisfies the following condition

$$(\bar{\Psi}'\Psi') = (\bar{\Psi}\bar{R}R\Psi) = (\bar{\Psi}\Psi)$$

whereby $\bar{R}R = I$, (17)

where

$\bar{R} = \gamma_0 R^+ \gamma_0$.

Then U(p,q) is the linear transformation group of (p+q)-dimensional complex space C^{p+q} with constraints

$$R^+ \gamma_0 R = \gamma_0 , \qquad (18)$$

where

$R \in U(p,q)$.

The number of independent parameters of U(p,q) group is $(p+q)^2$ and there are n^2 (n=p+q) related conserving local currents J_μ^{ik} ($\mu = 0,1$). Indeed, the difference, the k-th equation of system (2) multiplied by $\bar{\Psi}^{(i)}$ minus the i-th equation of system (3) multiplied by $\Psi^{(k)}$, gives continuity equations (conservation laws) $\partial_\mu J_\mu^{ik} = 0$ for a matrix current J_μ with the components

$$J_0^{ik} = \bar{\Psi}^{(i)} \Psi^{(k)} , \qquad (19a)$$

$$J_1^{ik} = i(\bar{\Psi}_x^{(i)} \Psi^{(k)} - \bar{\Psi}^{(i)} \Psi_x^{(k)}), \quad (i,k = 1,\dots,n) \qquad (19b)$$

or in the matrix form

$$J_0 = \bar{\Psi} \otimes \Psi , \quad J_1 = i(\bar{\Psi}_x \otimes \Psi - \bar{\Psi} \otimes \Psi_x).$$

The corresponding conserving charges form the (n×n) matrix with the components

$$Q^{ik} = \int_{-\infty}^{\infty} dx\, J_0^{ik}(x) = \int_{-\infty}^{\infty} dx\, \bar{\Psi}^{(i)}(x) \Psi^{(k)}(x) \qquad (20)$$

which Poisson brackets with the Hamiltonian vanish

$$\{ Q^{ik}, H \} = 0. \qquad (21)$$

Besides, they satisfy the commutation relations of the $g\ell(p+q,R)$ Lie algebra:

$$\{ Q_{ik} , Q_{j\ell} \} = \delta_{kj} Q_{i\ell} - \delta_{i\ell} Q_{jk} \qquad (22)$$

as well as the conjugation conditions:

$$Q^*_{ij} = \epsilon_{ij} Q_{ji} , \qquad (23)$$

where

$$\epsilon_{ij} = \begin{cases} +1 \text{ if } & 1 \le i \le p, \quad 1 \le j \le p \\ -1 \text{ if } & 1 \le i \le p < j \le p+q \\ -1 \text{ if } & 1 \le j \le p < i \le p+q \\ +1 \text{ if } & p+1 \le i \le p+q, \quad p+1 \le j \le p+q \end{cases} .$$

Equations (22) and (23) imply that Q^{ik} form the Lie algebra of U(p,q) group [14].

As shown in Appendix 1 n^2 conserving the Hermitian charges (Hermitian generators of U(p,q) group) may be constructed

$$N_i \doteq M_{ii} = \int_{-\infty}^{\infty} \bar{\Psi}_{(i)} \Psi_{(i)} \, dx \, , \quad (i = 1, \dots, n).$$

$$N_{ij} = \int_{-\infty}^{\infty} (\bar{\Psi}_{(i)} \Psi_{(j)} + \bar{\Psi}_{(j)} \Psi_{(i)}) dx, \quad C_{ij} = i \int_{-\infty}^{\infty} (\bar{\Psi}_{(i)} \Psi_{(j)} - \bar{\Psi}_{(j)} \Psi_{(i)}) dx,$$

(where $1 \leq i, j \leq p$, or $p+1 \leq i, j \leq p+q$)

$$T_{ij} = i \int_{-\infty}^{\infty} (\bar{\Psi}_{(i)} \Psi_{(j)} + \bar{\Psi}_{(j)} \Psi_{(i)}) dx, \quad K_{ij} = \int_{-\infty}^{\infty} (\bar{\Psi}_{(i)} \Psi_{(j)} - \bar{\Psi}_{(j)} \Psi_{(i)}) dx \qquad (24)$$

(where $1 \leq i \leq p < j \leq p+q$ or $1 \leq j \leq p < i \leq p+q$). Diagonal "charges" N_i are the numbers of "i" type particles which are positive

$$N_i = \int_{-\infty}^{\infty} dx \, |\Psi_{(i)}|^2 > 0 \qquad \text{when} \quad 1 \leq i \leq p$$

and negative

$$N_i = -\int_{-\infty}^{\infty} dx \, |\Psi_{(i)}|^2 < 0 \qquad \text{when} \quad p+1 \leq i \leq p+q = n.$$

The physical meaning of that will be given in Part 2. Nondiagonal charges generate transformations that mix different "pure" charge states. They belong to the subgroup $SU(p,q) \subset U(p,q)$. Taking particular solutions to the system (2)

$$\left\{ \begin{array}{ll} \Psi_{(i)} = \tilde{\Psi}_i & 1 \leq i \leq p, \\ \Psi_{(j)} = 0 & j \neq i \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{ll} \Psi_{(i)} = 0 & i \neq j \\ \Psi_{(j)} = \tilde{\Psi}_{(j)} & p+1 \leq j \leq p+q \end{array} \right. \qquad (25)$$

one may construct a whole set of solutions using transformations $R_1 \in SU(p,q)$ generated by N, C, T and K charges. But solutions (25) are those to the one-component nonlinear Schrödinger equations with positive and negative coupling constant, respectively, whereby the overall set of its solutions may be used to find solutions to system (2).

Consider, for example, single-soliton solution to the U(1,0) NLS

$$\tilde{\Psi}(x,t) = a \, e^{i\theta} \operatorname{sech} a\bar{x},$$

where $\bar{x} = x - vt - x_0$, $\theta = \frac{v}{2} x - \omega t$, $\omega = \frac{v^2}{4} - a^2$.

Making an isotopic rotation we get one-soliton solution to the U(p,q) NLS

$$\tilde{\Psi}_i = ac_i e^{i\theta} \operatorname{sech} a\bar{x}, \quad i=1,\ldots,n, \tag{26}$$

where c_i are the components of the unit vector

$$(\bar{c}c) = \sum_{i=1}^{p} |c_i|^2 - \sum_{i=p+1}^{n} |c_i|^2 = 1.$$

For the case $p=2$, $q=0$ we recover the solution obtained earlier by Manakov in ref.[6].

Under isogroup rotation R the linear problem is transformed by means of the $(n+1)\times(n+1)$ matrix

$$\mathcal{R} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}.$$

Whence the operators $\hat{L}$ and $\hat{A}$ in equations (6) and (7) are transformed as follows

$$\begin{aligned} \hat{L}(\Psi) &\to \hat{L}'(\Psi) = \hat{L}(\Psi') = \mathcal{R}\hat{L}(\Psi)\bar{\mathcal{R}}, \\ \hat{A}(\Psi) &\to \hat{A}'(\Psi) = \hat{A}(\Psi') = \mathcal{R}\hat{A}(\Psi)\bar{\mathcal{R}}, \end{aligned} \tag{27}$$

where $\bar{\mathcal{R}} = \begin{bmatrix} 1 & 0 \\ 0 & \bar{R} \end{bmatrix} = \Gamma_0 \mathcal{R}^+ \Gamma_0$ and $\Gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_0 \end{bmatrix}$.

The operators $\hat{U}$ and $\hat{V}$ of linear problem (8) are transformed in a similar way.

It follows from the structure of the operator

$$\hat{U}(x,t;\xi) = -i\xi\hat{\Sigma} + \hat{Q}, \tag{28}$$

where

$$\hat{\Sigma} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\frac{1}{n} I_n \end{array}\right], \quad \hat{Q} = \left[\begin{array}{c|c} 0 & i\bar{q} \\ \hline iq & 0\cdot I_n \end{array}\right],$$

that the n-fold degeneracy of the matrix $\hat{\Sigma}$ eigenvalues is directly connected to isogroup properties. Indeed, the condition

$$\hat{U}(R\,q(x,t)) = \mathcal{R}\,\hat{U}(q(x,t))\,\bar{\mathcal{R}}$$

may be fulfilled only if $\mathcal{R}\hat{\Sigma}\bar{\mathcal{R}} = \hat{\Sigma}$ or $\bar{R}R = I$. But this means that (see equation (17)) $R \in U(p,q)$. Therefore, the degeneracy of a linear problem implies an isotopic symmetry to be inherent in the system*

*The analogous fact has been stated independently for the particular case of U(2) symmetry in [15].

3. SCATTERING DATA AND CONSERVATION LAWS

Further investigation of equation (2) closely depends on the boundary conditions. The structure of system (2) admits the zero-valued boundary conditions for the fields (potentials), various constant-valued ones and the number of such combinations is large.

The simplest situation occurs for vanishing boundary conditions $q(\pm\infty)=0$. We examine this case in more detail. Let us consider two sets of Jost solutions $\phi_i(x,t)$ and $\Psi_i(x,t)$ $(i=1,\dots,n+1)$ to the system of equations

$$\frac{d}{dx}\phi_i(x,\xi)+i\xi\hat{\Sigma}\phi_i(x,\xi)=\hat{Q}\phi_i(x,\xi) \tag{29}$$

with the following asymptotic behaviour

$$(\phi_i)^{(k)}=\delta_{ik}\exp(-i\xi\Sigma_{ik}x) \qquad \text{as} \quad x\to-\infty \tag{30a}$$

and

$$(\Psi_i)^{(k)}=\delta_{ik}\exp(-i\xi\Sigma_{ik}x) \qquad \text{as} \quad x\to+\infty \tag{30b}$$

(for the definition of $\hat{Q}$ and $\hat{\Sigma}$ see equation (28)).

The system of equations (29) with boundary conditions (30a) and (30b) is equivalent to the system of integral equations for the matrix Jost solutions $\hat{\phi}(x,\xi)$ and $\hat{\Psi}(x,\xi)$ respectively

$$\hat{\phi}(x,\xi)=e^{-i\xi\hat{\Sigma}x}+\int_{-\infty}^{x}e^{-i\xi\hat{\Sigma}(x-y)}\hat{Q}(y)\hat{\phi}(y,\xi)\,dy, \tag{31a}$$

$$\hat{\Psi}(x,\xi)=e^{-i\xi\hat{\Sigma}x}-\int_{x}^{\infty}e^{-i\xi\hat{\Sigma}(x-y)}\hat{Q}(y)\hat{\Psi}(y,\xi)\,dy, \tag{31b}$$

where $\hat{Q}(y)=\begin{bmatrix}0 & i\bar{q}(y)\\ iq(y) & 0\end{bmatrix}$ is the $(n+1)\times(n+1)$ potential matrix. Matrices $\hat{\phi}$ and $\hat{\Psi}$ consist of $(n+1)$ columns of the corresponding Jost solutions.

The conjugate equation

$$\frac{d}{dx}\hat{\bar{\phi}}(x,\eta)-i\eta^*\hat{\bar{\phi}}(x,\eta)\hat{\Sigma}=-\hat{\bar{\phi}}(x,\eta)Q \tag{32}$$

for the matrix $\hat{\bar{\phi}}=\Gamma_0\hat{\phi}^+\Gamma_0$ follows from the matrix version of (29), i.e.,

$$\frac{d}{dx}\hat{\phi}(x,\xi)-i\xi\hat{\Sigma}\hat{\phi}(x,\xi)=\hat{Q}\hat{\phi}(x,\xi) \tag{33}$$

and the anti-pesudo-hermiticity condition

$$\bar{\hat{Q}} = -\hat{Q} \tag{34}$$

where

$$\bar{\hat{Q}} = \Gamma_0 \hat{Q}^+ \Gamma_0 \ .$$

Using equation (32) and (33) one can easily find

$$\frac{d}{dx}(\hat{\bar{\phi}}(x,\eta)\hat{\phi}(x,\xi)) = i(\eta^* - \xi)(\hat{\bar{\phi}}(x,\eta)\hat{\Sigma}\,\hat{\phi}\,(x,\xi)). \tag{35}$$

This equation, when $\eta^* = \xi$, is reduced to

$$\hat{\bar{\phi}}(x,\xi)\hat{\phi}(x,\xi) = \hat{I}. \tag{36}$$

The completeness of both the sets of Jost solutions $\{\phi_i\}$ and $\{\Psi_i\}$ means that they have to be linear dependent

$$\hat{\phi}(x,\xi) = \hat{\Psi}(x,\xi)\,\hat{S}(\xi). \tag{37}$$

Here $\hat{S}(\xi)$ is a transition matrix. Its matrix elements $(\hat{S}(\xi))_{ik} = S_{ik}(\xi)$ may be expressed through the Wronskians of various Jost solutions

$$S_{ik}(\xi) = \frac{W(\Psi_1 \dots \Psi_{i-1}\phi_k \Psi_{i+1} \dots \Psi_{n+1})}{W(\Psi_1 \dots \Psi_{n+1})} = W(\Psi_1 \dots \Psi_{i-1}\ \phi_k \Psi_{i+1} \dots \Psi_{n+1}). \tag{38}$$

The second equality in (38) follows from the relations $Sp\,\hat{\Sigma} = Sp\,\hat{U} = 0$ (i.e., in equation (32) Wronskians are x independent) and

$$W(\Psi_1 \dots \Psi_{n+1}) = \lim_{x \to +\infty} W(\Psi_1 \dots \Psi_{n+1}) = 1, \tag{39a}$$

$$W(\phi_1 \dots \phi_{n+1}) = \lim_{x \to -\infty} W(\phi_1 \dots \phi_{n+1}) = 1. \tag{39b}$$

It is clear that the transition matrix $\hat{S}(\xi)$ is also x independent. It satisfies the unimodularity condition

$$\det \hat{S}(\xi) = 1 \tag{40}$$

(which follows from equations (37), (38) and (39)), as well as the pseudounitarity one

$$\bar{\hat{S}}(\xi)\,\hat{S}(\xi) = \hat{I} \tag{41}$$

(the consequence of equations (36), (37) and the equation conjugate to the latter $\hat{\phi} = \hat{S}\hat{\Psi}$).

The Jost solutions $\phi_1(x,\xi), \Psi_2(x,\xi), \ldots, \Psi_{n+1}(x,\xi)$ allow analytical continuations into the upper half of ξ-plane and $\phi_2(x,\xi), \ldots, \phi_{n+1}(x,\xi), \Psi_1(x,\xi)$ into the lower half of ξ-plane. This provides the analyticity of $S_{11}(\xi) = W(\phi_1 \Psi_2 \ldots \Psi_{n+1})$ in the region $\mathrm{Im}\,\xi \geq 0$. The zeroes of $S_{11}(\xi)$ in the upper half of ξ-plane define the discrete spectrum of the linear problem and if $S_{11}(\zeta) = 0$ then

$$\phi_1(x,\zeta) = c_{12}\Psi_2(x,\zeta) + c_{13}\Psi_3(x,\zeta) + \ldots + c_{1,n+1}\Psi_{n+1}(x,\zeta).$$

The coefficients $c_{1\alpha}(\zeta)$ may be obtained using the relations

$$c_{1\alpha}(\zeta) = S_{1\alpha}(\zeta) = W(\phi_\alpha, \Psi_2, \Psi_3, \ldots, \Psi_{n+1})(\zeta), \quad (\alpha = 2, \ldots, n+1).$$

The set of scattering data $S_{ij}(\xi)$, $c_{1\alpha}$. describes completely the linear problem. Let us find their time dependence. The evolution of the transition matrix is given by the operator $\hat{V}$ (see (15)) according to the equation:

$$\hat{S}_t(\xi) = \hat{v}_+ \hat{S}(\xi) - \hat{S}(\xi)\hat{v}_- , \tag{42}$$

where $\hat{v}_\pm = \lim_{x \to \pm\infty} e^{i\xi\hat{\Sigma}x} \hat{V} e^{-i\xi\hat{\Sigma}x}$. Then the boundary conditions (30) for the Jost solutions are independent of t and the potential evolution governs that of transition matrix (42) according to non-linear equation (3). It follows from equation (15) that $\hat{v}_+ = \hat{v}_- \equiv \hat{v}_0$ and hence equation (42) goes to

$$\hat{S}_t(\xi) = [\hat{v}_0, \hat{S}(\xi)]. \tag{43}$$

Whereby the matrix elements

$$\hat{S}(\xi) = \begin{bmatrix} S_{11} & S_{1\beta} \\ S_{\alpha 1} & S_{\alpha\beta} \end{bmatrix}, \quad (\alpha, \beta = 2, 3, \ldots, n+1).$$

have the following time-dependence

$$\begin{gathered} S_{11}(\xi,t) = S_{11}(\xi,0), \qquad S_{\alpha\beta}(\xi,t) = S_{\alpha\beta}(\xi,0), \\ S_{1\beta}(\xi,t) = \exp\left(i\left(\frac{n+1}{n}\right)^2 \xi^2 t\right) S_{1\beta}(\xi,0), \\ S_{\alpha 1}(\xi,t) = \exp\left(-i\left(\frac{n+1}{n}\right)^2 \xi^2 t\right) S_{\alpha 1}(\xi,0). \end{gathered} \tag{44}$$

For the coefficients of the discrete spectrum we obtain

$$c_{1\alpha}(\zeta, t) = \exp(i(\frac{n+1}{n})^2 \zeta^2 t) c_{1\alpha}(\zeta, 0)$$
$$(\alpha = 2,3,\ldots,n+1). \qquad (45)$$

Such a simple time-dependence of scattering data allows us to state the problem of reconstructing potentials at every instant of time /6/ (see Part 2 of the present paper). However, we focus below on the conserving in time matrix elements. One of them $S_{11}(\xi)$ generates infinite series of local conservation laws. The conservation laws (n^2 infinite series) which are generated by the block $S_{\alpha\beta}(\xi)$ are nonlocal barring first n^2 of them which coincide with the isotopical currents J_μ^{ik} conservation.

The method for obtaining the integrals of motion is given in Appendix II. Thus, the functions $\ln S_{ii}(\xi) \equiv \Phi_i(+\infty) = \int_{-\infty}^{\infty} \Phi_{i_x}(x)\,dx$ are generating functions of "diagonal" integrals of motion, which numerable infinite set is related to asymptotic expansion as $\xi \to \infty$

$$\ln S_{ii}(\xi) \approx \sum_{k=1}^{\infty} \frac{I_{ii}^{(k)}}{(\frac{n+1}{n} i\xi)^k}. \qquad (46)$$

The condition $\det \hat{S} = 1$ gives rise to $\sum_{i=1}^{n+1} \ln S_{ii} = 0$, and then only n series of "diagonal" integrals of motion are independent since

$$\sum_{i=1}^{n+1} I_{ii}^{(k)} = 0, \qquad (47)$$

i.e., the k-th local integral of motion equals the negative sum of n nonlocal k-th integrals. The local integrals of motion $I_{11}^{(k)}$ are in involution with one another*

$$\{I_{11}^{(k)}, I_{11}^{(\ell)}\} = 0$$

and with all nonlocal integrals $I_{\alpha\beta}^{(\ell)}$:

$$\{I_{11}^{(k)}, I_{\alpha\beta}^{(\ell)}\} = 0,$$

where $k, \ell = 1,2,\ldots$ but $\alpha, \beta = 2,3,\ldots,n+1$. The nonlocal integrals of motion are obtained through the asymptotic expansion near $\xi \to \infty$

*This result would be published separately.

$$\frac{S_{\alpha\beta}(\xi)}{S_{\alpha\alpha}(\xi)} = \sum_{k=1}^{\infty} \frac{I_{\alpha\beta}^{(k)}}{(\frac{n+1}{n} i\xi)^k} \tag{48}$$

and hence are generated by the conserving block $S_{\alpha\beta}(\xi)$.

Infinite sets of involutative integrals of motion $I_{11}^{(k)}$ form an infinite-parameter Abelian group while nonlocal integrals of motion $I_{\alpha\beta}^{(k)}$ are not involutative and generate non-Abelian transformations of a rich algebraic structure. Thus, for example, the Poisson bracket

$$\{\ln S_{22}(\xi),\ S_{22}^{-1}(\xi) S_{23}(\xi)\} = -\frac{n}{n+1} \frac{d}{d\xi}(S_{22}^{-1}(\xi) S_{23}(\xi))$$

generates single translation along the spectral parameter ξ. Using equations (46) and (48) we get

$$\sum_{m=1}^{k} \{ I_{22}^{(k+1-m)}, I_{23}^{(m)} \} = ik I_{23}^{(k)} \quad (k=1,2,\ldots).$$

Let us write down some first terms

$$\{ I_{22}^{(1)}, I_{23}^{(1)} \} = i I_{23}^{(1)},$$

$$\{ I_{22}^{(1)}, I_{23}^{(2)} \} = 2i I_{23}^{(2)} - \{ I_{22}^{(2)}, I_{23}^{(1)} \},$$

$$\{ I_{22}^{(1)}, I_{23}^{(3)} \} = 3i I_{23}^{(3)} - \{ I_{22}^{(2)}, I_{23}^{(2)} \} - \{ I_{22}^{(3)}, I_{23}^{(1)} \},$$

. . . .

We have obtained the hierarchy of the Poisson brackets of which only one for $k=1$ contains the local integrals of motion coinciding with isotopical charges (20) of $U(p,q)$

$$I_{22}^{(1)} = iQ_{11} = i\int_{-\infty}^{\infty} \bar{q}_1 q_1 dx,$$

$$I_{23}^{(1)} = iQ_{12} = i\int_{-\infty}^{\infty} \bar{q}_1 q_2 dx$$

with their Poisson brackets being $\{Q_{11}, Q_{12}\} = Q_{12}$. The remaining Poisson brackets in (49) include higher IMs $I_{\alpha\alpha}^{(k)} (k>1)$ which correspond to the diagonal elements $S_{\alpha\alpha}$ of the transition matrix.

A method using the integral equations (31) for deriving $I_{\alpha\alpha}^{(k)}$ conserving densities is presented in Appendix II. Here we give the first five generated by $S_{11}(\xi)$:

$$I_{11}^{(1)} = \int_{-\infty}^{\infty} (\bar{q}q)(x,t)\,dx,$$

$$I^{(2)}_{11} = \int_{-\infty}^{\infty} (\bar{q} q_x)(x,t)\, dx,$$

$$I^{(3)}_{11} = \int_{-\infty}^{\infty} ((\bar{q} q_{xx}) + (\bar{q} q)^2)(x,t)\, dx,$$

$$I^{(4)}_{11} = \int_{-\infty}^{\infty} ((\bar{q} q_{xxx}) + 3(\bar{q} q)(\bar{q}_x q))(x,t) dx, \tag{49}$$

$$I^{(5)}_{11} = \int_{-\infty}^{\infty} ((\bar{q} q_{xxxx}) + (\bar{q} q)^3 - (\frac{\partial}{\partial x}(\bar{q} q))^2 - 6(\bar{q}_x q_x)(\bar{q} q))(x,t) dx,$$

where

$$(\bar{q} q) = \sum_{a=1}^{p} | q^{(a)} |^2 - \sum_{a=p+1}^{n} | q^{(a)} |^2 .$$

We also show in the Appendix that they are generalization of the familiar $U(1,0)$ integrals of motion [1] for the case of $U(p,q)$ isosymmetry. In the second part of this paper we shall present the recurrent formula for them when $p=1, q=1$.

Before describing some concrete systems we note that the particular variants of system (2) discussed earlier are the following:

1) nonlinear Schrödinger equations ($S3_+$) or $U(1,0)$ with positive coupling constant [1];

2) the same with negative coupling constant ($S3_-$) or $U(0,1)$ [2];

3) Manakov's system or $U(2,0)$ [6] and its generalization, the so-called vector version of NLS $U(p,0)$.

Then we shall see that our system even in the simplest case of $U(1,1)$ group possesses a rich variety of soliton-type solutions and manifests some characteristic properties of the general system (2).

APPENDIX I

Let us consider the block structure of matrix Q in the form

$$Q = \begin{bmatrix} Q^{(+)}_{p} & Q^{(-)}_{pq} \\ Q^{(-)}_{qp} & Q^{(+)}_{q} \end{bmatrix} \tag{A.1}$$

with

$$
Q_{ij} = \begin{cases} (Q_p^{(+)})_{ij} & \text{if} \quad 1 \le i, j \le p \\ (Q_q^{(+)})_{ij} & \text{if} \quad p+1 \le i, j \le p+q \\ (Q_{pq}^{(-)})_{ij} & \text{if} \quad 1 \le i \le p < j \le p+q \\ (Q_{qp}^{(-)})_{ij} & \text{if} \quad 1 \le j \le p < i \le p+q. \end{cases} \tag{A.2}
$$

The conjugation conditions (23) yield:

$$
(Q_p^{(+)})_{ij}^* = (Q_p^{(+)})_{ji}, \quad (Q_q^{(+)})_{ij}^* = (Q_q^{(+)})_{ji},
$$

$$
(Q_{pq}^{(-)})_{ij}^* = -(Q_{pq}^{(-)})_{ji}, \quad (Q_{qp}^{(-)})_{ij}^* = -(Q_{qp}^{(-)})_{ji},
$$

where the signs (+) or (-) imply that the corresponding generators are Hermitian and anti-Hermitian, respectively.

Using the matrix elements Q_{ij} one can construct n^2 Hermitian generators M_{ij} of the U(p,q) group:

$$
M_{ii} = \begin{cases} (Q_p^{(+)})_{ii} & 1 \le j \le p \\ (Q_q^{(+)})_{ii} & p+1 \le i \le p+q, \end{cases}
$$

$$
M_{ij} = (Q_p^{(+)})_{ij} + (Q_p^{(+)})_{ji}, \quad M_{ij} = i[(Q_p^{(+)})_{ij} - (Q_p^{(+)})_{ji}], \tag{A.3}
$$

$$
M_{ij} = (Q_q^{(+)})_{ij} + (Q_q^{(+)})_{ji}, \quad M_{ij} = i[(Q_q^{(+)})_{ij} - (Q_q^{(+)})_{ji}],
$$

$$
M_{ij} = i[(Q_{pq}^{(-)})_{ij} + (Q_{pq}^{(-)})_{ji}], \quad M_{ij} = (Q_{pq}^{(-)})_{ij} - (Q_{pq}^{(-)})_{ji},
$$

$$
M_{ij} = i[(Q_{qp}^{(-)})_{ij} + (Q_{qp}^{(-)})_{ji}], \quad M_{ij} = (Q_{qp}^{(-)})_{ij} - (Q_{qp}^{(-)})_{ji},
$$

where indices i, j lie in the domains defined by formula (A.2).

APPENDIX 2

Let us consider the problem of finding conserved densities. For this purpose we use the integral equations (31) rather than the system of differential equations (29) as usual. We introduce the following matrices

$$
\hat{\chi}(x,\xi) = \exp(i\xi \hat{\Sigma} x)\hat{\phi}(x,\xi) \tag{A.4}
$$

with $\hat{\phi}(x,\xi)$ being the Jost solution matrix and with the following asymptotic behaviours

$$\hat{\chi}(x,\xi) \to \hat{I} \quad \text{as} \quad x \to -\infty$$
$$\hat{\chi}(x,\xi) \to \hat{S}(\xi) \quad \text{as} \quad x \to +\infty . \tag{A.5}$$

Substituting (A.4) in equations (31a) we have an equation for $\hat{\chi}$ in the closed form

$$\hat{\chi}(x,\xi) = \hat{I} + \int_{-\infty}^{x} e^{i\xi\hat{\Sigma} y} \hat{Q}(y) e^{-i\xi\hat{\Sigma} y} \hat{\chi}(y,\xi)\, dy . \tag{A.6}$$

The integral equation (A.6) with boundary conditions (A.5) is the most convenient for our purposes. Conditions (A.5) provide the diagonal elements to be presented in the form

$$\chi_{ii}(x,\xi) = \exp(\Phi_i(x,\xi))$$

with

$$\Phi_i(-\infty) = 0, \quad \Phi_i(+\infty) = \ln S_{ii} .$$

Functions

$$\ln S_{ii}(\xi) = \Phi_i(+\infty) = \int_{-\infty}^{\infty} \Phi_{i_x}(x)dx$$

are the generating functions of the "diagonal" integrals of motion which numerable set consists of the coefficients of the series

$$\ln S_{jj}(\xi) \approx \sum_{k=1}^{\infty} \frac{I_{jj}^{(k)}}{(\frac{n+1}{n} i\xi)^k} \tag{A.7}$$

as $\xi \to \infty$.

The nonlocal integrals of motion $I_{\alpha\beta}^{(k)}$ are defined by the expansion

$$\frac{S_{\alpha\beta}(\xi)}{S_{\alpha\alpha}(\xi)} \approx \sum_{k=1}^{\infty} \frac{I_{\alpha\beta}^{(k)}}{(\frac{n+1}{n} i\xi)^k} \quad (\alpha, \beta = 2,3,\ldots,n+1) \tag{A.8}$$

as $\xi \to \infty$.

All integrals of motion (A.7), (A.8) can be obtained from equation (A.6). Most easily it is performed for local conservation laws generated by $S_{11}(\xi)$.

The matrix equation (A.6) for the first column components yields

$$\chi_{11}(x,\xi) = 1 + \int_{-\infty}^{x} dy\, e^{i\frac{n+1}{n}\xi y} (i\bar{q}_1\chi_{21} + i\bar{q}_2\chi_{31} + \ldots + i\bar{q}_n \chi_{n+1,1})$$

$$\chi_{21}(x,\xi) = \int_{-\infty}^{x} dy\, e^{-i\frac{n+1}{n}\xi y} i q_1 \chi_{11}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad (A.9)$$

$$\chi_{n+1,1}(x,\xi) = \int_{-\infty}^{x} dy\, e^{-i\frac{n+1}{n}\xi y} i q_n \chi_{11}.$$

Upon substituting $\chi_{21}, \ldots, \chi_{n+1,1}$ in the first equation we obtain an equation for χ_{11} in the closed form

$$\chi_{11}(x) = 1 - \int_{-\infty}^{x} dy \int_{-\infty}^{y} dz\, e^{i\frac{n+1}{n}\xi(y-z)} \sum_{a=1}^{n} \bar{q}_a(y) q_a(z) \chi_{11}(z). \qquad (A.10)$$

For a generating function $\Phi_1(x) = \ln \chi_{11}(x)$ this leads to the following integral equation

$$e^{\Phi_1(x)} = 1 - \int_{-\infty}^{x} dy \int_{-\infty}^{y} dz\, e^{i\frac{n+1}{n}\xi(y-z)} \sum_{a=1}^{n} \bar{q}_a(y) q_a(z) e^{\Phi_1(z)}. \qquad (A.11)$$

The equations for the generating functions of nonlocal conservation laws become more complicated and involve extra integral terms, which lead to nonlocality. A more detail study of these questions would be considered separately.

Differentiating $n+1$ times and eliminating n-variables

$$A_a = \int_{-\infty}^{y} dz\, e^{-i\frac{n+1}{n}\xi z} q_a(z) e^{\Phi_1(z)} \quad (a=1,\ldots,n)$$

one obtains a differential equation of the $(n+1)$ order for the function $\Phi_1(x)$. This equation leads to recurrent relations for integral densities $\phi_{11}^{(k)}(x)$ where

$$\Phi_{1x}(x) \approx \sum_{k=1}^{\infty} \frac{\phi_{11}^{(k)}(x)}{(\frac{n+1}{n} i\xi)} \qquad (A.12)$$

as $\xi \to \infty$.The corresponding integrals of motion follow from

$$I_{11}^{(k)} = \int_{-\infty}^{\infty} dx\, \phi_{11}^{(k)}(x). \qquad (A.13)$$

Let us illustrate the above procedure using as an example the case $n=1$, since, as we known, such an approach has not been considered yet. Then we show that going to the case of arbitrary n is straightforward if one redefines the inner product. In Part 2 of this paper we obtain recurrent formulas for $n=2$ (in the case of $U(1,1)$ isogroup).

Now, let us put $n=1$. Upon differentiating twice equation (A.11) we get

$$\Phi_{1_x} e^{\Phi_1} = -\bar q(x) e^{2i\xi x} A(x),$$

$$(\Phi_{1_{xx}} + (\Phi_{1_x})^2) e^{\Phi_1} = -\bar q(x) q(x) e^{\Phi_1(x)} - [(2i\xi)\bar q + \bar q_x] e^{2i\xi x} A(x), \quad (A.14)$$

with

$$A(x) = \int_{-\infty}^{x} dy\, e^{-2i\xi y} q(y) e^{\Phi_1(y)} .$$

Eliminating now the function $A(x)$ we obtain from (A.14) the equation

$$2i\xi \Phi_{1_x} = \bar q q + (\Phi_{1_x})^2 + \Phi_{1_{xx}} - \frac{\bar q_x}{\bar q} \Phi_{1_x} \quad (A.15)$$

which coincides with that obtained in ref./1/. This equation, already, allows us to compute the coefficients of the asymptotic expansion (in $1/2i\xi$) of Φ_{1_x} using (A.12) and recurrent relations

$$\phi^{(k+1)} = -\frac{\bar q_x}{\bar q} \phi^{(k)} + \frac{d}{dx} \phi^{(k)} + \sum_{k_1+k_2=k} \phi^{(k_1)} \phi^{(k_2)} \quad (A.16)$$

$$\phi^{(1)} = \bar q q$$

The integrals of motion are then governed by equation (A.13). The first four of them assume the form

$$I_{11}^{(1)} = \int_{-\infty}^{\infty} (\bar q q)(x,t)\, dx,$$

$$I_{11}^{(2)} = \int_{-\infty}^{\infty} (\bar q q_x)(x,t)\, dx, \quad (A.17)$$

$$I_{11}^{(3)} = \int_{-\infty}^{\infty} (\bar q q_{xx} + (\bar q q)^2)(x,t) dx,$$

$$I_{11}^{(4)} = \int_{-\infty}^{\infty} (\bar q q_{xxx} + 3(\bar q q)(\bar q_x q))(x,t)\, dx.$$

Now if in final equations (A.17) (but not in recurrent formulas (A.16)) we mean by $\bar q q = q^* q$ the $U(p,q)$ group inner product $(\bar q q) = \sum_{a=1}^{p} | q^{(a)} |^2 - \sum_{a=p+1}^{n} | q^{(a)} |^2$, then we produce exactly the polynomial integrals of motion for the case of arbitrary $n = p+q$. To show this let us return to the basic equation for generating function (A.11). Upon differentiating once we have

$$\Phi_{1_x}(x) e^{\Phi_1(x)} = -\int_{-\infty}^{x} dy\, e^{i\frac{n+1}{n}\xi(x-y)} \sum_{a=1}^{n} (\bar q_a(x) q_a(y)) e^{\Phi_1(y)} .$$

Integrating now the right-hand side of this equation k-times over y and using conventional formulas (see ref./16/)

$$J(\lambda) = \int_a^b e^{i\lambda\tau} f(\tau)\, d\tau = \sum_{m=1}^{k} \left(\frac{i}{\lambda}\right)^m (e^{i\lambda a} f^{(m-1)}(a) - e^{i\lambda b} f^{(m-1)}(b)) + \epsilon_k$$

with the remainder being

$$\epsilon_k = (\frac{i}{\lambda})^k \int_a^b e^{i\lambda\tau} f^{(k)}(\tau)\, d\tau$$

we obtain an asymptotic expansion

$$\Phi_{1_x}(x) \approx \sum_{k=1}^{\infty} \frac{1}{(\frac{n+1}{n} i\xi)^k} e^{-\Phi_1(x)} (\sum_{a=1}^{n} \bar{q}_a(x)(q_a(x) e^{\Phi_1(x)})^{(k-1)})$$

as $\xi \to \infty$. In this equation the (k–1)-th derivative may be rewritten as the (k–1) -th power of an operator $D_x = \frac{d}{dx} + \Phi_{1_x}$ leading to

$$\Phi_{1_x} \approx \sum_{k=1}^{\infty} \frac{1}{(\frac{n+1}{n} i\xi)^k} \sum_{a=1}^{n} (\bar{q}_a D_x^{k-1} q_a). \tag{A.18}$$

Using this relation the coefficients $\phi_{11}^{(k)}$ of the expansion into a power series of $1/\xi$ can be calculated. Substituting the sum of the ℓ first terms of the function Φ_{1x} expansion into the right-hand side of (A.18) and combining the (ℓ+1)-st power we obtain $\phi_{11}^{(\ell+1)}$. Starting with $\Phi_{1_x}^{(0)} = 0$, we should have the bilinear parts of the infinite set of conserving densities

$$\Phi_{1_x} \approx \sum_{k=1}^{\infty} \frac{1}{(i\frac{n+1}{n}\xi)^k} (\bar{q} q^{(k-1)})$$

which are related to higher integrals of motion of the linearized system (2).

The continuation of the recurrent procedure gives higher integrals of motion of nonlinear equation (2), those first four are given by (A.17).

The structure of relation (A.18) is such, that the integrals of motion associated with equal powers of $1/\xi$ at different n look identical. The only difference consists in the definition of the inner product. It is interesting to note that this fact also follows from the following remark: the transition matrix is transformed under the U(p,q) isotopical group as

$$\hat{S}(\xi) \to \hat{S}'(\xi) = \mathcal{R}\hat{S}(\xi)\bar{\mathcal{R}}$$

with $\mathcal{R} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$, $R \in U(p,q)$. The element $S_{11}(\xi)$ generating the polynomial conservation law is invariant under this transformation, so are the latters. Therefore, in our case the isotopic group commutes with the infinite-parameter Abelian group, generating high integrals of motion. As we could see, this is not true for the block $S_{\alpha\beta}(\xi)$ and, hence, for the nonlocal conservation laws associated with it.

REFERENCES

1. Zakharov V.E., Shabat A.B. ZhETF, 1971, vol.61, No.1, p.118.
2. Zakharov V.E., Shabat A.B. ZhETF, 1973, vol.64, No.5, p.1627.
3. Faddeev L.D. JINR, P2-12462, Dubna, 1979, p.249; Sklyanin E.K. Dokl. AN SSSR, 1979, vol.244, No.6, p.1337.
4. Beresin F.A., Pokhil G.P., Finkelberg V.M. Vestnik MGU, seria 1, 1964, No.1, p.21; Mc Guire J.B. Journ.Math. Phys., 1964, vol.5, No.5, p.439.
5. Lieb E.H., Liniger W. Phys.Rev., 1963, vol.130, No.4, p.1605; Lieb E.H., ibid., 1963, vol.130, No.4, p.1616.
6. Manakov S.V. ZhETF, 1973, vol.65, No.2, p.505.
7. Kulish P.P. Preprint LOMI P-3-79, Leningrad, 1979 (in Russian).
8. Teoriya solitonov, ed. by S.P.Novikov. "Nauka", M., 1980 (in Russian).
9. Zakharov V.E., Manakov S.V. Teor.Mat.Fiz., 1974, vol.19, No.3, p.332.
10. Lindner V., Fedyanin V.K. phys.stat.sol.(b), 1978, vol.89, p.123.
11. Makhankov V.G. Phys.Lett., 1981, vol.81A, No.2,3, p.156.
12. Makhankov V.G., Makhaldiani N.V., Pashaev O.K. JINR, E2-80-233, Dubna, 1980.
13. Makhankov V.G., Makhaldiani N.V., Pashaev O.K. Phys.Lett., 1981, vol.81A, No.2,3, p.161.
14. Barut A.O., Raczka R. Theory of Group Representations and Applications, PWN-Polish Scientific Publishers, Warszawa, 1977.
15. Alberty J.M., Koikawa T., Sasaki R. Preprint Utrecht, November, 1980.
16. Olver F.W.J. Introduction to Asymptotics and Special Functions. Academic Press, NY and London, 1974.
17. Ter Haar D. University of Oxford, 54/77, Oxford, 1977.

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