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POLYGONAL-PATH APPROXIMATIONS ON THE PATH SPACES OF QUANTUM-MECHANICAL SYSTEMS: EXTENDED FEYNMAN MAPS

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1. INTRODUCTION

In the first part of this paper[1] we have studied properties of the subset of polygonal paths in the Hilbert space $\mathcal{H} = AC_0[J^t; \boldsymbol{R}^d]$ referring to a d-dimensional quantum-mechanical system. In particular, we have shown that each element of $\mathcal{H}$ can be approximated in the $\mathcal{H}$-norm $\|.\|$ by polygonal paths, and that this approximation is uniform w.r.t. $\delta(\sigma)$, the maximal-subinterval length of partition σ of the time interval $J^t = [0,t]$.

Here we shall use the results of Ref.[1] for discussion of various types of polygonal-path approximations which appear in the functional-integration theory, in particular those of Cameron[2,3], Cameron and Storvick[4], Gel'fand and Yaglom[5], Babbitt[6], Nelson[7], Combe et al.[8], Truman[9-11], Elworthy and Truman[12], Johnson and Skoug[13], Tarski[14] and others. The uniform approximation of Cameron appears to be the "strongest" one of them; we shall apply it to extend domain of the Feynman maps introduced in ref.[16] and to prove consistency of this extension.

Two particular cases of these maps are especially interesting: the F_1-map or the Feynman integral and the F_{-i}-map. As to the latter, we shall give some sufficient conditions in the next section under which it can be identified with the Wiener integral. For $d=1$ the F_{-i}-map is closely related to the sequential W-integral of Cameron[2]. In the last section we shall show that the basic theorem concerning the latter must be improved; we shall give strengthened conditions on the order of growth under which its assertion holds. Application of the polygonally extended F-integrals to solving Schrödinger-type equations will be discussed in the subsequent paper[17].

Throughout this paper we used the notation of ref.[1] freely. The relations, theorems and propositions of that paper will be indicated with I, e.g., (I.4) means the relation (4) of ref.[1], etc.

2. POLYGONAL-PATH APPROXIMATIONS

The main purpose of the polygonal-path methods is to determine the Feynman integral and related objects. The term "approximation" is thus a little misleading, because it means at the same time definition of the "approximated quantity", too. The ideology of polygonal-path definitions of the F-integral is essentially that of the Riemann-integral theory, however, since there is no analogy to the Darboux sums [18] here the definitions must be formulated in terms of limits w.r.t. sequences or nets of partitions.

Let us consider now in more detail the problem how to define the F-maps, i.e., the one-parameter family of complex-valued maps $f \mapsto \Phi_s$ from a suitable set of functions f on $\mathcal{H}$, which correspond to the formal functional integrals

$$\Phi_s = \int_{\mathcal{H}} \exp\left(\frac{i}{2s}\|\gamma\|^2\right) f(\gamma)\, \mathcal{D}\gamma \qquad (1)$$

with s non-zero, $\mathrm{Im}\, s \le 0$ (this subset of s in $\mathbf{C}$ was denoted as $\mathbf{C}_F$ in ref.[16]). F-integral clearly refers to the case $s=1$, if we set for simplicity the Planck constant $\hbar$ as well as the mass(es) equal to 1. Each polygonal-path approach to this problem starts from a choice of mappings $\phi_s : \mathcal{P}' \to \mathbf{C}$ which assign to every partition σ of some subset $\mathcal{P}'$ in $\mathcal{P}(J^t)$ "finite-dimensional approximations" $\phi_s(\sigma)$ to the functional integrals (1). The following step consists of taking a limit of $\phi_s(\sigma)$, which corresponds in some sense to gradual refining of σ; if this limit exists it is identified with Φ_s.

It is therefore clear that there exist at least two points of view for classification of polygonal-path methods:

(i) according to a choice of ϕ_s,

(ii) according to a choice of the limiting procedure.

The first one will be discussed only briefly here (see also remarks in the following section); we limit ourselves to the case which is physically the most interesting, i.e., $s=1$ and

$$f(\gamma) = \exp\{-i\int_0^t V(\gamma(\tau)+x)\,d\tau\}\, u(\gamma(0)+x), \qquad (2)$$

where V is a potential on $\mathbf{R}^d$ and u belongs to some subset of $L^2(\mathbf{R}^d)$. Then different choices of $\phi_1(\sigma)$ are possible (of course, for those V, u for which the corresponding expressions make sense), e.g.,

$$\phi_1(\sigma) := I_1(f \circ P(\sigma)), \qquad (3a)$$

where $I_1(.)$ is the F-integral of Albeverio and Hoegh-Krohn [19,20], or

$$\phi_1(\sigma) := (2\pi i)^{-n/2} \int_{P(\sigma)\mathcal{H}} \exp\{\frac{i}{2}\|\gamma'\|^2\} f(\gamma')\, dm(\gamma') = \qquad (3b)$$
$$= (2\pi i)^{-n/2} \int_{P(\sigma)\mathcal{H}} \exp\{iS(\gamma'+x)\}\, u(\gamma(0)+x)\, dm(\gamma'),$$

where $\gamma' = P(\sigma)\gamma$, $n = \dim P(\sigma)$, further $S(\gamma)$ is the action along the path γ and the integral in (3b) is understood in the improper sense [9-12], or finally

$$\phi_1(\sigma) := (2\pi i)^{-n/2} \int_{P(\sigma)\mathcal{H}} \exp\{iS_\sigma(\gamma'+x)\}\, u(\gamma(0)+x)\, dm(\gamma'), \qquad (3c)$$

where the action S is replaced by a Riemannian approximation S_σ (in fact, it concerns the potential part only) corresponding to the partition σ, and the integral is again the improper one [6-8,14].

The last choice admits use of the Lie-Trotter-Kato formula and gives thus stronger results than the previous two (in the sense that the corresponding functional integral exists and expresses dynamics for a much wider class of potentials). The expression (3b) is in turn applicable to more potentials than (3a); if the latter exists and the integral in (3b) converges in the proper Lebesgue sense, then they equal each other [10,16]. On the other hand, (3c) does not correspond exactly to the heuristic prescription of Feynman [21] and choice of S_σ burdens the definition with an additional arbitrariness. Except for that, the improper integrals in (3b, c) are sensitive to the defining prescription [16,22] so it is desirable to get rid of them.

Let us turn now to discussion of possible limits w.r.t. the partitions σ. Again various possibilities arise:

(a) <u>the n^{-1}- and 2^{-n}-approximations:</u> the simplest choice is to take the sequence $\{\sigma_n^e\}_{n=1}^\infty$ of "equidistant" partitions [5-10], i.e., with $\tau_i^{(n)} = it/n$, and to set $\Phi_s := \lim_{n\to\infty} \phi_s(\sigma_n^e)$. The same can be performed with any subsequence of $\{\sigma_n^e\}_{n=1}^\infty$, in particular that one with $\delta_i^{(m)} = 2^{-m}t$.

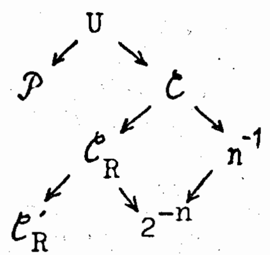
(b) <u>the $\mathcal{C}$-approximation:</u> one assumes all crumbling sequences and sets $\Phi_s := \lim_{m\to\infty} \phi_s(\sigma_m)$ if the limit exists for each $\{\sigma_m\}_{m=1}^\infty \in \mathcal{C}(J^t)$ and does not depend on a particular choice of the sequence.

(c) <u>the $\mathcal{C}_R$-and $\mathcal{C}_R'$-approximations:</u> are analogous to (b) with $\mathcal{C}(J^t)$ replaced by $\mathcal{C}_R$, the subset of refining sequences in $\mathcal{C}(J^t)$, i.e., with $\sigma_{m+1} \supset \sigma_m$ for all m, or by some subset $\mathcal{C}_R' \subset \mathcal{C}_R$. This is the method of Tarski [14] if we identify his path space with our $\mathcal{H}$ and his projections with $P(\sigma), \sigma \in \mathcal{P}(J^t)$. He employs increasing sequences of projections with unit limit, it means just the assumptions formulated above in view of Theorems I.1 and I.2. His reference families correspond, of course, to subsets $\mathcal{C}_R' \subset \mathcal{C}_R$.

(d) <u>the $\mathcal{P}$-approximation:</u> the set $\mathcal{P}(J^t)$ is partially ordered by $\supset$, further to each $\sigma, \sigma' \in \mathcal{P}(J^t)$ there exists $\sigma'' \in \mathcal{P}(J^t)$, say $\sigma'' = \sigma \cup \sigma'$ so that $\sigma \subset \sigma''$ and $\sigma' \subset \sigma''$. In other words, $\mathcal{P}(J^t)$ with $\supset$ is a directed set, and one can define Φ_s as a limit along it, $\Phi_s := \lim_{\mathcal{P}} \phi_s(\sigma)$. More explicitly, there exists $\sigma_\epsilon \in \mathcal{P}(J^t)$ to each $\epsilon > 0$ such that $|\Phi_s - \phi_s(\sigma)| < \epsilon$ for all $\sigma \supset \sigma_\epsilon$.

(e) the uniform approximation: one assumes again all sequences $\{\sigma_m\}_{m=1}^{\infty} \in \mathcal{C}(J^t)$, but requires now the convergence to be uniform w.r.t. the "norm" $\delta(\sigma)$ of σ, i.e., $\Phi_s := \lim_{\delta(\sigma)\to 0} \phi_s(\sigma)$, where the limit is understood in the same sense as in Theorem I.2. This approach belongs to Cameron[2,3] and was further used, e.g., by Johnson and Skoug[13]; a similar approximation, however, with a not very clearly specified subset of $\mathcal{C}(J^t)$ was recently used by Elworthy and Truman[12].

Proposition 1: Mutual relations of the above listed limiting prescriptions are given by the following diagram



where the arrows denote implications.

Proof: (U) $\Rightarrow$ ($\mathcal{P}$): there exists $\delta[\epsilon]$ to each $\epsilon > 0$; we choose $\sigma_\epsilon \in \mathcal{P}(J^t)$ such that $\delta(\sigma_\epsilon) < \delta[\epsilon]$, then we have $\delta(\sigma) < \delta[\epsilon]$ for all $\sigma \supset \sigma_\epsilon$, and therefore $|\Phi_s^u - \phi_s(\sigma)| < \epsilon$ for all $\sigma \supset \sigma_\epsilon$, i.e., $\lim_{\mathcal{P}} \phi_s(\sigma) = \Phi_s^u = \lim_{\delta(\sigma)\to 0} \phi_s(\sigma)$. (U) $\Rightarrow$ ($\mathcal{C}$): let us take $\epsilon > 0$, to which some $\delta[\epsilon]$ corresponds, and an arbitrary $\{\sigma_m\}_{m=1}^{\infty} \in \mathcal{C}(J^t)$; since the latter is crumbling there exists $m_0(\delta[\epsilon])$ such that $\delta(\sigma_m) < \delta[\epsilon]$ for all $m > m_0$, and consequently $|\Phi_s^u - \phi_s(\sigma_m)| < \epsilon$ for $m > m_0$, i.e., $\lim_{m\to\infty} \phi_s(\sigma_m) = \Phi_s^u$ independently of $\{\sigma_m\}_{m=1}^{\infty}$. The remaining implications are trivial. ■

We postpone commenting these relations to the next section. Now we shall use the uniform polygonal-path approximation in order to extend domain of the F-maps introduced in paper[16] to which we refer for the notation.

A function $f: \mathcal{H} \to \mathbf{C}$ is said to belong to $\mathcal{F}_s^u(\mathcal{H})$ for a given $s \in \mathbf{C}_F$ if the following conditions are fulfilled:

(i) the "cylindrical projections" $f \circ P(\sigma)$ belong to $\mathcal{F}(\mathcal{H})$, the B-algebra of F-integrable functions, for all $\sigma \in \mathcal{P}(J^t)$,

(ii) the uniform limit $\lim_{\delta(\sigma)\to 0} I_s(f \circ P(\sigma))$ exists. Then we define naturally the uniformly extended F_s-map in the following way

$$I_s^u : \mathcal{F}_s^u(\mathcal{H}) \to \mathbf{C}, \quad I_s^u(f) := \lim_{\delta(\sigma)\to 0} I_s(f \circ P(\sigma)), \tag{4}$$

in particular, $I_1^u(.)$ will be again called the Feynman integral.

Of course, one must check consistency of this extension. Another problem which arises here concerns relations between $I_{-i}^u(.)$ and the Wiener integral; it can be solved under suitable smoothness and boundedness assumptions. Let us denote $\mathcal{P}[J^t; \mathbf{R}^d] = \{\gamma_\sigma : \gamma_\sigma = P(\sigma)\gamma,\ \sigma \in \mathcal{P}(J^t),\ \gamma \in \mathcal{H}\}$, the set of all polygonal paths in $\mathcal{H}$, further w and $\|.\|_\infty$ will be the Wiener measure (understood as the n-fold product measure of W-measures with unit dispersion on $C_0[J^t; \mathbf{R}]$ – cf.[23]) and the uniform norm on $C_0[J^t; \mathbf{R}^d]$, respectively.

Theorem 1: (a) Let $s \in \mathbf{C}_F$, then $\mathcal{F}_s^u(\mathcal{H}) \supset \mathcal{F}(\mathcal{H})$ and $I_s^u(f) = I_s(f)$ for each $f \in \mathcal{F}(\mathcal{H})$.
(b) Let $f \in \mathcal{F}_{-i}^u(\mathcal{H})$ be a restriction to $\mathcal{H}$ of a w-measurable function $F: C_0[J^t; \mathbf{R}] \to \mathbf{C}$ which is uniformly continuous with exception of a w-zero subset of $C_0[J^t; \mathbf{R}^d] \setminus \mathcal{P}[J^t; \mathbf{R}^d]$. If there exists $K > 0$ such that $|f(\gamma)| \le K$ for all $\gamma \in \mathcal{P}[J^t; \mathbf{R}^d]$, then

$$I_{-i}^u(f) = \int_{C_0[J^t; \mathbf{R}^d]} F(\gamma)\, dw(\gamma) \tag{5}$$

and $|I_{-i}^u(f)| \le K$. In particular, if $f \in \mathcal{F}(\mathcal{H})$, $f(\gamma) = \int_{\mathcal{H}} \exp(i(\gamma, \gamma'))\, d\mu(\gamma')$, $\mu \in \mathcal{M}(\mathcal{H})$, then

$$\int_{C_0[J^t; \mathbf{R}^d]} F(\gamma)\, dw(\gamma) = \int_{\mathcal{H}} \exp(-\tfrac{1}{2}\|\gamma\|^2)\, d\mu(\gamma). \tag{6}$$

Proof: (a) Let $f \in \mathcal{F}(\mathcal{H})$, then the same argument as in the proof of Theorem 2 in[16] shows that the condition (i) is fulfilled. As to (ii), it follows immediately from Theorem I.2 together with Theorem 1 (c) of[16].
(b) The set $\mathcal{P}[J^t; \mathbf{R}^d]$ is $\|.\|_\infty$-dense in $C_0[J^t; \mathbf{R}^d]$: assume some $\gamma \in C_0[J^t; \mathbf{R}^d]$ and an arbitrary $\epsilon > 0$, then according to the Weierstrass theorem there exist polynomials π_j such that $|\pi_j(\tau) - \gamma_j(\tau)| < \frac{1}{3} d^{-1/2} \epsilon$ for all $\tau \in J^t$, $j = 1, \ldots, d$, so the path $\pi^\epsilon : \pi^\epsilon(\tau) = (\pi_1(\tau), \ldots, \pi_d(\tau))$ obeys

$$\|\pi^\epsilon - \gamma\|_\infty < \frac{1}{3}\epsilon. \tag{*}$$

Further each π_j can be approximated by $P^c(\sigma)\pi_j$ (cf. (I.4)): if $\pi_j(\tau)=\sum_{k=0}^{n} a_k \tau^k$, one obtains easily $|\pi_j(\tau)-(P^c(\sigma)\pi_j)(\tau)| \le$ $\le 2\delta(\sigma)\sum_{k=1}^{n} k|a_k| t^{k-1}$, thus there exists $\delta_0(\epsilon)$ such that

$$\|\pi^\epsilon - P^c(\sigma)\pi^\epsilon\|_\infty < \frac{1}{3}\epsilon \qquad (**)$$

for all σ with $\delta(\sigma) < \delta_0(\epsilon)$. Finally, the inequality

$$\|P^c(\sigma)\gamma\|_\infty \le \|\gamma\|_\infty, \quad \gamma \in C_0[J^t; R^d]. \qquad (7)$$

together with (*) give $\|P^c(\sigma)\pi^\epsilon - P^c(\sigma)\gamma\|_\infty < \frac{1}{3}\epsilon$; combining it with (*), (**) we arrive at the relation

$$\lim_{\delta(\sigma)\to 0} \|\gamma - P^c(\sigma)\gamma\|_\infty = 0, \quad \gamma \in C_0[J^t; R^d]. \qquad (8)$$

Now the assumed continuity of F implies

$$\lim_{\delta(\sigma)\to 0} F(P^c(\sigma)\gamma) = F(\gamma) \quad w\text{-a.e. in } C_0[J^t; R^d]. \qquad (9)$$

and

$$|F(\gamma)| \le K \quad \text{for } w\text{-almost all } \gamma. \qquad (10)$$

Further we take $F(P^c(\sigma)\gamma) = F(P^c(\sigma)\gamma, \ldots, P^c(\sigma)\gamma_d)$; owing to the definition of w as a product measure and using the Fubini theorem we get

$$\int_{C_0[J^t; R^d]} F(P^c(\sigma)\gamma)\, dw(\gamma) =$$

$$= \int_{C_0[J^t; R]} dw_1(\gamma_1)\ldots \int_{C_0[J^t; R]} dw_1(\gamma_d) F(P^c(\sigma)\gamma_1, \ldots, P^c(\sigma)\gamma_d).$$

Applying now d-times the standard formula to the above cylindrical integrals w.r.t. w_1 and using the Fubini theorem once more we can rewrite the last expression as follows

$$[(2\pi)^n \prod_{k=0}^{n-1} \delta_k]^{-d/2} \int_{R^{nd}} \exp\{-\frac{1}{2}\sum_{k=0}^{n-1} |\gamma^{k+1} - \gamma^k|^2 \delta_k^{-1}\} f_\sigma(\gamma^0, \ldots$$

$$\ldots, \gamma^{n-1})\, d\gamma^0 \ldots d\gamma^{n-1},$$

where $f_\sigma(\gamma^0, \ldots, \gamma^{n-1}) := F(P^c(\sigma)\gamma)$. On the other hand, one has $f_\sigma(\gamma^0, \ldots, \gamma^{n-1}) = f(P(\sigma)\gamma)$ for $\gamma \in \mathcal{H}$. Further the relation (I.10) makes it possible to express $P(\sigma)\gamma$ in terms of the orthonormal basis (I.17):

$$P(\sigma)\gamma = \sum_{k=0}^{n-1} \sum_{j=1}^{d} \delta_k^{-1/2} (\gamma^{k+1} - \gamma^k)_j \gamma_{kj}$$

so we can make the substitution $(\gamma^0, \ldots, \gamma^{n-1}) \to \gamma_\sigma \equiv P(\sigma)\gamma$ in the last integral and obtain

$$\int_{C_0[J^t; R^d]} F(P^c(\sigma)\gamma)\, dw(\gamma) = (2\pi)^{-nd/2} \int_{P(\sigma)\mathcal{H}} \exp\{-\frac{1}{2}\|\gamma_\sigma\|^2\} f(\gamma_\sigma)\, dm(\gamma_\sigma)$$
$$= I_{-i}(f \circ P(\sigma)), \qquad (11)$$

where m is the Lebesgue measure on $P(\sigma)\mathcal{H}$; the last equality follows from sec. 3 (ii, iv) in [16]. Due to the assumption the rhs of (11) tends to $I^u_{-i}(f)$ with $\delta(\sigma) \to 0$, further (5) and the related bound follow from the dominated convergence theorem, (8), (9) and the normalization of w. Finally, if $f \in \mathcal{F}(\mathcal{H})$, then $|f(\gamma)| \le \|f\|_0$ for all $\gamma \in \mathcal{H}$ due to Proposition 2 of [16] and the assertion (a) together with (5) and the definition of $I_{-i}(.)$ prove (6). ■

3. CONCLUDING REMARKS

The polygonal path-methods are not, of course, the only tool of the F-integral theory. Their results must be compared with the results of other approaches, in particular with the methods of Itô[24], DeWitt-Morette[22,25,26] and those based on analytic continuation of the Wiener integral (e.g., Cameron[2,3,27], Cameron and Storvick[4,28], Johnson and Skoug [13]). Anyhow we feel that, though the situation on this field is a little better now than that described by Dyson[29] nine years ago, the existence of different "weakly interacting" concepts and of many scattered results represents the challenge to deal with for both mathematicians and physicists.

In conclusion, let us make some comments on the matters discussed in the previous section:

(a) <u>On the choice of $\phi_s(\sigma)$:</u> starting from (3a) one can avoid complications with improper integrals in definition of the F-integral. Except of that, the analogous approach to the F-maps (cf. condition (i) of the above definition) allows to treat them on the equal footing for all $s \in C_F$ including the real ones. On the other hand, the definition under consideration applies to those f only for which all $f \circ P(\sigma)$ are continuous (cf. Proposition 2 of ref. [16]), and this seems to be too restrictive from the viewpoint of physical interest. An alternative way is to consider the case of real s, i.e. the F-integrals, separately; it was pursued, e.g., by Cameron[2,27] Cameron and Storvick[4] or by Johnson and Skoug[13]. In this approach one defines the F_s-maps for $\mathrm{Im}\, s < 0$ by polygonal-path approximation based on $\phi_s(\sigma)$ defined in the analogy with (3b); obviously improper integrals are not needed for a reasonable class of functions f. The F-integral Φ_1 of f is then defined as $\lim_{\epsilon\to 0+}\Phi_{1-i\epsilon}$. The idea of this definition is thus near to that of Gel'fand and Yaglom[5], however, with replacement of the erroneous measure-theoretical determination of $\phi_s(\sigma)$ by the sequential one. Let us mention finally that a similar procedure can be applied to $\phi_s(\sigma)$ defined in the analogy with (3c). Such a method could be promising, if only independence on a chosen Riemannian approximation to the action has been established. A certain progress in this direction was achieved by Cameron[3].

(b) On the limiting prescriptions: Proposition 1 illustrates the dominating role of the uniform approximation. As to the $\mathcal{P}$-approximation, we have not found it used in the literature, however, it represents one of the natural choices. Let us remind in this connection the Itô's definition[24] of the F-integral, where the limit is taken along the directed set of all trace-class covariance operators. Let us further stress out that there is no direct correspondence between the $\mathcal{P}$- and $\mathcal{C}_R$-approximations, because in general convergence of a net, the index set of which is not fully ordered, does not imply convergence of its subnets (in particular, subsequences) and vice versa. In the same sense one cannot assert that the Itô's definition yields a sequential approximation (cf. P.6, sec.2 in ref.[20]) without an extra proof.

(c) On Theorem 3: a somewhat stronger assertion can be formulated, namely instead of bounded functions in part (b) one can assume those with limited order of growth. For this purpose one has to know the distribution of $\|\gamma\|_\infty$ w.r.t. w: if $d=1$ then the deduction of Cameron[2] can be adapted (see below), the general case will be discussed elsewhere. However, as presently states, the assumptions of the part (b) cover most of the physically interesting functions (cf. the rhs of the Feynman-Kac formula [30]). Let us further notice that the function F can be discontinuous (on a w-zero set), but only outside $\mathcal{P}[J^t;R^d]$. The analogous assumption is not stated explicitly in the mentioned paper of Cameron, however, it is clear from the proof. Finally the relation (6) was first obtained (for $d=1$, and in a slightly weaker form) by Truman[10].

(d) On the Cameron's sequential Wiener integral: it is defined (for $d=1$) by the uniform polygonal-path limit of $\phi_{-i}(\sigma)$ analogous to (3b). Cameron[2] deduced sufficient conditions under which it can be identified with the usual Wiener integral; they are alike our Theorem 1(b),

however, the modulus $|F(\gamma)|$ is allowed to grow w.r.t. $\|\gamma\|_\infty$ polynomially or even exponentially (cf. Theorem 1 of the mentioned paper). Proof of this assertion depends essentially on the distribution ω of $\|\gamma\|_\infty$ w.r.t. the Wiener measure w_1 borrowed from Erdös and Kac[31]. The Cameron's argument is wrong at this place: one can check easily that it is not Theorem I, but Theorem II of Erdös and Kac which gives ω. Consequently, only a much weaker assertion can be proved (and, of course, the assumptions of Theorems 2,3 and 5 of ref.[2] must be correspondingly strengthened):

Proposition 2: Let $d=1$ and F be the same as in Theorem 1(b) with exception of the boundedness condition which is replaced by the following one: there exists a measurable non-decreasing $\phi: R_+\to R_+$ such that $|F(\gamma)|\le\phi(\|\gamma\|_\infty)$ for all $\gamma\in\mathcal{P}[J^t;R]$ and $\Phi: \Phi(u)=\phi(u)u^{-3}$ belongs to $L(u_0,\infty)$ for some $u_0\ge 0$. Then the sequential W-integral of F (which is equal to $I^u_{-i}(f\restriction\mathcal{H})$, if the latter exists) exists and equals to the W-integral of F (the rhs of (5) with $d=1, w=w_1$).

The core of the proof is to justify use of the dominated convergence theorem for

$$\int_{C_0[J^t;R]} F(P^c(\sigma)\gamma)\,dw_1(\gamma).$$

The inequality (7) together with the assumed monotony of ϕ imply $|F(P^c(\sigma)\gamma)|\le\phi(\|\gamma\|_\infty)$ for all $\gamma\in C_0[J^t;R]$ so one has to check that the integral

$$I\equiv\int_{C_0[J^t;R]}\phi(\|\gamma\|_\infty)\,dw_1(\gamma)$$

is finite. The image-measure theorem gives $I=\int_0^\infty\phi(u)\omega(u)\,du$ and the mentioned Theorem II of Erdös and Kac asserts particularly that

$$\lim_{n\to\infty} w_1\{\gamma: |\gamma(kt/n)|\le\alpha,\ k=0,1,\dots,n-1\}=\int_0^\alpha\omega(u)\,du=$$
$$=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\exp(-(2m+1)\pi^2 t/8\alpha^2).$$

Now one has to sum the last series and to take its derivative w.r.t. α in order to obtain

$$\omega(u)=\frac{1}{2}\pi t[u^3\,\mathrm{ch}(\pi^2 t/8u^2)]^{-1}.$$

It holds $0<\omega(u)<\frac{1}{2}\pi t u^{-3}$ for $u>0$ and the rhs of this inequality gives the asymptotics of ω for large u; thus using the assumptions about ϕ we obtain finally

$$I<\int_0^{u_0}\phi(u)\omega(u)\,du+\frac{1}{2}\pi t\int_{u_0}^{\infty}\phi(u)u^{-3}\,du<\infty.$$

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