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ON PHASE SPACE REPRESENTATIONS. II

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Phase space representations (PSR's)[x] translate quantum theory into a language, close to the classical theory one: c-number coordinate and momentum variables x and p (e.g., such as the coherent state expectation values of the coordinate and momentum operators) participate simultaneously. Further uniformity is achieved by using as an equation of motion the Liouville equation (a linear partial differential equation) for a phase space distribution function instead of the Newton (or Hamilton) equations in the classical theory and the Schrödinger equation in the quantum one. This equation is of the first order in the classical mechanics, and, in the quantum one (in PSR's) it includes, in addition, higher derivatives multiplied by the Planck constant $\hbar$ ($\hbar$ enters like a "coupling constant"). The quantum PSR-2 (Wigner representation) Liouville equation, proposed by Wigner /1/, looks as the most immediate generalization of the classical Liouville equation.

In the classics a general solution of the Cauchy problem of the Liouville equation is expressible in terms of characteristics(in fact, its particular solutions). The Hamilton equations appear as equations for them, which hence are trajectories. In the quantum case characteristic manifolds are more complex. In the classical mechanics the phase space distribution function ρ is positive definite, and in the quantum mechanics it may be positive definite too (as in PSR-1[xx]) or not (as in PSR-2). In any case these PSR distribution functions contain however the same total information as the wave function does.

In what follows we expose evolution laws of quantum mechanics in terms of the PSR's using a formalism of nonoperator and left and right operator representatives. Besides the quantum Liouville equation and its classical limit ($\hbar \to 0$), other forms of quantum equations of motion are considered in the Schrödinger, Heisenberg and interaction pictures in terms of PSR's. For many other aspects of the Liouville equation see refs./1-6/. We use terms of ref./7/, where some other references may be found.

[x] There are many ways to define PSR's in quantum theory unlike the classics.

[xx] Recall that it results from the coherent state representation if one keeps only diagonal matrix elements.

Further it is shown that the PSR distribution function carries the same total information as the wave function does. In Appendix multiplication rules for nonoperator representatives are derived in some PSR's.

Evolution in the Schrödinger and Heisenberg pictures. Starting with the usual quantum evolution laws of a density operator("matrix") $\hat{\rho}(t)$ in the Schrödinger picture and any operator $\mathbb{F}(t)$, which does not depend explicitly on time, in the Heisenberg picture

$$\hat{\rho}(t)=e^{-i\hbar^{-1}\hat{H}(t-t_0)}\hat{\rho}(t_0)e^{i\hbar^{-1}\hat{H}(t-t_0)}, \tag{1}$$

$$\hat{\mathbb{F}}(t)=e^{i\hbar^{-1}\hat{H}(t-t_0)}\hat{\mathbb{F}}(t_0)e^{-i\hbar^{-1}\hat{H}(t-t_0)}, \tag{2}$$

we translate them into PSR's. We shall imply mainly two of them: PSR-1 (based on the coherent state representation)

$$\rho_1(xpt) = \mathrm{Tr}\left(|xp\rangle\langle xp|\hat{\rho}(t)\right)=\langle xp|\hat{\rho}(t)|xp\rangle, \tag{3}$$

and PSR-2 (the Wigner representation)

$$\rho_2(xpt)=\mathrm{Tr}(\Lambda^{-1}|xp\rangle\langle xp|\hat{\rho}(t))=\Lambda^{-1}\langle xp|\hat{\rho}(t)|xp\rangle= \tag{4.a}$$

$$=\hbar^n\int d^n\alpha\, e^{-ip\alpha}\langle x+\tfrac{\hbar}{2}\alpha|\hat{\rho}(t)|x-\tfrac{\hbar}{2}\alpha\rangle= \tag{4.b}$$

$$=\hbar^n\int d^n\beta\, e^{ix\beta}\langle p+\tfrac{\hbar}{2}\beta|\hat{\rho}(t)|p-\tfrac{\hbar}{2}\beta\rangle. \tag{4.c}$$

In the same fashion we define $\mathbb{F}_1(xpt)$ and $\mathbb{F}_2(xpt)$. See ref./7/ for definition of the coherent states $|xp\rangle$ and the inverse Gauss transformation operator Λ^{-1}. The state vectors $|x\pm\frac{\hbar}{2}\alpha\rangle$ and $|p\pm\frac{\hbar}{2}\beta\rangle$ in (4.6) and (4.c) are eigenvectors of the standard coordinate and momentum operators, respectively. Any number n of degrees of freedom is implied (any number of particles and dimensions of space). For identical particles symmetrization (antisymmetrization) must be taken into account.

If we insert eq. (1) into eqs. (3) or (4) and do analogously with $\hat{F}(t)$ and use the Dirac representation theory x), we obtain the Schrödinger picture evolution law of the density matrix and the

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x) Somewhat extended: a) both the left and right representatives of operators participate simultaneously, b) either expectation values (cf. eq. (3)) or transformed ones (cf.eq. (4)) are taken as nonoperator representatives, so that the operators act simultaneously on two state vectors (unlike the situation in the x- or p-representations). The off-diagonal matrix elements constitute a redundant information (due to overcompleteness of the coherent state set). They are deducible from the diagonal ones.

Heisenberg picture evolution law of any operator, which does not depend explicitly on time, in PSR

$$\rho(xpt)=e^{-i\hbar^{-1}(H^{\ell}-H^{r})(t-t_0)}\rho(xpt_0)=e^{-\mathcal{L}(t-t_0)}\rho(xpt_0), \tag{5}$$

$$\mathbb{F}(xpt)=e^{i\hbar^{-1}(H^{\ell}-H^{r})(t-t_0)}\mathbb{F}(xpt_0)=e^{\mathcal{L}(t-t_0)}\mathbb{F}(xpt_0). \tag{6}$$

These forms of evolution laws are valid not only for all PSR's, but for all representations in general x) (e.g., for x- or p-representations); only variables x and p in $\rho(xpt)$ and $\mathbb{F}(xpt)$ must be substituted by the corresponding ones. The left and right representatives of the Hamiltonian H^{ℓ} and H^{r} (which are differential operators acting on x and p) are obtained by substituting in $\hat{H}$ the operators $\hat{x}$ and $\hat{p}$ by their corresponding left and right representatives x^{ℓ}, p^{ℓ} and x^{r}, p^{r}. The left representatives are multiplied in the same order, as the original operators, while the right ones in the inverse order. Note that the right representatives are complex conjugate to the left ones. The left representatives commute with the right ones xx), and operators of one kind satisfy the usual commutation relations (except for the sign for r-representatives)

$$[x_k^{\ell,r}, x_m^{\ell,r}]=0, \quad [x_k^{\ell,r}, p_m^{\ell,r}]=\pm i\hbar\delta_{km}, \quad [p_k^{\ell,r}, p_m^{\ell,r}]=0. \tag{7}$$

The Hamiltonian $\hat{H}$ may be any function of $\hat{x}$ and $\hat{p}$, however, we usually shall exploit the Hamiltonian of the special form

$$\hat{H}=\sum_{i=1}^{n}\frac{\hat{p}_i^2}{2m_i}+V(\hat{x}), \qquad H^{\ell,r}=\sum\frac{p^{\ell,r\,2}}{2m}+V(x^{\ell,r}). \tag{8}$$

Various representations differ in form of the operators x^l, p^l, x^r, p^r. In PSR-2

$$x_k^{\ell,r}=x_k\pm\frac{i\hbar}{2}\frac{\partial}{\partial x_k}, \quad p_k^{\ell,r}=p_k\mp\frac{i\hbar}{2}\frac{\partial}{\partial x_k}. \tag{9}$$

For the representatives in PSR-1 and in some other cases see in ref./7/xxx). For the evolution generator in eqs. (5) and (6) we get

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x) That is, with other bases instead of $|xp\rangle\langle xp|$ or $\Lambda^{-1}|xp\rangle\langle xp|$.

xx) This is a general rule for all associative theories, i.e., when products of operators are associative. In nonassociative cases left and right representatives become noncommutative with each other (see ref./8/).

xxx) In the x-representation, i.e., for

$$\rho(xx't)=\mathrm{Tr}\left(|x'\rangle\langle x|\hat{\rho}(t)\right)=\langle x|\hat{\rho}(t)|x'\rangle, \; \mathbb{F}(xx't)=\langle x|\hat{\mathbb{F}}(t)|x'\rangle,$$

we have the familiar representatives

$$x^{\ell}=x, \; p^{\ell}=-i\hbar\frac{\partial}{\partial x}, \; x^{r}=x', \; p^{r}=i\hbar\frac{\partial}{\partial x'}.$$

$$\mathcal{L}=i\hbar^{-1}(H^{\ell}-H^{r})=\mathcal{L}_0+\mathcal{L}_I\ , \tag{10}$$

$$\mathcal{L}_0=i\hbar^{-1}(H_0^{\ell}-H_0^{r})=i\hbar^{-1}\left(\sum\frac{p^{\ell 2}}{2m}-\sum\frac{p^{r2}}{2m}\right)=$$

$$=i\hbar^{-1}\sum\frac{\left(p-\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)^2}{2m}-i\hbar^{-1}\sum\frac{\left(p+\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)^2}{2m}=\sum\frac{p}{m}\frac{\partial}{\partial x}\ , \tag{11}$$

$$\mathcal{L}_I=i\hbar^{-1}(H_I^{\ell}-H_I^{r})=i\hbar^{-1}\left[V(x^{\ell})-V(x^{r})\right]=$$

$$=i\hbar^{-1}\left[V\left(x+\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)-V\left(x-\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)\right]. \tag{12}$$

The operators $\mathcal{L},\mathcal{L}_0$ and $\mathcal{L}_I$ may be called the generalized total, free and interaction Liouvillians (by analogy with the classics, see below). Last expressions (11) and (12) are written down for a Hamiltonian of the form (8) in PSR-2. Note, that the Planck constant falls out from the free PSR-2 Liouvillian and the latter turns out to be the same as in the classics.

If initial and final states are described equally by operators

$$\hat{\rho}_{1i}=|x_0p_0\rangle\langle x_0p_0|\ ,\qquad \hat{\rho}_{1f}=|xp\rangle\langle xp|\ ,\qquad \text{in PSR-1} \tag{13}$$

$$\hat{\rho}_{2i}=\Lambda_0^{-1}|x_0p_0\rangle\langle x_0p_0|\ ,\quad \hat{\rho}_{2f}=\Lambda^{-1}|xp\rangle\langle xp|\ ,\qquad \text{in PSR-2} \tag{14}$$

and, therefore, ρ_1 and ρ_2 are chosen to satisfy the initial conditions

$$\rho_1(xpt_0,x_0p_0t_0)=\mathrm{Tr}(\hat{\rho}_{1f}\hat{\rho}_{1i})=|\langle xp|x_0p_0\rangle|^2=e^{-(2\hbar)^{-1}\left(A(x-x_0)^2+A^{-1}(p-p_0)^2\right)} \tag{15}$$

$$\rho_2(xpt_0,x_0p_0t_0)=\mathrm{Tr}(\hat{\rho}_{2f}\hat{\rho}_{2i})=\Lambda^{-1}\Lambda_0^{-1}|\langle xp|x_0p_0\rangle|^2=$$

$$=(2\pi\hbar)^n\delta\ (x-x_0)\delta\ (p-p_0)\ ^{x)}, \tag{16}$$

respectively, we obtain the distribution functions

$$\rho_1(xpt,x_0p_0t_0)=|\langle xp|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|x_0p_0\rangle|^2= \tag{17.a}$$

$$=e^{-i\hbar^{-1}(H^{\ell}-H^{r})(t-t_0)}|\langle xp|x_0p_0\rangle|^2= \tag{17.b}$$

$$=e^{i\hbar^{-1}(H^{\ell}-H^{r})^{0}(t-t_0)}|\langle xp|x_0p_0\rangle|^2\ , \tag{17.c}$$

$$\rho_2(xpt,x_0p_0t_0)=\Lambda^{-1}\Lambda_0^{-1}|\langle xp|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|x_0p_0\rangle|^2= \tag{18.a}$$

$$=e^{-i\hbar^{-1}(H^{\ell}-H^{r})(t-t_0)}\Lambda^{-1}\Lambda_0^{-1}|\langle xp|x_0p_0\rangle|^2= \tag{18.b}$$

$$=e^{-i\hbar^{-1}(H^{\ell}-H^{r})(t-t_0)}(2\pi\hbar)^n\delta\ (x-x_0)\delta\ (p-p_0)= \tag{18.c}$$

$$=e^{i\hbar^{-1}(H^{\ell}-H^{r})^{0}(t-t_0)}\Lambda^{-1}\Lambda_0^{-1}|\langle xp|x_0p_0\rangle|^2= \tag{18.d}$$

$$=e^{i\hbar^{-1}(H^{\ell}-H^{r})^{0}(t-t_0)}(2\pi\hbar)^n\delta\ (x-x_0)\delta\ (p-p_0)\ . \tag{18.e}$$

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x) Here and in what follows δ's stand for n-dimesional δ-functions δ^n.

In eq. (17.c), (18.d) and (18.e) the Hamiltonian $\hat{H}$ is converted into the operators H^{ℓ} and H^{r}, acting on the variables x_0, p_0, what is indicated by the sign $\circ$ of $(H^{\ell}-H^{r})^{\circ}$. H^{ℓ} and H^{r} in the variables x_0, p_0 have the old structure (only $t-t_0 \to t_0 - t$ in eqs. (17.c) and (18.d),(18.e)).

The density $\rho_1(xpt)$ ($\rho_1(xpt, x_0p_0t_0)$) is positive definite and can be considered as a probability density (however, for transitions into (between) the nonorthogonal states (13)). As to the expressions (14), they cannot be considered as true density matrices, since their diagonal matrix elements are not positive definite (this is easily seen, using an oscillator basis, see ref. /I.24.f/x), Appendix C), and hence the phase space density $\rho_2(cpt)$ $(\rho_2(xpt, x_0p_0t_0))$ has the same defect. However, despite this fact ρ_2 permits us to obtain all the usual positive definite probabilities of the quantum mechanics (see pp. 17, 18 below). In particular, one can, in principle, transform ρ_2 into ρ_1 as follows

$$\rho_1(xpt) = \Lambda \rho_2(xpt)\,, \tag{19}$$

$$\rho_1(xpt, x_0p_0t_0) = \Lambda\Lambda_0 \rho_2(xpt, x_0p_0t_0)\,, \tag{20}$$

i.e., using the Gauss transformation, or, in other words, by a convolution with a given normal distribution. Let us stress that both ρ_1 and ρ_2 contain the same comprehensive information as the wave function does. ρ_2 is more convenient for obtaining some transition probabilities (e.g., for transitions $x_0 \to x, p_0 \to p$, $x_0 \to p, p_0 \to x$) and ρ_1 for some others (e.g., for transitions between oscillator states, and, therefore, in quantum field theory), see pp. 16–18 below.

<u>Equations of motion.</u> Expressions (1),(2),(5),(6),(17), and (18) can be considered as solutions of the following equations of motion: the von Neumann equation for evolution of the density matrix (in the Schrödinger picture) and Heisenberg-Born-Jordan-Dirac equation for evolution for any operator $\hat{\mathbb{F}}(t)$, which does not depend on time explicitly, (in the Heisenberg picture)

$$\frac{d}{dt}\hat{\rho}(t) = -i\hbar^{-1}[\hat{H}, \hat{\rho}(t)]\,, \tag{21}$$

$$\frac{d}{dt}\hat{\mathbb{F}}(t) = i\hbar^{-1}[\hat{H}, \hat{\mathbb{F}}(t)]\,, \tag{22}$$

and their equivalents in PSR's xx)

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x) The reference in the list of references of Part I (ref. /7/).

xx) And in any other representations, if suitable variables are used instead of x and p.

$$\frac{\partial}{\partial t}\rho(xpt) = -i\hbar^{-1}(H^{\ell}-H^{r})\rho(xpt)\,, \tag{23.a}$$

$$\frac{\partial}{\partial t}\rho(xpt,x_0p_0t_0) = i\hbar^{-1}(H^{\ell}-H^{r})^{0}\rho(xpt,x_0p_0t_0)\,, \tag{23.b}$$

$$\frac{\partial}{\partial t}\mathbb{F}(xpt) = i\hbar^{-1}(H^{\ell}-H^{r})\mathbb{F}(xpt)\,. \tag{24}$$

With the special Hamiltonian (8) these became

$$\frac{\partial}{\partial t}\rho(xpt) = -i\hbar^{-1}\Big(\sum\frac{p^{\ell 2}}{2m}+V(x^{\ell})-\sum\frac{p^{r2}}{2m}-V(x^{r})\Big)\rho(xpt)\,, \tag{25.a}$$

$$\frac{\partial}{\partial t}\rho(xpt,x_0p_0t_0) = i\hbar^{-1}\Big(\sum\frac{p_0^{\ell 2}}{2m}+V(x_0^{\ell})-\sum\frac{p_0^{r2}}{2m}-V(x_0^{r})\Big)\rho(xpt,x_0p_0t_0)\,, \tag{25.b}$$

$$\frac{\partial}{\partial t}\mathbb{F}(xpt) = i\hbar^{-1}\Big(\sum\frac{p^{\ell 2}}{2m}+V(x^{\ell})-\sum\frac{p^{r2}}{2m}-V(x^{r})\Big)\mathbb{F}(xpt) \tag{26}$$

and in PSR-2 explicitly

$$\frac{\partial}{\partial t}\rho_2(xpt) = -i\hbar^{-1}\Big\{\sum\frac{\big(p-\frac{i\hbar}{2}\frac{\partial}{\partial x}\big)^2}{2m}+V\Big(x+\frac{i\hbar}{2}\frac{\partial}{\partial p}\Big)-\sum\frac{\big(p+\frac{i\hbar}{2}\frac{\partial}{\partial x}\big)^2}{2m}-V\Big(x-\frac{i\hbar}{2}\frac{\partial}{\partial p}\Big)\Big\}\rho_2(xpt) =$$

$$= -\Big\{\sum\frac{p}{m}\frac{\partial}{\partial x}+i\hbar^{-1}\Big[V\Big(x+\frac{i\hbar}{2}\frac{\partial}{\partial p}\Big)-V\Big(x-\frac{i\hbar}{2}\frac{\partial}{\partial p}\Big)\Big]\Big\}\rho_2(xpt)\,, \tag{27.a}$$

$$\frac{\partial}{\partial t}\rho_2(xpt,x_0p_0t_0) =$$

$$= i\hbar^{-1}\Big\{\sum\frac{\big(p_0-\frac{i\hbar}{2}\frac{\partial}{\partial x_0}\big)^2}{2m}+V\Big(x_0+\frac{i\hbar}{2}\frac{\partial}{\partial p_0}\Big)-\sum\frac{\big(p_0+\frac{i\hbar}{2}\frac{\partial}{\partial x_0}\big)^2}{2m}-V\Big(x_0-\frac{i\hbar}{2}\frac{\partial}{\partial p_0}\Big)\Big\}\rho_2(xpt,x_0p_0t_0) =$$

$$= \Big\{\sum\frac{p_0}{m}\frac{\partial}{\partial x_0}+i\hbar^{-1}\Big[V\Big(x_0+\frac{i\hbar}{2}\frac{\partial}{\partial p_0}\Big)-V\Big(x_0-\frac{i\hbar}{2}\frac{\partial}{\partial p_0}\Big)\Big]\Big\}\rho_2(xpt,x_0p_0t_0)\,, \tag{27.b}$$

$$\frac{\partial}{\partial t}\mathbb{F}_2(xpt) = i\hbar^{-1}\Big\{\sum\frac{\big(p-\frac{i\hbar}{2}\frac{\partial}{\partial x}\big)^2}{2m}+V\Big(x+\frac{i\hbar}{2}\frac{\partial}{\partial p}\Big)-\sum\frac{\big(p+\frac{i\hbar}{2}\frac{\partial}{\partial x}\big)^2}{2m}-V\Big(x-\frac{i\hbar}{2}\frac{\partial}{\partial p}\Big)\Big\}\mathbb{F}_2(xpt) =$$

$$= \Big\{\sum\frac{p}{m}\frac{\partial}{\partial x}+i\hbar^{-1}\Big[V\Big(x+\frac{i\hbar}{2}\frac{\partial}{\partial p}\Big)-V\Big(x-\frac{i\hbar}{2}\frac{\partial}{\partial p}\Big)\Big]\Big\}\mathbb{F}_2(xpt)\,. \tag{28}$$

Equation (27.a) is the Wigner equation /1/ (in a somewhat different form: Wigner has not used terms of the left and right representatives). Equations (23), (25), and (27) generalize the Liouville equation of classical mechanics to the quantum case. Equations (24), (26), and (28) also generalize classical equations of motion for dynamical variables.

<u>The norm conservation</u> easily follows from the Liouville equation,

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\rho_k(xpt) = \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\rho_k(xpt_0) = \mathrm{Tr}(\hat{\rho}(t_0))^{x)} \tag{29.a}$$

(k=1 or 2). For densities (17) and (18) we have

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\rho_k(xpt,x_0p_0t_0) = \frac{1}{(2\pi\hbar)^n}\int dx_0\,dp_0\,\rho_k(xpt,x_0p_0t_0) = 1\,. \tag{29.b}$$

However, it is easily to obtain both (29.a) and (29.b) without the Liouville equation, but using definitions (3), (4.a), (17), and (18) and the completeness relations /7/

x) Here and in what follows dx and dp stand for d^nx and d^np.

$$(2\pi\hbar)^{-n}\int dx\,dp|xp\rangle\langle xp| = (2\pi\hbar)^{-n}\int dx\,dp\,\Lambda^{-1}|xp\rangle\langle xp| = 1\,. \tag{30}$$

Note, that integrating the Liouville equation only over p, one can deduce the usual continuity equations both in classical and in quantum cases.

<u>In linear cases</u> (equations of motion for coordinates are linear, and hence the Hamiltonian is at most bilinear) the Planck constant $\hbar$ falls out from the total Liouvillian $i\hbar^{-1}(H^{\ell}-H^{r})$ in PSR-2 completely and equation (27.a) exactly coincides with the classical Liouville equation (the same is true for equation (28)). We give some examples.

<u>Free particles</u>. Hamiltonian: $\hat{H}=\sum\frac{\hat{p}^2}{2m}$,

Liouvillian: $$\mathcal{L} = i\hbar^{-1}(H^{\ell}-H^{r}) = \sum\frac{p}{m}\frac{\partial}{\partial x}\,, \tag{31}$$

equation: $$\frac{\partial}{\partial t}\rho_2(xpt) = -\sum\frac{p}{m}\frac{\partial}{\partial x}\rho_2(xpt) = \sum\frac{p_0}{m}\frac{\partial}{\partial x_0}\rho_2(xpt)\,, \tag{32}$$

solution: $$\rho_2(xpt,x_0p_0t_0) = e^{-\tau\sum\frac{p}{m}\frac{\partial}{\partial x}}(2\pi\hbar)^n\delta(x-x_0)\delta(p-p_0) =$$
$$= e^{\tau\sum\frac{p_0}{m}\frac{\partial}{\partial x_0}}(2\pi\hbar)^n\delta(x-x_0)\delta(p-p_0) = (2\pi\hbar)^n\delta(x-x_0-p_0(t-t_0))\delta(p-p_0)\,, \tag{33}$$

subjected to initial condition (16), $\tau = t-t_0$.

<u>Motion under a constant force</u>. Hamiltonian: $\hat{H}=\sum\left(\frac{\hat{p}^2}{2m}-F\hat{x}\right)$,

Liouvillian: $$\mathcal{L} = \sum\left(\frac{p}{m}\frac{\partial}{\partial x}+F\frac{\partial}{\partial p}\right), \tag{34}$$

equation: $$\frac{\partial}{\partial t}\rho_2(xpt) = -\sum\left(\frac{p}{m}\frac{\partial}{\partial x}+F\frac{\partial}{\partial p}\right)\rho_2(xpt)\,, \tag{35}$$

solution: $$\rho_2(xpt,x_0p_0t_0) = (2\pi\hbar)^n\delta\left(x-\frac{p}{m}(t-t_0)+\frac{F}{2m}(t-t_0)^2-x_0\right)\delta(p-F(t-t_0)-p_0) =$$
$$= (2\pi\hbar)^n\delta\left(x-x_0-\frac{p_0}{m}(t-t_0)-\frac{F}{2m}(t-t_0)^2\right)\delta(p-p_0-F(t-t_0)) \tag{36}$$

with the same initial condition (16).

<u>Harmonic oscillators</u> (isotropic or nonisotropic, rotation, any Lissajous curves, etc.). Hamiltonian: $\hat{H}=\sum\left(\frac{\hat{p}^2}{2m}+\frac{m\omega^2}{2}\hat{x}^2\right)$,

Liouvillian: $$\mathcal{L} = i\hbar^{-1}(H^{\ell}-H^{r}) = \sum\left(\frac{p}{m}\frac{\partial}{\partial x}-m\omega^2x\frac{\partial}{\partial p}\right), \tag{37}$$

equation: $$\frac{\partial}{\partial t}\rho_2(xpt) = -\sum\left(\frac{p}{m}\frac{\partial}{\partial x}-m\omega^2x\frac{\partial}{\partial p}\right)\rho_2(xpt)\,, \tag{38}$$

solution: $$\rho_2(xpt,x_0p_0t_0) =$$
$$= (2\pi\hbar)^n\delta\left(x-x_0\cos\omega\tau-\frac{p_0}{m\omega}\sin\omega\tau\right)\delta(p-p_0\cos\omega\tau+m\omega x_0\sin\omega\tau) =$$
$$= (2\pi\hbar)^n\delta\left(x\cos\omega\tau-\frac{p}{m\omega}\sin\omega\tau-x_0\right)\delta(p\cos\omega\tau+m\omega x\sin\omega\tau-p_0)\,. \tag{39}$$

. Here again the initial solution (16) was assumed.

Thus, in all linear cases the Liouville equation in the Wigner representation (PSR-2) and its solution subjected to initial condition (16) coincide with their classical limits. However, the above phase space densities $\rho_2(xpt,x_0p_0t_0)$ permit us to get all the usual quantum-mechanical probabilities and other quantities (see pp.16-18 below).

The classical limit $\hbar=0$ in general case gives the usual classical Liouvillian

$$\mathcal{L}_c=\mathcal{L}|_{\hbar=0}=i\hbar^{-1}(H^{\ell}-H^{r})|_{\hbar=0}=\sum\left(\frac{\partial H}{\partial p}\frac{\partial}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial}{\partial p}\right)=\sum\left(\frac{p}{m}\frac{\partial}{\partial x}-\frac{\partial V}{\partial x}\frac{\partial}{\partial p}\right) \quad (40)$$

(the last expression for the Hamiltonian of form (8)) and the usual classical Liouville equation

$$\frac{\partial}{\partial t}\rho(xpt)=-i\hbar^{-1}(H^{\ell}-H^{r})|_{\hbar=0}\rho(xpt)=-\sum\left(\frac{\partial H}{\partial p}\frac{\partial}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial}{\partial p}\right)\rho(xpt) \quad (41.a)$$

or

$$\frac{\partial}{\partial t}\rho(xpt,x_0p_0t_0)=i\hbar^{-1}(H^{\ell}-H^{r})^{0}|_{\hbar=0}\rho(xpt,x_0p_0t_0)=\sum\left(\frac{\partial H^{0}}{\partial p_0}\frac{\partial}{\partial x_0}-\frac{\partial H^{0}}{\partial x_0}\frac{\partial}{\partial p_0}\right)\rho(xpt,x_0p_0t_0). \quad (41.b)$$

Its formal solution with the initial condition (16) is

$$\begin{aligned}
&(2\pi\hbar)^{-n}\rho(xpt,x_0p_0t_0)|_{\hbar=0}=\rho_c(xpt,x_0p_0t_0)=\\
&=e^{-i\hbar^{-1}\tau(H^{\ell}-H^{r})}|_{\hbar=0}\delta(x-x_0)\delta(p-p_0)=e^{-\tau\sum\left(\frac{\partial H}{\partial p}\frac{\partial}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial}{\partial p}\right)}\delta(x-x_0)\delta(p-p_0)=\\
&=e^{-\tau\mathcal{L}_c}\delta(x-x_0)\delta(p-p_0)=\delta(\mathbb{x}(x,p,t_0-t)-x_0)\delta(\mathbb{p}(x,p,t_0-t)-p_0)= \quad (42.a)\\
&=e^{i\hbar^{-1}\tau(H^{\ell}-H^{r})^{0}}|_{\hbar=0}\delta(x-x_0)\delta(p-p_0)=e^{\tau\sum\left(\frac{\partial H^{0}}{\partial p_0}\frac{\partial}{\partial x_0}-\frac{\partial H^{0}}{\partial x_0}\frac{\partial}{\partial p_0}\right)}\delta(x-x_0)\delta(p-p_0)=\\
&=e^{\tau\mathcal{L}_c^{0}}\delta(x-x_0)\delta(p-p_0)=\delta(x-\mathbb{x}(x_0,p_0,t-t_0))\delta(p-\mathbb{p}(x_0,p_0,t-t_0)) \quad (42.b)
\end{aligned}$$

where

$$\mathbb{x}(x_0,p_0,t-t_0)=\Lambda_0^{-1}\langle x_0p_0|\hat{\mathbb{x}}(t)|x_0p_0\rangle|_{\hbar=0}=e^{\tau\mathcal{L}_c^{0}}x_0\cdot 1=e^{\tau\mathcal{L}_c^{0}}x_0e^{-\tau\mathcal{L}_c^{0}} \quad (43)$$

$$\mathbb{p}(x_0,p_0,t-t_0)=\Lambda_0^{-1}\langle x_0p_0|\hat{\mathbb{p}}(t)|x_0p_0\rangle|_{\hbar=0}=e^{\tau\mathcal{L}_c^{0}}p_0\cdot 1=e^{\tau\mathcal{L}_c^{0}}p_0e^{-\tau\mathcal{L}_c^{0}} \quad (44)$$

$$\mathbb{x}(x,p,t_0-t)=\Lambda^{-1}\langle xp|\hat{\mathbb{x}}(t_0-\tau)|xp\rangle|_{\hbar=0}=e^{-\tau\mathcal{L}_c}x\cdot 1=e^{-\tau\mathcal{L}_c}x\,e^{\tau\mathcal{L}_c} \quad (45)$$

$$\mathbb{p}(x,p,t_0-t)=\Lambda^{-1}\langle xp|\hat{\mathbb{p}}(t_0-\tau)|xp\rangle|_{\hbar=0}=e^{-\tau\mathcal{L}_c}p\cdot 1=e^{-\tau\mathcal{L}_c}p\,e^{\tau\mathcal{L}_c} \quad (46)$$

Last expressions (42.a) and (42.b) are obtained from the preceding ones by the following procedure. Let $\rho(x,p,t_0)$ be an arbitrary initial function of x and p, then the general Cauchy problem solution of the Liouville equation (41.a) is given by [x)]

$$\rho(xpt)=e^{-\tau L_c}\rho(xpt_0)\cdot 1=e^{-\tau L_c}\rho(xpt_0)e^{\tau L_c}=$$

$$=\rho\left(e^{-\tau L_c}xe^{\tau L_c},e^{-\tau L_c}pe^{\tau L_c},t_0\right)=\rho\left(\mathbb{X}(x,p,t_0-t),\mathbb{P}(x,p,t_0-t),t_0\right). \quad (47)$$

Because L_c is a first order partial differential operator, the expressions $e^{-\tau L_c}f(x,p)e^{\tau L_c}$ and, in particular, $e^{-\tau L_c}xe^{\tau L_c}$ and $e^{-\tau L_c}pe^{\tau L_c}$ are functions (not operators), unlike general situation in quantum mechanics. This is clear from series expansions in $\tau=t-t_0$, e.g.,

$$\mathbb{X}(x,p,t_0-t)=x-\tau\left[\sum\left(\frac{\partial H}{\partial p}\frac{\partial}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial}{\partial p}\right),x\right]+\frac{\tau^2}{2}\left[L_c\left[L_c,x\right]\right]+\cdots \quad (48)$$

(each commutator is the Poisson bracket) [xx).]

Let us differentiate the last but one expressions (43) and (44) with respect to t, and once place the operator L_c^0 on the left of $e^{\tau L_c^0}$ and then on the right of it. The first way shows that $\mathbb{X}(x_0,p_0,t-t_0)$ and $\mathbb{P}(x_0,p_0,t-t_0)$ are particular solutions of the Liouville equation (41.b)[xxx)]

$$\frac{\partial}{\partial t}\mathbb{X}(x_0,p_0,t-t_0)=L_c^0\,\mathbb{X}(x_0,p_0,t-t_0),\quad \mathbb{X}(x_0,p_0,0)=x_0, \quad (49)$$

$$\frac{\partial}{\partial t}\mathbb{P}(x_0,p_0,t-t_0)=L_c^0\,\mathbb{P}(x_0,p_0,t-t_0),\quad \mathbb{P}(x_0,p_0,0)=p_0. \quad (50)$$

Solutions (42.b), (43), and (44) of eq. (41.b) differ only in initial values. The same is true for solutions (42.a), (45), and (46) of eq. (41.a), the general solution (47) being constructed in terms of particular ones (45) and (46). The second way demonstrates that $\mathbb{X}(x_0,p_0,t-t_0)$ and $\mathbb{P}(x_0,p_0,t-t_0)$ are the general solution of the Hamilton equations

$$\frac{\partial}{\partial t}\mathbb{X}(x_0,p_0,t-t_0)=\frac{\partial H(\mathbb{X},\mathbb{P})}{\partial\mathbb{P}(x_0,p_0,t-t_0)},\ \frac{\partial}{\partial t}\mathbb{P}(x_0,p_0,t-t_0)=-\frac{\partial H(\mathbb{X},\mathbb{P})}{\partial\mathbb{X}(x_0,p_0,t-t_0)} \quad (51)$$

---

x) For eq. (41.b) with $\rho(xpt_0)=\delta(x-x_0)\delta(p-p_0)$ $e^{\tau L_c^0}\delta(x-x_0)\delta(p-p_0)=$

$$=e^{\tau L_c^0}\delta(x-x_0)\delta(p-p_0)e^{-\tau L_c^0}=\delta\left(x-e^{\tau L_c^0}x_0e^{-\tau L_c^0}\right)\delta\left(p-e^{\tau L^0}p_0e^{-\tau L^0}\right),$$

and we get the last expression (42.b).

xx) This way gives easily the above densities (36) and (39).

xxx) Analogously for $\mathbb{X}(x,p,t_0-t)$ and $\mathbb{P}(x,p,t_0-t)$.

(and hence $\mathbf{x}(x_0, p_0, t-t_0)$ is the general solution of the Newton equations).

The Liouville equation is a linear partial differential equation both in the classical and quantum mechanics. The classical Liouville equation is only of the first order. The integration problem of such an equation is well known to be reduced to integration of a set of ordinary first order differential equations (equations for characteristics of the Liouville equation and also of the Hamilton-Jacobi one), i.e., Hamilton equations (51), the characteristics being the trajectories (45),(46). Eq. (47) for the general solution of the Cauchy problem of eq. (41.a) gives a constructive proof of this theorem. Its usual proof in analysis is given in terms of implicit functions.

In quantum mechanics in nonlinear cases the PSR-2 Liouville equations contain, in addition to the first derivatives, the third derivatives with respect to p (and to x for general H's) as for the anharmonic oscillator $\left(V(x)=\frac{m\omega^2}{2}x^2+fx^3+gx^4\right)$

$$\frac{\partial}{\partial t}\rho(xpt, x_0p_0t_0)=\left[\frac{p_0}{m}\frac{\partial}{\partial x_0}-m\omega^2 x\frac{\partial}{\partial p}-f\left(3x_0^2\frac{\partial}{\partial p_0}-\frac{\hbar^2}{4}\frac{\partial^3}{\partial p_0^3}\right)-g\left(4x_0^3\frac{\partial}{\partial p_0}-\hbar^2 x_0\frac{\partial^3}{\partial p_0^3}\right)\right]\rho \tag{52}$$

higher odd derivatives for other polynomial cases, and derivatives of all odd orders for nonpolynomial cases, e.g., for the Coulomb problem

$$\frac{\partial}{\partial t}\rho=\left\{\frac{\vec{p}_0}{m}\frac{\partial}{\partial\vec{x}_0}+i\hbar^{-1}\left[\frac{e^2}{\sqrt{\left(\vec{x}_0+\frac{i\hbar}{2}\frac{\partial}{\partial\vec{p}_0}\right)^2}}-\frac{e^2}{\sqrt{\left(\vec{x}_0-\frac{i\hbar}{2}\frac{\partial}{\partial\vec{p}_0}\right)^2}}\right]\right\}\rho . \tag{53}$$

A transformation into the x-representation reduces these equations to second order ones.[x)] In any case characteristic manifolds are more complex than in the classics.

Determinism in the classics is explicitly demonstrated by the δ-functions of phase space trajectories (eqs. (42.a),(42.b) and (51)). In quantum mechanics in PSR-2 for the linear cases (see above) also formally there are trajectories (see above eqs. (33),(36) and (39)), however, the variables x and p have meaning expectation values. Other cases (i.e.,nonlinear ones) in PSR-2 suffer from indeterminism (nonuniqueness of predictions): the initial phase space δ-function (16) spreads out instantaneously, and loses the δ-function form and positivity. Note that in classical mechanics the Laplace determinism (the phase space δ-function remains a δ-function always) is also not realized in general.

[x)] $\partial/\partial t\; \rho(xx't)=-i\hbar^{-1}\left[-\hbar^2\Delta+\hbar^2\Delta'+V(x)-V(x')\right]\rho(xx't)$.

Nonoperator representatives of the coordinate and momentum operators undergo the evolution similar to the above classical one:[x)]

$$\mathbb{x}(x_0,p_0,t-t_0)=\Lambda_0^{-1}\langle x_0p_0|\hat{\mathbb{x}}(t)|x_0p_0\rangle=e^{\mathbb{L}^0\tau}x_0\cdot 1=e^{\mathbb{L}^0\tau}x_0e^{-\mathbb{L}^0\tau}\cdot 1 \quad (54.a)$$

$$\mathbb{p}(x_0,p_0,t-t_0)=\Lambda_0^{-1}\langle x_0p_0|\hat{\mathbb{p}}(t)|x_0p_0\rangle=e^{\mathbb{L}^0\tau}p_0\cdot 1=e^{\mathbb{L}^0\tau}p_0e^{-\mathbb{L}^0\tau}\cdot 1 \quad (54.b)$$

$$\mathbb{x}(x,p,t_0-t)=\Lambda^{-1}\langle xp|\hat{\mathbb{x}}(t_0-\tau)|xp\rangle=e^{-\mathbb{L}\tau}x\cdot 1=e^{-\mathbb{L}\tau}x\,e^{\mathbb{L}\tau}\cdot 1 \quad (54.c)$$

$$\mathbb{p}(x,p,t_0-t)=\Lambda^{-1}\langle xp|\hat{\mathbb{p}}(t_0-\tau)|xp\rangle=e^{-\mathbb{L}\tau}p\cdot 1=e^{-\mathbb{L}\tau}pe^{\mathbb{L}\tau}\cdot 1 \quad (54.d)$$

and also are the solution of the Liouville equation

$$\frac{\partial}{\partial t}\mathbb{x}(x_0,p_0,t-t_0)=\mathbb{L}^0\mathbb{x}(x_0,p_0,t-t_0)\cdot 1=[\mathbb{L}^0,\mathbb{x}]\cdot 1,\quad \mathbb{x}(x_0,p_0,0)=x_0 \quad (55.a)$$

$$\frac{\partial}{\partial t}\mathbb{p}(x_0,p_0,t-t_0)=\mathbb{L}^0\mathbb{p}(x_0,p_0,t-t_0)\cdot 1=[\mathbb{L}^0,\mathbb{p}]\cdot 1,\quad \mathbb{p}(x_0,p_0,0)=p_0 \quad (55.b)$$

$$\frac{\partial}{\partial t}\mathbb{x}(x,p,t_0-t)=\mathbb{L}\mathbb{x}(x,p,t_0-t)\cdot 1=[\mathbb{L},\mathbb{x}]\cdot 1,\quad \mathbb{x}(x,p,0)=x \quad (55.c)$$

$$\frac{\partial}{\partial t}\mathbb{p}(x,p,t_0-t)=\mathbb{L}\mathbb{p}(x,p,t_0-t)\cdot 1=[\mathbb{L},\mathbb{p}]\cdot 1,\quad \mathbb{p}(x,p,0)=p \quad (55.d)$$

although we cannot omit the units in last expressions (54) and (55). However, except for the linear cases the Hamilton equations are no longer satisfied. Incidentally, with the special Hamiltonian (8) due to the bilinearity of the kinetic energy one can write the first set of Hamilton equations[xx)]

$$\frac{\partial}{\partial t}\mathbb{x}(x_0,p_0,t-t_0)=m^{-1}\mathbb{p}(x_0,p_0,t-t_0)=\frac{\partial H(\mathbb{x},\mathbb{p})}{\partial\mathbb{p}(x_0,p_0,t-t_0)} \quad (56)$$

in general (beyond the linear case) not the second one.

One can suppose some meaning for the phase space density

$$\delta(x-\mathbb{x}(x_0,p_0,t-t_0))\delta(p-\mathbb{p}(x_0,p_0,t-t_0))=\delta(\mathbb{x}(x,p,t_0-t)-x_0)\delta(\mathbb{p}(x,p,t_0-t)-p_0), \quad (57)$$

however, it does not longer satisfy the Liouville equation.

The functions $\mathbb{x}(x_0,p_0,t-t_0)$ and $\mathbb{p}(x_0,p_0,t-t_0)$ do not define $\varrho(xpt,x_0p_0t_0)$ so simply as in classics, because characteristic manifolds of the quantum Liouville equation are of a more complex nature.

Evolution of the left and right operator representatives of the coordinate operators in the Heisenberg picture may be written down as follows

$$\mathbb{x}^\ell(t)=e^{i\hbar^{-1}H^\ell\tau}\mathbb{x}^\ell(t_0)\,e^{-i\hbar^{-1}H^\ell\tau}= \quad (58.a)$$

$$=e^{\mathbb{L}\tau}\mathbb{x}^\ell(t_0)e^{-\mathbb{L}\tau}= \quad (58.b)$$

$$=U^{\ell-1}(t,t_0)x^\ell(t)U^\ell(t,t_0)= \quad (58.c)$$

$$=W^{-1}(t,t_0)x^\ell(t)W(t,t_0) \quad (58.d)$$

[x)] Eqs. (54.a) and (54.b) are special cases of eq. (6) (where also x_0 and p_0 are more appropriate than x and p).

[xx)] The usual connection between momentum and velocity.

$$\mathbb{x}^{r}(t)=e^{-i\hbar^{-1}H^{r}\tau}\,\mathbb{x}^{r}(t_0)\,e^{i\hbar^{-1}H^{r}\tau}= \tag{59.a}$$

$$=e^{\mathcal{L}\tau}\,\mathbb{x}^{r}(t_0)\,e^{-\mathcal{L}\tau}= \tag{59.b}$$

$$=U^{r}(t,t_0)\,x^{r}(t)\,U^{r-1}(t,t_0)= \tag{59.c}$$

$$=W^{-1}(t,t_0)\,x^{r}(t)\,W(t,t_0), \tag{59.d}$$

where $x^{\substack{\ell\\ r}}(t)$ are left and right operator representatives of the coordinate operators in the interaction picture

$$x^{\substack{\ell\\ r}}(t)=e^{\pm i\hbar^{-1}H_0^{\substack{\ell\\ r}}\tau}\,\mathbb{x}^{\substack{\ell\\ r}}(t_0)\,e^{\mp i\hbar^{-1}H_0^{\substack{\ell\\ r}}}=e^{\mathcal{L}_0\tau}\,\mathbb{x}^{\substack{\ell\\ r}}(t_0)\,e^{-\mathcal{L}_0\tau}=$$
$$=x(t)\pm\frac{i\hbar}{2}\frac{\delta}{\delta f(t)}\ , \tag{60}$$

$$U^{\substack{\ell\\ r}}(t,t_0)=\left.\begin{matrix}T\\ \tilde{T}\end{matrix}\right\}\exp\left\{-i\hbar^{-1}\int_{t_0}^{t}dt'\,H_I\big(x^{\substack{\ell\\ r}}(t')\big)\right\}=$$
$$=\left.\begin{matrix}T\\ \tilde{T}\end{matrix}\right\}\exp\left\{-i\hbar^{-1}\int_{t_0}^{t}dt'\,H_I\Big(x(t')\pm\frac{i\hbar}{2}\frac{\delta}{\delta f(t')}\Big)\right\}, \tag{61}$$

$$\frac{d}{dt}U^{\ell}(t,t_0)=-i\hbar^{-1}H_I\big(x^{\ell}(t)\big)\,U^{\ell}(t,t_0)\ , \tag{62}$$

$$\frac{d}{dt}U^{r}(t,t_0)=-i\hbar^{-1}U^{r}(t,t_0)\,H_I\big(x^{r}(t)\big), \tag{63}$$

$$W(t,t_0)=U^{\ell}(t,t_0)\,U^{r-1}(t,t_0)=T\exp\left\{-i\hbar^{-1}\int_{t_0}^{t}dt'\big[H_I(x^{\ell}(t'))-H_I(x^{r}(t'))\big]\right\}$$
$$\equiv T\exp\left\{-\int_{t_0}^{t}dt'\,\mathcal{L}_I(t')\right\}$$
$$=T\exp\left\{-i\hbar^{-1}\int_{t_0}^{t}dt'\Big[H_I\Big(x(t')+\frac{i\hbar}{2}\frac{\delta}{\delta f(t')}\Big)-H_I\Big(x(t')-\frac{i\hbar}{2}\frac{\delta}{\delta f(t')}\Big)\Big]\right\}, \tag{64}$$

$$\frac{d}{dt}W(t,t_0)=-\mathcal{L}_I(t)\,W(t,t_0),\quad \mathcal{L}_I(t)\equiv i\hbar^{-1}\big[H_I(x^{\ell}(t))-H_I(x^{r}(t))\big]. \tag{65}$$

Last expressions (60),(61), and (64) are given in PSR-2. Definitions of x(t) and f(t) /7/ will be recalled below. Eqs. (58) and (59) are solutions of the equations of type (2)

$$\frac{d}{dt}\mathbb{x}^{\substack{\ell\\ r}}(t)=\pm i\hbar^{-1}\big[H^{\substack{\ell\\ r}},\mathbb{x}^{\substack{\ell\\ r}}(t)\big],\ \Big(H^{\substack{\ell\\ r}}=H\big(\mathbb{x}^{\substack{\ell\\ r}}(t),\mathbb{p}^{\substack{\ell\\ r}}(t)\big)\Big) \tag{66}$$

or of ones

$$\frac{d}{dt}\mathbb{x}^{\substack{\ell\\ r}}(t)=\big[\mathcal{L},\mathbb{x}^{\substack{\ell\\ r}}(t)\big],\qquad \Big(\mathcal{L}=i\hbar^{-1}\big(H^{\ell}-H^{r}\big)\Big) \tag{67}$$

(a simple consequence of eq. (66)); similarly, for eq. (60). From eq. (67) one can easily see how commutators turn into Poisson brackets as $\hbar \to 0$ (the transition is especially simple in PSR-2).

Let us stress that equations of evolution (58),(59),(60),(66), and (67) are valid for any Heisenberg operator, which does not depend explicitly on time, i.e., one can substitute $\mathbb{x}^{\ell,r}(t) \to \mathbb{F}^{\ell,r}(t)$, $x^{\ell,r}(t) \to F^{\ell,r}(t)$ there.

Starting with eqs. (58),(59) or (66) one can write equations of motion for $\mathbb{x}^{\ell,r}(t)$ also in Newton's form

$$m\ddot{\mathbb{x}}^{\ell}_{r}(t) = F(\mathbb{x}^{\ell}_{r}(t)) \qquad (68)$$

or in Hamilton's one

$$\frac{d}{dt}\mathbb{x}^{\ell}_{r}(t) = \frac{\partial H(\mathbb{x}^{\ell}_{r}, \mathbb{p}^{\ell}_{r})}{\partial \mathbb{p}^{\ell}_{r}(t)}, \quad \frac{d}{dt}\mathbb{p}^{\ell}_{r}(t) = -\frac{\partial H(\mathbb{x}^{\ell}_{r}, \mathbb{p}^{\ell}_{r})}{\partial \mathbb{x}^{\ell}_{r}(t)} \qquad (69)$$

with the nonusual in classics operatorvalued initial conditions

$$\mathbb{x}^{\ell}_{r}(t_0) = x_0 \pm \frac{i\hbar}{2}\frac{\partial}{\partial p_0}, \quad \mathbb{p}^{\ell}_{r}(t_0) = p_0 \mp \frac{i\hbar}{2}\frac{\partial}{\partial x_0} \quad \text{(for PSR-2)}. \qquad (70)$$

One can convert eqs. (68) with initial conditions (70) into the integral equation (of Yang-Feldman's form)[x)]

$$\mathbb{x}^{\ell}_{r}(t) = x^{\ell}_{r}(t) + \int_{t_0} dt'\, G_{ret}(t,t')\, F(\mathbb{x}^{\ell}_{r}(t')), \qquad (71)$$

where $x^{\ell}_{r}(t)$ are interaction-picture operators, i.e., solutions of free equations (68) (or any other linear equations instead, e.g., the oscillator ones, provided F in eq. (71) is modified accordingly). In particular, in PSR-2

$$\mathbb{x}^{\ell}_{r}(t) = x(t) \pm \frac{i\hbar}{2}\frac{\delta}{\delta f(t)} + \int_{t_0} dt'\, G_{ret}(t,t')\, F(\mathbb{x}^{\ell}_{r}(t')), \qquad (72)$$

where x(t) is the general solution of the free (or other linear) Newton equations in the classics[xx)]

[x)] See ref. /I.24.c,e/ for another course of derivation.

[xx)] The derivative $\delta/\delta f(t)$ in eq. (72) is only an abbreviation for

$$\frac{\delta}{\delta f(t)} = \left(\frac{\partial}{\partial t'} D(t,t') - D(t,t')\frac{\partial}{\partial t'}\right)\frac{\delta}{\delta f(t')}.$$

However, when applying to functionals of x(t), it is legitimate to take it literally.

$$x(t)=\frac{\partial}{\partial t_0}D(t,t_0)x(t_0)-D(t,t_0)\dot{x}(t_0)=-\int dt' D(t,t')f(t'). \qquad (73)$$

For $\mathrm{x}^{\ell}_{r}(t)$ in PSR-1 see /7/, eq. (106).

In PSR-2 the constant $\hbar$ enters in an especially simple way: for equations (68) or (69) only in initial conditions (70); in eq. (72) only as a coefficient of $\delta/\delta f(t)$.

In the linear cases equation (72) (and (71) too) may be readily reduced to the classical one, e.g., for the oscillator $F(\mathbb{x}^{\ell,r}(t))=g\,\mathbb{x}^{\ell,r}(t)$. According to the general rule, any (left or right) operator representative, being applied to unity, passes into its nonoperator representative:

$$\mathbb{F}^{\ell,r}\cdot 1=\mathbb{F}\ , \text{ so that } \ \mathbb{x}^{\ell,r}(t)\cdot 1=\mathbb{x}(t),\ x^{\ell,r}(t)\cdot 1=x(t), \qquad (74)$$

eq. (72)(and (71) too) reduces to its classical analog

$$\mathbb{x}(t)=x(t)+g\int_{t_0} dt' G_{ret}(t,t')\,\mathbb{x}(t'). \qquad (75)$$

When $\mathbb{x}(t)\equiv\mathbb{x}(x_0,p_0,t)$ in PSR-2 is found, then, if one likes, the operator itself can easily be reconstructed as

$$\hat{\mathbb{x}}(t)=\mathrm{sym}\ \mathbb{x}(\hat{x}\ ,\hat{p}\ ,t) \qquad (76)$$

and expressed in any representations (including PSR-2). This is true for any operator (see Appendix, eqs. (112)).

One can obtain a solution of eq. (72) (and (71) too) by infinitely iterating it (see refs. /1.24.c,e/ for a field theoretical example corresponding to an anharmonic oscillator), all the Neumann series being written down symbolically by eqs. (58.c) or (59.c).

Let us return to the phase space density, and point out its evolution in the interaction picture

$$\rho_2^{int}(xpt)=\Lambda^{-1}\langle xp|\hat{\rho}^{int}(t)|xp\rangle=\Lambda^{-1}\langle xp|\hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^{-1}(t,t_0)|xp\rangle=$$

$$=U^{\ell}(t,t_0)U^{r-1}(t,t_0)\Lambda^{-1}\langle xp|\hat{\rho}(t_0)|xp\rangle=W(t,t_0)\rho_2(xpt_0) \qquad (77)$$

$$\left\{\frac{\partial}{\partial t}+i\hbar^{-1}\left[H_I\left(x(t)+\frac{i\hbar}{2}\frac{\delta}{\delta f(t)}\right)-H_I\left(x(t)-\frac{i\hbar}{2}\frac{\delta}{\delta f(t)}\right)\right]\right\}\rho_2^{int}(xpt)=0 \qquad (78)$$

(see ref. /4/ for another treatment of this picture).

Let us note that the Liouville equation (23.a) (including eq. (27.a)) may be written via the classical Hamiltonian

$H(x,p)\equiv\Lambda^{-1}\langle xp|\mathrm{sym}\,H(\hat{x},\hat{p})|xp\rangle$ as follows [2]

$$\frac{d}{dt}\rho_2(xpt)=-i\hbar^{-1}[H(x,p),\rho_2(xpt)]_M, \tag{79}$$

where $[\ ]_M$ is a Moyal bracket [2] (a generalized Poisson bracket) using some complicated nonassociative multiplication rule (see Appendix, eq. (114.a)), inherent to all the quantum mechanics[I.22]. However, the above operator formalism (of the left and right representatives), seems more compact and flexible.

Operator representatives of the density matrix ρ can also be useful

$$\rho^{\ell}(t)=e^{-i\hbar^{-1}H^{\ell}\tau}\rho^{\ell}(t_o)e^{i\hbar^{-1}H^{\ell}\tau}=e^{-L\tau}\rho^{\ell}(t_o)e^{L\tau}, \tag{80}$$

$$\rho^{int\,\ell}(t)=U^{\ell}(t,t_o)\rho^{\ell}(t_o)U^{\ell-1}(t,t_o)=W(t,t_o)\rho^{\ell}(t_o)W^{-1}(t,t_o), \tag{81}$$

$$\rho^{r}(t)=e^{i\hbar^{-1}H^{r}\tau}\rho^{r}(t_o)e^{-i\hbar^{-1}H^{r}\tau}=e^{-L\tau}\rho^{r}(t_o)e^{L\tau}, \tag{82}$$

$$\rho^{int\,r}(t)=U^{r-1}(t,t_o)\rho^{r}(t_o)U^{r}(t,t_o)=W(t,t_o)\rho^{r}(t_o)W^{-1}(t,t_o), \tag{83}$$

$$\dot{\rho}^{\ell}_{r}(t)=\mp i\hbar^{-1}[H^{\ell}_{r},\rho^{\ell}_{r}(t)]=-[L,\rho^{\ell}_{r}(t)], \tag{84}$$

$$\dot{\rho}^{int\,\ell}_{\ \ \ r}(t)=\mp i\hbar^{-1}[H_I^{int\,\ell}_{\ \ \ r}(t),\rho^{int\,\ell}_{\ \ \ r}(t)]=-[L_I^{int}(t),\rho^{int\,\ell}_{\ \ \ r}(t)] \tag{85}$$

and

$$\rho(xpt)=\rho^{\ell}_{r}(t)\cdot 1\,. \tag{86}$$

<u>The PSR distribution functions ρ_1 and ρ_2 contain total information.</u> Expression (17) is, on the one hand, a squared modulus of the amplitude $a(xpt,x_op_ot_o)=\langle xp|\exp(-i\hbar^{-1}\hat{H}\tau)|x_op_o\rangle$ and, on the other hand, it may be considered as expectation values of the operators

$$e^{-i\hbar^{-1}\hat{H}\tau}|x_op_o\rangle\langle x_op_o|e^{i\hbar^{-1}\hat{H}\tau}=:\rho_1(\hat{x}\,\hat{p}t,x_op_ot_o): \tag{87.a}$$

$$e^{i\hbar^{-1}\hat{H}\tau}|xp\rangle\langle xp|e^{-i\hbar^{-1}\hat{H}\tau}=:\rho_1(xpt,\hat{x}\,\hat{p}\,t_o): \tag{87.b}$$

in the coherent states $|xp\rangle$ and $|x_o p_o\rangle$, respectively. Similarly

$$e^{-i\hbar^{-1}\hat{H}\tau}\Lambda_o^{-1}|x_op_o\rangle\langle x_op_o|e^{i\hbar^{-1}\hat{H}\tau}=\mathrm{sym}\,\rho_2(\hat{x}\hat{p}t,x_op_ot_o), \tag{88.a}$$

$$e^{i\hbar^{-1}\hat{H}\tau}\Lambda^{-1}|xp\rangle\langle xp|e^{-i\hbar^{-1}\hat{H}\tau}=\mathrm{sym}\,\rho_2(xpt,\hat{x}\,\hat{p}\,t_o). \tag{88.b}$$

When the function $\rho_1(xpt,x_op_ot_o)$(17) (or $\rho_2(xpt,x_op_ot_o)$(18)) is known, we can reconstruct these operators in the above N-ordered (symmetrized) forms. Furthermore using the completeness relations of ref.[7](eq. (1)) we have, e.g.,

$$\| e^{-i\hbar^{-1}\hat{H}} \| \otimes \| e^{i\hbar^{-1}\hat{H}\tau} \| =$$

$$= \frac{1}{(2\pi\hbar)^{2n}} \int dx\,dp\,dx_0\,dp_0 \left(\Lambda^{-1} \| xp \rangle\langle xp \|\right) \otimes \left(\Lambda_0^{-1} \| x_0p_0\rangle\langle x_0p_0 \|\right) \underbrace{\Lambda^{-1}\Lambda_0^{-1} |\langle xp| e^{-i\hbar^{-1}\hat{H}\tau} |x_0p_0\rangle|^2}_{\rho_2(xpt,\, x_0p_0t_0)} . \quad (89)$$

Hence we can obtain a product of independent amplitudes (a "density matrix") in any representation, e.g., in the coherent state representation

$$\langle x_4p_4| e^{-i\hbar^{-1}\hat{H}\tau} |x_3p_3\rangle\langle x_2p_2| e^{i\hbar^{-1}\hat{H}\tau} |x_1p_1\rangle = \frac{1}{(2\pi\hbar)^{2n}} \int dx\,dp\,dx_0\,dp_0$$

$$\Lambda^{-1} \langle x_4p_4|xp\rangle\langle xp|x_1p_1\rangle \; \Lambda_0^{-1} \langle x_2p_2|x_0p_0\rangle\langle x_0p_0|x_3p_3\rangle \underbrace{\Lambda^{-1}\Lambda_0^{-1} |\langle xp| e^{-i\hbar^{-1}\hat{H}\tau} |x_0p_0\rangle|^2}_{\rho_2(xpt,\, x_0p_0t_0)} . \quad (90)$$

Other important formulas for density matrices in terms of ρ_2 and ρ_1 are

$$\langle x_4| e^{-i\hbar^{-1}\hat{H}\tau} |x_3\rangle\langle x_2| e^{i\hbar^{-1}\hat{H}\tau} |x_1\rangle \equiv$$

$$\equiv \langle q + \tfrac{\hbar}{2}\alpha| e^{-i\hbar^{-1}\hat{H}\tau} |q' - \tfrac{\hbar}{2}\alpha'\rangle\langle q' + \tfrac{\hbar}{2}\alpha'| e^{i\hbar^{-1}\hat{H}\tau} |q - \tfrac{\hbar}{2}\alpha\rangle =$$

$$= \frac{1}{(2\pi\hbar)^{2n}} \int d\,p\, e^{ip\alpha} \int d\,p'\, e^{ip'\alpha'} \underbrace{\Lambda^{-1}\Lambda'^{-1} |\langle qp| e^{-i\hbar^{-1}\hat{H}\tau} |q'p'\rangle|^2}_{\rho_2(qpt,\, q'p't_0)} \quad (91)$$

(similarly in the p-representation, see eq. (86) in ref./7/) and

$$\langle \ell| e^{-i\hbar^{-1}\hat{H}\tau} |\ell_0\rangle\langle m_0| e^{i\hbar^{-1}\hat{H}\tau} |m\rangle = \quad (92)$$

$$= \left(\frac{\partial}{\partial\alpha^*}\right)^{\ell} \left(\frac{\partial}{\partial\alpha}\right)^{m} \left(\frac{\partial}{\partial\alpha_0^*}\right)^{m_0} \left(\frac{\partial}{\partial\alpha_0}\right)^{\ell_0} \frac{|\langle xp| e^{-i\hbar^{-1}\hat{H}\tau} |x_0p_0\rangle|^2}{|\langle xp|0\rangle|^2 \, |\langle x_0p_0|0\rangle|^2} \Bigg|_{\substack{\alpha_0=\alpha_0^*=0,\\ \alpha=\alpha^*=0}}$$

where $|\ell\rangle, |\ell'\rangle, |m'\rangle, |m\rangle$ are the oscillator states, and directly the function $\rho_1(xpt, x_0p_0t_0)$ (17) is used. These formulas mean, in fact, that we can return from the densities ρ_1 (17) and ρ_2 (18) to the level of amplitudes (i.e., they carry total information). On the contrary, when constructing the densities

$$\rho(xt, x_0t_0) = |\langle x| e^{-i\hbar^{-1}\hat{H}\tau} |x_0\rangle|^2 , \; \rho(mt, m_0t_0) = |\langle m| e^{-i\hbar^{-1}\hat{H}\tau} |m_0\rangle|^2 ,$$

$$\rho(xt, x_0p_0t_0) = |\langle x| e^{-i\hbar^{-1}\hat{H}\tau} |x_0p_0\rangle|^2$$

or $$\langle x_2| e^{-i\hbar^{-1}\hat{H}\tau} |x_0\rangle\langle x_0| e^{i\hbar^{-1}\hat{H}\tau} |x_1\rangle$$

we lose some information irreversibly. This is because a complete matrix basis is formed by $|x\rangle\langle x'|$ or $|m\rangle\langle m'|$, but not by $|x\rangle\langle x|$ or $|m\rangle\langle m|$ (unlike $|xp\rangle\langle xp|$ or $\Lambda^{-1}|xp\rangle\langle xp|$).

We can express the usual quantum-mechanical densities directly in terms of ρ_1 and ρ_2 without any reference to amplitudes, and that is demonstrated by the next formulas. We give the general definitions and then final results, obtained with eqs. (33) and (39) for the particular cases of the free particles (a) and the oscillators (b)

$$\rho(xt,x_0p_0t_0)\equiv|\langle x|e^{-i\hbar^{-1}\hat{H}\tau}|x_0p_0\rangle|^2=$$

$$=\frac{1}{(2\pi\hbar)^n}\Lambda_0\int dp\,\rho_2(xpt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^n}\int dp\,\Lambda^{-1}\rho_1(xpt,x_0p_0t_0)= \qquad (93)$$

$$=\begin{cases}(\pi\hbar)^{-\frac{n}{2}}[\det C(\tau)]^{-\frac{1}{2}}\exp\left[-\hbar^{-1}C^{-1}(\tau)(x-x_0-\frac{p_0}{m}\tau)^2\right]\ \left(C(\tau)=A^{-1}+A\frac{\tau^2}{m^2}\right) & (93.a)\\ (\pi\hbar)^{-\frac{n}{2}}[\det C(\tau)]^{-\frac{1}{2}}\exp\left[-\hbar^{-1}C^{-1}(\tau)(x-x_0\cos\omega\tau-\frac{p_0}{m\omega}\sin\omega\tau)^2\right]^{x)}, & (93.b)\end{cases}$$

$$\rho(pt,x_0p_0t_0)\equiv|\langle p|e^{-i\hbar^{-1}\hat{H}\tau}|x_0p_0\rangle|^2=$$

$$=\frac{1}{(2\pi\hbar)^n}\Lambda_0\int dx\,\rho_2(xpt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^n}\int dx\,\Lambda^{-1}\rho_1(xpt,x_0p_0t_0)= \qquad (94)$$

$$=\begin{cases}(\pi\hbar)^{-\frac{n}{2}}[\det A]^{-\frac{1}{2}}\exp\left[-\hbar^{-1}A^{-1}(p-p_0)^2\right] & (94.a)\\ (\pi\hbar)^{-\frac{n}{2}}[\det A(\tau)]^{-\frac{1}{2}}\exp\left[-\hbar^{-1}A^{-1}(\tau)(p-p_0\cos\omega\tau+\omega m x_0\sin\omega\tau)^2\right], & (94.b)\end{cases}$$

$$\rho(xpt,x_0t_0)\equiv|\langle xp|e^{-i\hbar^{-1}\hat{H}\tau}|x_0\rangle|^2=$$

$$=\frac{1}{(2\pi\hbar)^n}\Lambda\int dp_0\,\rho_2(xpt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^n}\int dp_0\,\Lambda_0^{-1}\rho_1(xpt,x_0p_0t_0)= \qquad (95)$$

$$=\begin{cases}(\pi\hbar)^{-\frac{n}{2}}[\det C(\tau)]^{-\frac{1}{2}}\exp\left[-\hbar^{-1}C^{-1}(\tau)(x-x_0-\frac{p}{m}\tau)^2\right] & (95.a)\\ (\pi\hbar)^{-\frac{n}{2}}[\det C(\tau)]^{-\frac{1}{2}}\exp\left[-\hbar^{-1}C^{-1}(\tau)(x\cos\omega\tau-\frac{p}{m\omega}\sin\omega\tau-x_0)^2\right], & (95.b)\end{cases}$$

$$\rho(xpt,p_0t_0)\equiv|\langle xp|e^{-i\hbar^{-1}\hat{H}\tau}|p_0\rangle|^2=$$

$$=\frac{1}{(2\pi\hbar)^n}\Lambda\int dx_0\,\rho_2(xpt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^n}\int dx_0\,\Lambda_0^{-1}\rho_1(xpt,x_0p_0t_0)= \qquad (96)$$

$$=\begin{cases}(\pi\hbar)^{-\frac{n}{2}}[\det A]^{-\frac{1}{2}}\exp\left[-\hbar^{-1}A^{-1}(p-p_0)^2\right] & (96.a)\\ (\pi\hbar)^{-\frac{n}{2}}[\det A(\tau)]^{-\frac{1}{2}}\exp\left[-\hbar^{-1}A^{-1}(\tau)(p\cos\omega\tau+m\omega x\sin\omega\tau-p_0)^2\right], & (96.b)\end{cases}$$

$$\rho(xt,x_0t_0)\equiv|\langle x|e^{-i\hbar^{-1}\hat{H}\tau}|x_0\rangle|^2=$$

$$=\frac{1}{(2\pi\hbar)^{2n}}\int dp\,dp_0\,\rho_2(xpt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^{2n}}\int dp\,dp_0\,\Lambda^{-1}\Lambda_0^{-1}\rho_1(xpt,x_0p_0t_0) \qquad (97)$$

$$=\begin{cases}[m/2\pi\hbar(t-t_0)]^n & (97.a)\\ [m\omega/2\pi\hbar\sin\omega(t-t_0)]^n, & (97.b)\end{cases}$$

x) Equation (93b) corresponds to the Kennard nonspreading wave packets with the oscillating dispersion $C(\tau)$ (99) unlike infinitely growing one in eq. (93.a).

$$\rho(pt,p_0t_0) \equiv |\langle p|e^{-i\hbar^{-1}\hat{H}\tau}|p_0\rangle|^2$$
$$=\frac{1}{(2\pi\hbar)^{2n}}\int dx\,dx_0\rho_2(xpt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^{2n}}\int dx\,dx_0\Lambda^{-1}\Lambda_0^{-1}\rho_1(xpt,x_0p_0t_0)= \quad (98)$$
$$=\begin{cases} \frac{1}{(2\pi\hbar)^n}V_n\delta(p-p_0) \qquad \left(V_n=\int dx_0=\int dx=\infty^n\right) & (98.a) \\ \left[2\pi\hbar m\omega\sin\omega(t-t_0)\right]^{-n}, & (98.b)\end{cases}$$
where, e.g., $C^{-1}x^2 = (C^{-1})_{ij}x_ix_j$, and in expressions (b)
$$A(\tau)=A\cos^2\omega\tau+A^{-1}m^2\omega^2\sin^2\omega\tau, \; C(\tau)=A^{-1}\cos^2\omega\tau+A\frac{\sin^2\omega\tau}{m^2\omega^2}. \quad (99)$$
The densities (97) and (98) are calculated in terms of $\rho_2(xpt,x_0p_0t_0)$ as if we calculated them in classics, i.e., simply integrating over variables of no interest. In the same way one can calculate
$$|\langle x|\exp(-i\hbar^{-1}\hat{H}\tau)|p_0\rangle|^2 \text{ and } |\langle p|\exp(-i\hbar^{-1}\hat{H}\tau)|x_0\rangle|^2.$$
The quantum-mechanical densities (93)-(96) were calculated by applying the Gauss transformations Λ or Λ_0 to the densities
$$\tilde\rho(xt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^n}\int dp\,\rho_2(xpt,x_0p_0t_0)=\delta\left(x-x_0\cos\omega\tau-\frac{p_0}{m\omega}\sin\omega\tau\right), \quad (100)$$
$$\tilde\rho(pt,x_0p_0t_0)=\frac{1}{(2\pi\hbar)^n}\int dx\,\rho_2(xpt,x_0p_0t_0)=\delta(p-p_0\cos\omega\tau+m\omega x_0\sin\omega\tau), \quad (101)$$
$$\tilde\rho(xpt,x_0t_0)=\frac{1}{(2\pi\hbar)^n}\int dp_0\,\rho_2(xpt,x_0p_0t_0)=\delta\left(x\cos\omega\tau-\frac{p}{m\omega}\sin\omega\tau-x_0\right), \quad (102)$$
$$\tilde\rho(xpt,p_0t_0)=\frac{1}{(2\pi\hbar)^n}\int dx_0\,\rho_2(xpt,p_0t_0)=\delta(p\cos\omega\tau+m\omega x\sin\omega\tau-p_0) \quad (103)$$
the latter expressions corresponding to the oscillators (with $\omega = 0$ we get the free case).

In addition to examples of $\rho_2(xpt,x_0p_0t_0)$ on p. 7 we give examples of $\rho_1(xpt,x_0p_0t_0)$ (17) and distribution functions in mixed representations: the initial density matrix $\Lambda_0^{-1}|x_0p_0\rangle\langle x_0p_0|$ ($|x_0p_0\rangle\langle x_0p_0|$) and the final one $|xp\rangle\langle xp|$ ($\Lambda^{-1}|xp\rangle\langle xp|$), i.e., with the initial condition
$$\Lambda^{-1}|\langle xp|x_0p_0\rangle|^2 = \Lambda_0^{-1}|\langle xp|x_0p_0\rangle|^2=2^n e^{-\hbar^{-1}(A(x-x_0)^2+A^{-1}(p-p_0)^2)} \quad (104)$$
Free particles. $\rho_1(xpt,x_0p_0t_0) = \left\{\det\left[\left(A^{-1}+A\frac{\tau^2}{4m^2}\right)A\right]\right\}^{-\frac{1}{2}}\cdot$
$$\cdot\exp\left\{-(2\hbar)^{-1}\left[\left(A^{-1}+A\frac{\tau^2}{4m^2}\right)^{-1}\left(x-x_0-\frac{p+p_0}{2m}\tau\right)^2+A^{-1}(p-p_0)^2\right]\right\}, \quad (105)$$
$$\Lambda^{-1}\rho_1(xpt,x_0p_0t_0)=\Lambda_0\rho_2(xpt,x_0p_0t_0)=\Lambda^{-1}|\langle xp|e^{-i\hbar^{-1}\hat{H}\tau}|x_0p_0\rangle|^2=$$
$$=2^n\exp\left\{-\hbar^{-1}\left[A\left(x-\frac{p}{m}\tau-x_0\right)^2+A^{-1}(p-p_0)^2\right]\right\}, \quad (106)$$
$$\Lambda_0^{-1}\rho_1(xpt,x_0p_0t_0)=\Lambda\rho_2(xpt,x_0p_0t_0)=\Lambda_0^{-1}|\langle xp|e^{-i\hbar^{-1}\hat{H}\tau}|x_0p_0\rangle|^2=$$
$$=2^n\exp\left\{-\hbar^{-1}\left[A\left(x-x_0-\frac{p_0}{m}\tau\right)^2+A^{-1}(p-p_0)^2\right]\right\}. \quad (107)$$

Oscillators. $\rho_1(xpt,x_0p_0t_0) = 2^n(\det D)^{-\frac{1}{2}}\cdot$

$$\cdot\exp\{-\hbar^{-1}[(D^{-1}A)(x-x_0\cos\omega\tau-\tfrac{p_0}{m\omega}\sin\omega\tau)^2+(D^{-1}A^{-1})(p-p_0\cos\omega\tau+m\omega x_0\sin\omega\tau)^2+$$
$$+(D^{-1}A)(x_0-x\cos\omega\tau+\tfrac{p}{m\omega}\sin\omega\tau)^2+(D^{-1}A^{-1})(p_0-p\cos\omega\tau-m\omega x\sin\omega\tau)^2]\},$$

$$/D = 2+A(\tau)A^{-1}+AC(\tau)/,\qquad (108)$$

$$\Lambda^{-1}\rho_1(xpt,x_0p_0t_0)=\Lambda_0\rho_2(xpt,x_0p_0t_0)=$$
$$=2^n\exp\{-\hbar^{-1}[A(x_0-x\cos\omega\tau+\tfrac{p}{m\omega}\sin\omega\tau)^2+A^{-1}(p_0-p\cos\omega\tau-m\omega\sin\omega\tau)^2]\}, \qquad (109)$$

$$\Lambda_0^{-1}\rho_1(xpt,x_0p_0t_0)=\Lambda\rho_2(xpt,x_0p_0t_0)=$$
$$=2^n\exp\{-\hbar^{-1}[A(x-x_0\cos\omega\tau-\tfrac{p_0}{m\omega}\sin\omega\tau)^2+A^{-1}(p-p_0\cos\omega\tau+m\omega x_0\sin\omega\tau)^2]\}. \qquad (110)$$

Let us give also the formula

$$\Lambda^{(\nu)}\langle xp|x'p'\rangle\langle x'p'|x_0p_0\rangle=\left(\frac{2}{\nu+2}\right)^n\exp\{i(2\hbar)^{-1}[(x-x_0)p'-(p-p_0)x'+$$
$$+\frac{\nu}{\nu+2}(x'-\frac{x+x_0}{2})(p-p_0)-\frac{\nu}{\nu+2}(p'-\frac{p+p_0}{2})(x-x_0)]-(4\hbar)^{-1}\frac{\nu+1}{\nu+2}[A(x-x_0)^2+A^{-1}(p-p_0)^2]-$$
$$-(\hbar(\nu+2))^{-1}[A(x'-\frac{x+x_0}{2})^2+A^{-1}(p'-\frac{p+p_0}{2})^2]\}. \qquad (111)$$

With $\nu = -1$ we obtain the expression, encountered in eq. (90).

Appendix. Nonoperator representatives of a product of operators in terms of their representatives may be obtained starting with

$$\hat{\mathbb{F}} = \vdots\mathbb{F}_1(\hat x,\hat p)\vdots = \vdots e^{\hat x\frac{\partial}{\partial x}+\hat p\frac{\partial}{\partial p}}\vdots\,\mathbb{F}_1(x,p)\Big|_{\substack{x=0\\p=0}}=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\vdots e^{-i\hbar^{-1}(\hat x p-\hat p x)}\vdots\,\widetilde{\mathbb{F}}_1(x,p) \qquad (112.a)$$

$$=\mathrm{sym}\,\mathbb{F}_2(\hat x,\hat p)=e^{\hat x\frac{\partial}{\partial x}+\hat p\frac{\partial}{\partial p}}\mathbb{F}_2(x,p)\Big|_{\substack{x=0\\p=0}}=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,e^{-i\hbar^{-1}(\hat x p-\hat p x)}\widetilde{\mathbb{F}}_2(x,p) \qquad (112.b)$$

$$=:\mathbb{F}_3(\hat x,\hat p):=:e^{\hat x\frac{\partial}{\partial x}+\hat p\frac{\partial}{\partial p}}:\mathbb{F}_3(x,p)\Big|_{\substack{x=0\\p=0}}=\frac{1}{(2\pi\hbar)^n}\int dx\,dp:e^{-i\hbar^{-1}(\hat x p-\hat p x)}:\widetilde{\mathbb{F}}_3(x,p), \qquad (112.c)$$

where $\vdots\ \vdots$ means anti-N-ordering. Then

$$\hat{\mathbb{F}}\hat{\mathbb{G}} = e^{\hat x\frac{\partial}{\partial x''}+\hat p\frac{\partial}{\partial p''}}\,e^{\hat x\frac{\partial}{\partial x'}+\hat p\frac{\partial}{\partial p'}}\,\mathbb{F}_2(x'',p'')\mathbb{G}_2(x',p')\Big|_{x'=p'=x''=p''=0}=$$
$$=e^{\hat x\left(\frac{\partial}{\partial x''}+\frac{\partial}{\partial x'}\right)+\hat p\left(\frac{\partial}{\partial p''}+\frac{\partial}{\partial p'}\right)}\,e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial x''}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p''}\frac{\partial}{\partial x'}\right)}\mathbb{F}_2(x'',p'')\mathbb{G}_2(x',p')\Big|_{\substack{x'=p'=0\\x''=p''=0}} \qquad (113)$$

and the representative of the product in PSR-2 is given by

$$\Lambda^{-1}\langle xp|\hat{\mathbb{F}}\hat{\mathbb{G}}|xp\rangle=e^{x\left(\frac{\partial}{\partial x''}+\frac{\partial}{\partial x'}\right)+p\left(\frac{\partial}{\partial p''}+\frac{\partial}{\partial p'}\right)}\,e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial x''}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p''}\frac{\partial}{\partial x'}\right)}\mathbb{F}_2(x'',p'')\mathbb{G}_2(x',p')\Big|_{\substack{x'=p'=0\\x''=p''=0}}=$$
$$=e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial x''}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p''}\frac{\partial}{\partial x'}\right)}\mathbb{F}_2(x+x'',p+p'')\mathbb{G}_2(x+x',p+p')\Big|_{\substack{x'=p'=0\\x''=p''=0}}= \qquad (114.a)$$
$$=\mathbb{F}_2\left(x+\frac{i\hbar}{2}\frac{\partial}{\partial p'},p-\frac{i\hbar}{2}\frac{\partial}{\partial x'}\right)\mathbb{G}_2(x+x',p+p')\Big|_{x'=p'=0}=$$
$$=\mathbb{F}_2^{ord}\left(x+\frac{i\hbar}{2}\frac{\partial}{\partial p},p-\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)\mathbb{G}_2(x,p)=\mathbb{G}_2^{ord}\left(x-\frac{i\hbar}{2}\frac{\partial}{\partial p},p+\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)\mathbb{F}_2(x,p), \qquad (114.b)$$

where ord (ordered) means that the derivatives $\partial/\partial x$ and $\partial/\partial p$ are placed on the right of all x and p in $\mathbb{F}_2^{ord}$ and $\mathbb{G}_2^{ord}$.

Similarly in PSR-1 and PSR-3

$$\hat{\mathbb{F}}\hat{\mathbb{G}} = :e^{\hat{x}\frac{\partial}{\partial x''}+\hat{p}\frac{\partial}{\partial p''}}::e^{\hat{x}\frac{\partial}{\partial x'}+\hat{p}\frac{\partial}{\partial p'}}:\mathbb{F}_1(x'',p'')\mathbb{G}_1(x',p')\Big|_{x'=p'=x''=p''=0} =$$

$$=:e^{\hat{x}\left(\frac{\partial}{\partial x''}+\frac{\partial}{\partial x'}\right)+\hat{p}\left(\frac{\partial}{\partial p''}+\frac{\partial}{\partial p'}\right)}:e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial x''}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p''}\frac{\partial}{\partial x'}\right)}\tilde{\Lambda}\Lambda''^{-1}\Lambda'^{-1}\mathbb{F}_1(x'',p'')\mathbb{G}_1(x',p')\Big|_{\substack{x'=p'=0\\ x''=p''=0}} \quad (115)$$

$$\langle xp|\hat{\mathbb{F}}\hat{\mathbb{G}}|xp\rangle = e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial x''}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p''}\frac{\partial}{\partial x'}\right)}\tilde{\Lambda}\Lambda''^{-1}\Lambda'^{-1}\mathbb{F}_1(x+x'',p+p'')\mathbb{G}_1(x+x',p+p')\Big|_{\substack{x'=p'=0\\ x''=p''=0}} \quad (116)$$

$$\hat{\mathbb{F}}\hat{\mathbb{G}} = :e^{\hat{x}\frac{\partial}{\partial x''}+\hat{p}\frac{\partial}{\partial p''}}::e^{\hat{x}\frac{\partial}{\partial x'}+\hat{p}\frac{\partial}{\partial p'}}:\mathbb{F}_3(x'',p'')\mathbb{G}_3(x',p')\Big|_{x'=p'=x''=p''=0} =$$

$$=:e^{\hat{x}\left(\frac{\partial}{\partial x''}+\frac{\partial}{\partial x'}\right)+\hat{p}\left(\frac{\partial}{\partial p''}+\frac{\partial}{\partial p'}\right)}:e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial x''}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p''}\frac{\partial}{\partial x'}\right)}\tilde{\Lambda}^{-1}\Lambda''\Lambda'\,\mathbb{F}_3(x'',p'')\mathbb{G}_3(x',p')\Big|_{\substack{x'=p'=0\\ x''=p''=0}} \quad (117)$$

$$\Lambda^{-2}\langle xp|\hat{\mathbb{F}}\hat{\mathbb{G}}|xp\rangle = e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial x''}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p''}\frac{\partial}{\partial x'}\right)}\tilde{\Lambda}^{-1}\Lambda''\Lambda'\,\mathbb{F}_3(x+x'',p+p'')\mathbb{G}_3(x+x',p+p')\Big|_{\substack{x'=p'=0\\ x''=p''=0}} \quad (118)$$

where

$$\tilde{\Lambda}\Lambda''^{-1}\Lambda'^{-1} = \exp\left[+\frac{\hbar}{2}\left(A^{-1}\frac{\partial}{\partial x''}\frac{\partial}{\partial x'}+A\frac{\partial}{\partial p''}\frac{\partial}{\partial p'}\right)\right], \quad \tilde{\Lambda}^{-1}\Lambda''\Lambda' = \exp\left[-\frac{\hbar}{2}(\cdots)\right]. \quad (119)$$

Forms $\mathbb{F}^{ord}_{\substack{1\\3}}\mathbb{G}_{\substack{1\\3}} = \mathbb{G}^{ord}_{\substack{1\\3}}\mathbb{F}_{\substack{1\\3}}$ in the above sense are also possible with eqs.(104), (105) (in PSR-1) and (133),(134) of ref.[7] as arguments of $\mathbb{F}^{ord}$ and $\mathbb{G}^{ord}$.

In terms of the product (114) the Liouville equation looks like

$$\frac{\partial}{\partial t}\rho(xpt) = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\left(\frac{\partial}{\partial x''}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p''}\frac{\partial}{\partial x'}\right)\right)H(x+x'',p+p'')\rho(x+x',p+p')\Big|_{\substack{x'=p'=0\\ x''=p''=0}} \quad (120.a)$$

$$= -i\hbar\left(H^{ord}\left(x+\frac{i\hbar}{2}\frac{\partial}{\partial p},p-\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)-H^{ord}\left(x-\frac{i\hbar}{2}\frac{\partial}{\partial p},p+\frac{i\hbar}{2}\frac{\partial}{\partial x}\right)\right)\rho(xpt) \quad (120.b)$$

where the first row contain the Moyal bracket[2] (a generalization of the Poisson one), and the second one demonstrate the identity of this equation with eq. (27).

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