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UNIFORM PRODUCT FORMULAE WITH APPLICATION TO THE FEYNMAN-NELSON INTEGRAL FOR OPEN SYSTEMS

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1. INTRODUCTION

Among approaches to defining the Feynman integral, various sequential methods are probably the most popular ones. Here we consider that one based on product formulae, the original idea of which belongs to Nelson[1]. This method is powerful in the sense that it enables to prove the strict version of Feynman's basic dynamical formula for a wide class of potentials[2,3] including time-dependent ones[4] in comparison with other known methods(see,e.g.,refs.[5,6]).On the other hand,the method does not correspond exactly to Feynman's heuristic prescription because it replaces the exact action along polygonal paths by its Riemannian approximation. Moreover, one can regard as an unsatisfactory feature that it employs the "equidistant" time-interval partitions only.

In the present paper we show that the last mentioned source of arbitrariness can be removed: the product formula for perturbations of propagators by Faris[4] remains valid if the limit is carried out with any "crumbling" sequence of partitions uniformly w.r.t. the partition "norm" given by the maximal subinterval length. In particular, it gives the uniform version of the Trotter formula[7] in the important special case considered in ref.[1] when sum of the generators of continuous contractive semigroups (CCSG) involved is closed and generates itself a CCSG.

Applying these results to the F-integral we make one more generalization: we assume potentials not only time-dependent but complex (obeying the dissipativity condition). This is an alternative way how to treat non-isolated systems; such a description is not merely useful phenomenologically but can be embedded into the standard quantum-mechanical framework within the pseudo-Hamiltonian approach[8].Our goal is to establish validity of the Feynman-Itô formula with the F-integral defined in terms of the mentioned product formula for three classes of complex potentials (cf. Theorems 2, 3 below). Let us remark that for some particular cases analogous results have been recently obtained using other definitions of the F-integral[9].

The last introductory item concerns notation: we shall use

$J^t = [0,t]$, $t > 0$,

$\sigma = \{r_i : 0 = r_0 < r_1 < \ldots < r_n = t\}$ partition of J^t; the set of all these partitions is denoted as $\mathcal{P}(J^t)$

$\delta_k = r_{k+1} - r_k$, $k = 0,1,\ldots,n-1$,

$\delta(\sigma) = \max\{\delta_k : k = 0,1,\ldots,n-1\}$.

2. THE PRODUCT FORMULAE

Theorem 1. Let X with a norm $\|.\|$ be a Banach space, and assume that for each $t \in J^b$, $A(t)$ and $B(t)$ are generators of CCSG's $\{e^{-A(t)s} : s \geq 0\}$ and $\{e^{-B(t)s} : s \geq 0\}$, respectively. Let further $C(t) = A(t) + B(t)$ be a closed operator for each $t \in J^b$, the domain of which is a dense subspace D in X independent of t. Assume that for every $u \in D$, $A(.)u$ and $B(.)u$ are C^0 on J^b. Let exist a contraction-valued propagator $V(.,.)$ on X (cf.[4]) such that $V(t,s)D \subset D$ for all $t,s \in J^b$. Let us denote $u(t) = V(t,0)$ for $u \in D$ and assume that $C(.)u(.)$ is C^0 on J^b and that $u(.)$ is C^1 on $(0,b)$ and there obeys

$$\frac{du(t)}{dt} + C(t)u(t) = 0. \tag{1}$$

Finally, let $u(.)$ be C^0 on J^b w.r.t. the Banach norm $\|.\|_0$ in D, $\|u\|_0 = \|u\| + \|C(t_0)u\|$ for some $t_0 \in J^b$. Then, for any $t \in J^b$,

$$V(t,0) = \operatorname*{s-lim}_{\delta(\sigma)\to 0} R(r_{n-1}, \delta_{n-1}) R(r_{n-2}, \delta_{n-2}) \ldots R(0, \delta_0), \tag{2}$$

where $R(r,\delta) = e^{-A(r)\delta} e^{-B(r)\delta}$ for $r \in J^b$, $\delta \geq 0$ and r_k, δ_k in (2) refer to a partition $\sigma \in \mathcal{P}(J^t)$.

Remark. The operator $C(t_0)$ above can be replaced by any closed operator C on X, the domain of which is D. If $\rho(C)$ is nonempty, then in order to check the $\|.\|_0$ -continuity of $u(.)$ it is enough to show that $Cu(.)$ is $\|.\|$ -continuous on J^b (cf.[4], Prop.1 and the remark following Th.1).

Proof of the theorem: We abbreviate $P_k = \exp(-A(r_k)\delta_k)$, $Q_k = \exp(-B(r_k)\delta_k)$, $R_k = P_k Q_k = R(r_k, \delta_k)$, $k = 0,1,\ldots,n-1$, and $S(\sigma) = R_{n-1} R_{n-2} \ldots R_0 - V(t,0)$. Relation (2) now reads $\operatorname{s-lim} S(\sigma) = 0$. We shall show first that for any $u \in D$, $S(\sigma)u \to 0$ with $\delta(\sigma) \to 0$. Using the equality

$$S(\sigma) = \sum_{k=0}^{n-1} R_{n-1} \ldots R_{k+1}(R_k - V(r_{k+1}, r_k))V(r_k, 0)$$

together with $\|R_k\| \leq 1$ and $A(r_k) + B(r_k) = C(r_k)$ we obtain

$$\|S(\sigma)u\| \leq \sum_{k=0}^{n-1} \delta_k \|(E_1(r_k,\delta_k) + E_2(r_k,\delta_k) - E_3(r_k,\delta_k))V(r_k,0)u\|,$$

where

$$E_1(r,\delta) = (e^{-A(r)\delta} - I)\delta^{-1} + A(r),$$

$$E_2(r,\delta) = e^{-A(r)\delta}(e^{-B(r)\delta} - I)\delta^{-1} + B(r),$$

$$E_3(r,\delta) = (V(r+\delta, r) - I)\delta^{-1} + C(r).$$

Now $\sum_{k=0}^{n-1} \delta_k = t$ so that we have

$$\|S(\sigma)u\| \leq t \sum_{j=1}^{3} \sup\{\|E_j(r,\delta)V(r,0)u\| : (r,\delta) \in M(\sigma)\},$$

where $M(\sigma) = \{(r,\delta) : 0 < \delta < \delta(\sigma),\ 0 \leq r \leq t-\delta\}$. Faris[4] proved that under stated continuity assumptions

$$\lim_{\delta\to 0} E_j(r,\delta)V(r,0)u = 0, \quad j = 1,2,3,$$

for each $u \in D$ uniformly in $r \in J^b$. This implies easily

$$\lim_{\delta(\sigma)\to 0} \sup_{M(\sigma)} \|E_j(r,\delta)V(r,0)u\| = 0, \quad j = 1,2,3,$$

i.e., $\lim_{\delta(\sigma)\to 0} S(\sigma)u = 0$. Finally, D is assumed to be dense in X and $\|S(\sigma)\| \leq \|R_{n-1} \ldots R_0\| + \|V(t,0)\| \leq 2$ for each partition σ; thus $\lim_{\delta(\sigma)\to 0} S(\sigma)u = 0$ for $u \in X$ as well, i.e., relation (2) holds. ∎

In particular, if A, B are t-independent, we obtain the following uniform version of the "special" Trotter formula:

Corollary. Let A,B be generators of CCSG's on a Banach space X. If the sum $C = A+B$ generates a CCSG, then for each $t > 0$,

$$e^{-Ct} = \operatorname*{s-lim}_{\delta(\sigma)\to 0} R(\delta_{n-1}) R(\delta_{n-2}) \ldots R(\delta_0), \tag{3}$$

where $R(s) = e^{-As} e^{-Bs}$ and δ_k refer to a partition $\sigma \in \mathcal{P}(J^t)$.

3. APPLICATION TO THE FEYNMAN INTEGRAL

Now we specify the Banach space X to be $\mathcal{H}_c = L^2(\mathbb{R}^d)$ and set $A = -\frac{i}{2}\Delta$ (independently of t) and $B(t) = iV_t$ for each $t \in J^b$, where $(V_t\phi)(x) = v(x,t)\phi(x)$. We assume that

(a) the Borel function $v(.,.)$ on $\mathbb{R}^d \times J^v$ is complex-valued and such that $v(.,t)$ is almost regular on $\mathbb{R}^d$ for each $t \in J^b$ (cf.[8]), $t \to v(.,t)$ is $\|.\|_\infty$ -continuous, and the dissipativity condition

$$\operatorname{Im} v(x,t) \leq 0 \tag{4}$$

holds for each $t \in J^b$ and almost all $x \in \mathbb{R}^d$.

The rhs of (2) can be expressed now more explicitly: using the free propagator corresponding to $H_0 = -\frac{1}{2}\Delta$ we obtain for $\psi_t = V(t,0)\phi$ the following relation

$$\psi_t(x) = \lim_{\delta(\sigma)\to 0} ((2\pi i)^n \delta_0 \delta_1 \dots \delta_{n-1})^{-d/2} \int_{R^d} \dots \int_{R^d} \exp\{iS(\sigma; \gamma_0, \dots, \gamma_n)\} \phi(\gamma_0)\, d\gamma_0\, d\gamma_1 \dots d\gamma_{n-1}, \tag{5}$$

where $\gamma_n = x$, the integrals are in general improper ones, $\int_{R^d} d\gamma = \lim_{m\to\infty} \int_{|\gamma|\le m} d\gamma$, all limits are understood in the L^2-sense, and

$$S(\sigma; \gamma_0, \dots, \gamma_n) = \sum_{k=0}^{n-1} \left[\frac{1}{2}\left(\frac{\gamma_{k+1} - \gamma_k}{\delta_k}\right)^2 - v(\gamma_k, \tau_k)\right]\delta_k . \tag{6}$$

One can interpret the last expression as the Riemannian approximation to

$$S_v(\gamma) = \int_0^t \left[\frac{1}{2}|\dot{\gamma}(\tau)|^2 - v(\gamma(\tau), \tau)\right] d\tau \tag{7}$$

if the latter makes sense. A particular case of the relation (5) appeared first in ref. /1/ ; this is why we call the rhs of (5) the <u>uniform Feynman-Nelson integral</u> and abbreviate it as $\int^{ufn} \exp\{iS_v(\gamma)\}\phi(\gamma(0))\,\mathfrak{D}\gamma$ (for discussion of relations to other definitions of the F-integral cf. /10/).

With these prerequisites Theorem 1 can be reformulated for the considered particular case. Clearly $\{\exp(\frac{is}{2}\Delta) : s \ge 0\}$ is a CCSG and the same is true for $\{\exp(-isV_t) : s \ge 0\}$ due to (4), further the assumption (a) implies that $t \to V_t \phi$ is continuous for each $\phi \in D$. Thus we have:

<u>Proposition</u>. In addition to (a), assume that
(b) for each $t \in J^b$, the domain D of $H(t) = H_0 + V_t$ is dense in $\mathcal{H}_c$, independent of t, and the operator $H(t)$ is closed;
(c) there exist a contraction-valued propagator $V(.,.)$ on $\mathcal{H}_c$, which preserves D for all $t,s \in J^b$, and with the following property: let $\psi_t = V(t,0)\phi$ for an arbitrary $\phi \in D$, then the function $t \to H(t)\psi_t$ is C^0 on J^b, further $t \to \psi_t$ is C^1 on $(0,b)$ and there satisfies

$$i\frac{d}{dt}\psi_t = H(t)\psi_t = -\frac{1}{2}\Delta\psi_t + V_t\psi_t ; \tag{8}$$

(d) the function $t \to \psi_t$ is continuous w.r.t. $\|.\|_0$ corresponding to $H(t_0)$ for some $t_0 \in J^b$.

Then for each $\phi \in \mathcal{H}_c$, ψ_t is given by (5), i.e.,

$$\psi_t(x) = \int^{ufn} \exp\{iS_v(\gamma)\}\phi(\gamma(0))\,\mathfrak{D}\gamma . \tag{9}$$

Let us exhibit now some classes of potentials for which the conditions (b)-(d) are fulfilled. As to (b), it can be easily verified in the following cases:

$$v(.,t) \text{ bounded for each } t \in J^b \tag{10}$$

trivially with $D = D(H_0)$, further

$$v(.,t) \in L^2(R^d) + L^\infty(R^d) \text{ for each } t \text{ and } d \le 3, \tag{11}$$

where again $D = D(H_0)$ (for proof see ref. /8/, th.7), and finally

$$v(x,t) = x \cdot B(t)x, \quad B(t) = A(t) - iW(t), \tag{12}$$

are positive symmetric $d \times d$ matrices (linear operators on R^d) for each $t \in J^b$, $A(t)$ strictly positive. As to the last case, one verifies first that (for fixed t) $H_0 + x.A(t)x$ is self-adjoint on $D(H_0) \cap D(Q^2)$, Q^2 being the operator of multiplication by $r^2 = x_1^2 + \dots + x_d^2$ on $\mathcal{H}_c$, then the Kato-Rellich-type lemma (cf. /2/ sec. X.8; ref. /8/) is applied successively to prove that $iH(t)$ is closed (on $D = D(H_0) \cap D(Q^2)$) and generates a CCSG; details of this proof will be given elsewhere.

Each operator $iH(t_0)$ corresponding to some of the potentials (10)-(12) with fixed $t_0 \in J^b$ generates a CCSG, and therefore its resolvent set is nonempty due to Hille-Yosida theorem. Using now the remark following Theorem 1 together with the fact that the operator $V_{t_0} \restriction D$ is closable one can show easily that the condition (d) is satisfied for the above classes of potentials, whenever (c) is satisfied.

As to the condition (c), let us consider first the case (10), where its validity can be established under a slightly strengthened smoothness assumptions:

<u>Theorem 2</u>. Assume (a) and (10), further let $t \to v(.,t)$ be a $\|.\|_\infty$ - continuously differentiable function with the derivative bounded in $(0,b)$. Then for each $\phi \in D(H_0)$, the rhs of (9) makes sense and expresses the solution $t \to \psi_t$ of eq. (8) corresponding to the initial data ϕ.

<u>Proof</u>: In view of Proposition and the above considerations it is sufficient to verify (c). This can be accomplished by virtue of Theorem X.70 from ref. /2/; standard arguments show that its assumptions are fulfilled under the stated requirements on v. ■

As to the unbounded potentials (11), (12), we limit ourselves in the present paper to the simplest possibility when they are time-independent. The above-mentioned results imply that $H=H_0+V$ generates a CCSG in this case, and therefore the condition is fulfilled automatically. We obtain thus the following result:

Theorem 3. Let V be an operator of multiplication by a complex-valued almost regular Borel function v which obeys the dissipativity condition (4) a.e. in $\mathbb{R}^d$. Assume further that either $d \leq 3$ and $v \in L^2(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)$ or $v(x)=x.(A-iW)x$ with A strictly positive and W positive. Then the rhs of (8) makes sense and expresses the solution $t \to \psi_t$ of the equation

$$i\frac{d}{dt}\psi_t = -\frac{1}{2}\Delta\psi_t + V\psi_t \tag{13}$$

corresponding to initial data $\phi \in D$, where $D=D(H_0)$ in the first case and $D=D(H_0)\cap D(Q^2)$ in the second one.

4. CONCLUDING REMARKS

The assertions formulated in the previous section remain valid if we use some other reasonable polygonal-path approximation instead of the uniform one, because the latter is the "strongest" one among them[10]. There are several ways in which the presented results could be generalized, e.g.:

- to treat unbounded time-dependent potentials of the type (11) under suitably strengthened t-smoothness conditions,
- to drop the restriction $d \leq 3$ in (11) with replacement of L^2 by a suitable L^p,
- to prove (and apply) the uniform version of the general Trotter formula (where the sum is not assumed closed - cf.[7]) and of its generalizations[11,12].

Let us finally remark that according to the mentioned results of ref.[9] the UFN-integral coincides with the F-integral in the sense of refs.[5,6] for potentials $v \in \mathcal{F}(R^d)$ as well as for the damped harmonic oscillator. It is desirable to find some weaker sufficient conditions under which a value of the F-integral will be independent of the used Riemannian approximation to the action.

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