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POLYGONAL-PATH APPROXIMATIONS ON THE PATH SPACES OF QUANTUM-MECHANICAL SYSTEMS: PROPERTIES OF THE POLYGONAL PATHS

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1. INTRODUCTION

Functional integration often employs methods in which the path space under consideration is replaced by the subset of polygonal paths. In this way, e.g., the Wiener integral of sufficiently smooth function(al)s can be evaluated [1,2]. However, while in the mentioned case the polygonal-path approximations represent a useful calculation technique, they are of conceptual importance for the Feynman integral because of absence of its sufficiently general and widely accepted definition.

The polygonal paths were connected closely with the very beginning of the concept of F-integral [3,4]. Later they have appeared in various attempts to develop a rigorous F-integral theory, to say nothing of numerous non-rigorous calculations. We have listed some of these attempts in the introduction of our previous paper [5]; among them Nelson's variant of Feynman's heuristic definition [6,7] and its generalization [8] as well as the method of Truman [2,9,10] are based on variously modified polygonal-path approximation. Let us remind some other treatments in which this idea played a central role: the mentioned paper of Cameron [1] and those of Gel'fand and Yaglom [11], Babbitt [12] (for a more complete bibliography till the middle of seventies we refer to [13], and further to [10] and other papers contained in [14]), or recently the treatment of F-integrals on Riemannian manifolds [15] and the general cylindrical approximation [16] with a particular choice of the path space and the reference family.

The present paper and its sequel [17] are devoted to study of polygonal-path approximations on the Hilbert space of paths which refers to a quantum-mechanical system with d degrees of freedom. In the following section we examine in detail properties of time-interval partitions and of the corresponding polygonal paths. Results of this treatment will serve in ref. [17] for discussion of different types of polygonal-path approximations, in particular those used by the above-named authors. Further we shall apply there the "strongest" one of them, the uniform polygonal-path approximation, to extend domain of the F-maps introduced in [5] and to prove consistency of this extension. Among these maps the F_{-i}-map is particularly interesting: we shall give some sufficient conditions under which it can be identified with the Wiener integral. For d=1 the F_{-i}-map is closely related to the sequential Wiener integral of Cameron/1/. We shall show that the basic theorem concerning the latter must be improved: strenghthened conditions on the order of growth under which the assertion holds are given in the conclusion of ref./17/. Applications of the polygonally extended F-integrals to solving Schrödinger-type equations will be discussed elsewhere/18/.

2. PARTITIONS AND POLYGONAL PATHS

We shall consider a d-dimensional quantum-mechanical system referring to the configuration space $\boldsymbol{R}^d$ the elements of which will be abbreviated as $x=(x_1,\dots,x_d)$. Let us introduce first some notation:

$J^t=[0,t],\quad t>0$,

$\gamma:\ J^t\to\boldsymbol{R}^d$ is a $\boldsymbol{R}^d$-valued function, $\gamma(\tau)=(\gamma_1(\tau),\dots,\gamma_d(\tau))$, conventionally $\gamma(\tau)\cdot\tilde\gamma(\tau)=\sum_{j=1}^{d}\gamma_j(\tau)\tilde\gamma_j(\tau)$, further $\gamma^2(\tau)=\gamma(\tau)\cdot\gamma(\tau)$ and $|\gamma(\tau)|=\sqrt{\gamma^2(\tau)}$,

$C_0[J^t;\boldsymbol{R}^d]=\{\gamma: J^t\to\boldsymbol{R}^d:\gamma$ continuous in J^t, $\gamma(t)=0\}$,

γ is said to be absolutely continuous in J^t iff γ_i, $j=1,\dots,d$, are absolutely continuous in J^t,

$AC_0[J^t;\boldsymbol{R}^d]=\{\gamma\in C_0[J^t;\boldsymbol{R}^d]:\gamma$ absolutely continuous in $J^t,\dot\gamma\in L^2(J^t;\boldsymbol{R}^d)\}$,

clearly $\dot\gamma\in L^2(J^t;\boldsymbol{R}^d)$ iff $\dot\gamma_j\in L^2(J^t;\boldsymbol{R}^d)$, $j=1,\dots,d$,

$$(\gamma,\tilde\gamma)=\int_{J^t}\dot\gamma(\tau)\cdot\dot{\tilde\gamma}(\tau)\,d\tau=\sum_{j=1}^{d}\int_{J^t}\dot\gamma_j(\tau)\dot{\tilde\gamma}_j(\tau)\,d\tau\,.$$

We shall addopt in the following $AC_0[J^t;\boldsymbol{R}^d]$ as the path space; for the sake of simplicity we shall denote it often as $\mathcal{H}$. The following assertion is valid/19/:

Proposition 1: (a) $AC_0[J^t;\boldsymbol{R}^d]$ equipped with the inner product $(.,.)$ is the real separable Hilbert space.
(b) The elements of $AC_0[J^t;\boldsymbol{R}^d]$ can be expressed by means of trigonometric series: if γ is an arbitrary element of $AC_0[J^t;\boldsymbol{R}^d]$, then there exist $\alpha_0\in\boldsymbol{R}^d$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}\subset\boldsymbol{R}^d$ which obey

$$\sum_{n=1}^{\infty}(\alpha_n^2+\beta_n^2)=\sum_{n=1}^{\infty}\sum_{j=1}^{d}(\alpha_{nj}^2+\beta_{nj}^2)<\infty \qquad (1)$$

and such that

$$\gamma(\tau)=\alpha_0(\tau-t)+\sum_{n=1}^{\infty}\frac{\alpha_n t}{2\pi n}\sin\left(\frac{2\pi n\tau}{t}\right)+\sum_{n=1}^{\infty}\frac{\beta_n t}{2\pi n}\left(1-\cos\left(\frac{2\pi n\tau}{t}\right)\right) \qquad (2)$$

for all $\tau\in J^t$. Conversely, any sequence of $\boldsymbol{R}^d$-valued coefficients which fulfils (1) determines through (2) some element of $AC_0[J^t;\boldsymbol{R}^d]$. Finally, if $\tilde\alpha_0$, $\{\tilde\alpha_n\}$, $\{\tilde\beta_n\}$ refer to $\tilde\gamma\in AC[J^t;\boldsymbol{R}^d]$, the inner product is given by

$$(\gamma,\tilde\gamma)=t\alpha_0\cdot\tilde\alpha_0+\frac{t}{2}\sum_{n=1}^{\infty}(\alpha_n\cdot\tilde\alpha_n+\beta_n\cdot\tilde\beta_n)\,. \qquad (3)$$

As mentioned above we shall deal with polygonal approximations to the elements of $\mathcal{H}$. To this purpose we introduce first some more notions. Partition of J^t is a set $\sigma=\{\tau_i: i=0,\dots,n\}$, $0=\tau_0<\tau_1<\dots<\tau_n=t$. The family of all these partitions is denoted as $\mathcal{P}(J^t)$, further we introduce $\Delta_i=[\tau_i,\tau_{i+1}]$ and $\delta_i=\tau_{i+1}-\tau_i$. A partition σ' is said to be refinement of σ, $\sigma'\supset\sigma$, if each Δ'_k is contained in some Δ_j; clearly $\supset$ defines a partial ordering on $\mathcal{P}(J^t)$ without maximal elements. The symbols $\sigma\cap\sigma'$ and $\sigma\cup\sigma'$ mean the partitions obtained by natural ordering of the intersection and union of σ,σ', respectively. A partition is said to decompose to subpartitions $\sigma^{(1)},\dots,\sigma^{(r)}$, $\sigma=\{\sigma^{(1)},\dots,\sigma^{(r)}\}$ if $\sigma^{(1)}=\{\tau_i: i=0,\dots,i_1\}$, $\sigma^{(2)}=\{\tau_i: i=i_1,\dots,i_2\},\dots,\sigma^{(r)}=\{\tau_i: i_{r-1},\dots,n\}$ (endpoints of the neighbouring partitions coincide). Let $\sigma\supset\sigma'$ so that $\tau'_k=\tau_{i_k}$ for each $k=0,1,\dots,n'$, then the decomposition $\sigma=\{\sigma^{(1)},\dots,\sigma^{(n')}\}$, $\sigma^{(k)}=\{\tau_i: i=i_{k-1},\dots,i_k\}$ is said to be generated by σ'. Decompositions $\sigma=\{\sigma^{(1)},\dots,\sigma^{(r)}\}$, $\tilde\sigma=\{\tilde\sigma^{(1)},\dots,\tilde\sigma^{(r)}\}$ are comparable if $\tau_{i_j}=\tilde\tau_{k_j}$, $j=1,\dots,r$; in other words, if the subpartitions of σ and $\tilde\sigma$ refer to the same subintervals of J^t. Partitions $\sigma,\tilde\sigma\in\mathcal{P}(J^t)$ are said to be commuting if there exist comparable decompositions $\sigma=\{\sigma^{(1)},\dots,\sigma^{(r)}\}$ and $\tilde\sigma=\{\tilde\sigma^{(1)},\dots,\tilde\sigma^{(r)}\}$ such that $\sigma^{(j)}\subset\tilde\sigma^{(j)}$ or $\sigma^{(j)}\supset\tilde\sigma^{(j)}$ for each $j=1,\dots,r$. Clearly, σ and σ' commute if one of them refines the other. Finally, we introduce $\sigma'*\sigma=\{0,t\}\cup\{\tau'_k:[\tau'_{k-1},\tau'_{k+1}]\not\subset\Delta_i, i=1,\dots,n\}$. The following auxiliary statement holds:

Proposition 2: Partitions $\sigma,\sigma'\in\mathcal{P}(J^t)$ commute iff $\sigma*\sigma'=\sigma'*\sigma$; then $\sigma*\sigma'=\sigma'*\sigma=\sigma\cap\sigma'$.

Proof: (a) Let $\sigma,\tilde\sigma$ commute so that there exist comparable decompositions $\sigma=\{\sigma^{(1)},\dots,\sigma^{(r)}\}$, $\tilde\sigma=\{\tilde\sigma^{(1)},\dots,\tilde\sigma^{(r)}\}$. Assume an arbitrary $j=1,\dots,r$. The endpoints $\tau_{i_j}=\tilde\tau_{k_j}$ and $\tau_{i_{j+1}}=\tilde\tau_{k_{j+1}}$ of the subpartitions $\sigma^{(j)},\tilde\sigma^{(j)}$ belong clearly to $\sigma\cap\tilde\sigma$ and also to $\sigma*\tilde\sigma$ and $\tilde\sigma*\sigma$ since their neighbouring points belong to different intervals of the other partition. Let, e.g., $\sigma^{(j)}\supset\tilde\sigma^{(j)}$, then the points of $\sigma^{(j)}\setminus\tilde\sigma^{(j)}$ are contained neither in $\sigma\cap\tilde\sigma$, nor in $\sigma*\tilde\sigma$, nor of course in $\tilde\sigma*\sigma$. On the other hand, it can be seen easily that the subpartition $\tilde\sigma^{(j)}$ belongs wholy to the sets $\sigma\cap\tilde\sigma$, $\sigma*\tilde\sigma$, $\tilde\sigma*\sigma$. Thus these sets coincide with the more rough subpartition in every particular interval, and therefore they equal each other.

(b1) Conversely, let $\sigma * \sigma' = \sigma' * \sigma$. Assume any two neighbouring points r_{i_0}, r_{i_0+1} of σ. If there is some $r'_k \in \sigma'$, $r_{i_0} < r'_k < r_{i_0+1}$, then the following possibilities arise: either $r_{i_0} \le r'_{k-1} < r'_{k+1} \le r_{i_0+1}$ or $r'_k \in \sigma' * \sigma$ and in the same time $r'_k \notin \sigma \supset \sigma * \sigma'$, however, the latter contradicts to the assumption. As to the former one: either $r'_{k-1} = r_{i_0}$ and $r'_{k+1} = r_{i_0+1}$ or at least one of them belongs to (r_{i_0}, r_{i_0+1}) and the same argument as above can be applied. Since the partition σ' is finite we arrive to the following result: there exist some k_1, k_2, $k_1 < k < k_2$, such that $r'_{k_1} = r_{i_0}$ and $r'_{k_2} = r_{i_0+1}$. Analogously, if $r'_{k_0} < r_i < r'_{k_0+1}$, then there exist $i_1 < i < i_2$ such that $r_{i_1} = r'_{k_0}$ and $r_{i_2} = r'_{k_0+1}$.

(b2) Assume further the decomposition of σ and σ' generated by $\sigma \cap \sigma'$ which are clearly comparable. Let $\Delta = [\xi, \eta]$ be an arbitrary subinterval of $\sigma \cap \sigma'$, $\xi = r_{i_0} = r'_{k_0}$, then the interior of Δ contains points of at most one of the partitions σ, σ'. Suppose that this is not true, then either $\xi = r_{i_0} < r'_{k_0+1} < r_{i_0+1} < \eta$ or $\xi = r'_{k_0} < r_{i_0+1} < r'_{k_0+1} < \eta$; in both these cases, however, (b1) implies existence of $\zeta \in (\xi, \eta)$ which belongs to $\sigma \cap \sigma'$, but according to the assumption ξ, η are neighbouring points of $\sigma \cap \sigma'$. Consequently, one of the subpartitions referring to Δ is trivial (consisting of ξ, η only) and the other is therefore its refinement, i.e., σ and σ' commute. ■

Further we introduce for any fixed $\sigma \in \mathcal{P}(J^t)$ the mapping $P^c(\sigma): C_0[J^t; \mathbf{R}^d] \to AC_0[J^t; \mathbf{R}^d]$ by

$$(P^c(\sigma)\gamma)(r) = \gamma^i + (\gamma^{i+1} - \gamma^i)\delta_i^{-1}(r - r_i) \tag{4}$$

for $r \in \Delta_i$, $i = 0,1,\dots,n-1$, where $\gamma^i \equiv \gamma(r_i)$. It assigns obviously to each continuous path $\gamma \in C_0[J^t; \mathbf{R}^d]$ the polygonal path going through the points γ^i, $i = 0,1,\dots,n$. In what follows we shall deal mainly with the restrictions

$$P(\sigma) = P^c(\sigma) \restriction AC_0[J^t; \mathbf{R}^d]. \tag{5}$$

Properties of the operators $P(\sigma)$ can be derived easily by means of the reproduction kernel technique. Let us denote

$$g: J^t \times J^t \to \mathbf{R}^d: \; g(r, \xi) = t - \max(r, \xi), \tag{6}$$

$$G: J^t \times J^t \to \mathcal{L}(\mathbf{R}^d): \; G(r, \xi) = g(r, \xi) I_d, \tag{7}$$

where I_d is the unit operator on $\mathbf{R}^d$. As noticed by Albeverio and Hoegh-Krohn [13], $G(.,.)$ represents kernel of the operator $-\frac{d^2}{dr^2} I_d$ with boundary conditions $\phi(t) = \dot{\phi}(0) = 0$. For our purpose the following property is important:

Proposition 3: $G(.,.)$ is a reproducing kernel of $AC_0[J^t; \mathbf{R}^d]$ in the sense that $G(r,.)\beta \in AC_0[J^t; \mathbf{R}^d]$ for all $r \in J^t$, $\beta \in \mathbf{R}^d$ and the relation

$$(\gamma, G(r,.)\beta) = \beta . \gamma(r) \tag{8}$$

holds for each $\gamma \in AC_0[J^t; \mathbf{R}^d]$.

Proof: If $\beta \in \mathbf{R}^d$, then $G(r,.)\beta = \beta g(r,.)$ belongs obviously to $AC_0[J^t; \mathbf{R}^d]$. Further

$$(\gamma, G(r,.)\beta) = \int_{J^t} \dot{\gamma}(\xi) . \beta \frac{\partial g(r, \xi)}{\partial \xi} d\xi = -\int_r^t \dot{\gamma}(\xi) . \beta \; d\xi = \gamma(r) . \beta$$

because $\gamma(t) = 0$. ■

Theorem 1: (a) $P(\sigma)$ is an orthogonal projection for any $\sigma \in \mathcal{P}(J^t)$
(b) $P(\sigma)$ commutes with $P(\sigma')$ iff the partitions σ, σ' commute.
(c) $P(\sigma) \ge P(\sigma')$ iff $\sigma \supset \sigma'$.
(d) $\dim P(\sigma) = dn(\sigma)$, especially the d-dimensional subspace of linear paths ending in the origin corresponds to the trivial partition $\sigma_0 = \{0, t\}$.

Proof: (a) For an arbitrary $\gamma \in \mathcal{H}$ we have $(\mathcal{P}(\sigma)\gamma)(r_i) = \gamma^i = \gamma(r_i)$ so $(P(\sigma))^2 = P(\sigma)$. The relation

$$g(r_{i+1}, r) - g(r_i, r) = \begin{cases} -\delta_i & \dots r \le r_i \\ r - r_{i+1} & \dots r \in \Delta_i \\ 0 & \dots r \ge r_{i+1} \end{cases} \tag{9}$$

implies easily the following identity

$$(P(\sigma)\gamma)(r) = \sum_{i=0}^{n-1} (G(r_{i+1}, r) - G(r_i, r))\delta_i^{-1}(\gamma^{i+1} - \gamma^i). \tag{10}$$

Thus we can write

$$(\tilde{\gamma}, P(\sigma)\gamma) = \sum_{i=0}^{n-1} (\tilde{\gamma}, G(r_{i+1}, .)(\gamma^{i+1} - \gamma^i))\delta_i^{-1} - \sum_{i=0}^{n-1} (\tilde{\gamma}, G(r_i, .)(\gamma^{i+1} - \gamma^i))\delta_i^{-1}$$

and the reproducing kernel property (8) yields

$$(\tilde\gamma, P(\sigma)\gamma) = \sum_{i=0}^{n-1} (\tilde\gamma^{i+1} - \tilde\gamma^{i}).(\gamma^{i+1}-\gamma^{i}) = (P(\sigma)\tilde\gamma,\gamma) \qquad (11)$$

for all $\gamma,\tilde\gamma\in\mathcal{H}$, where $\tilde\gamma^k$ denotes again $\tilde\gamma(r_k)$. Consequently, the operator $P(\sigma)$ is symmetric, idempotent and defined everywhere in $\mathcal{H}$, i.e., an orthogonal projection.

(b) The mapping $\sigma\mapsto P(\sigma)$ is obviously injective. If σ,σ' do not commute, then $\sigma*\sigma' \neq \sigma'*\sigma$ due to Proposition 2, and the mentioned injectivity together with the relation $\mathrm{Ran}\,P(\sigma)P(\sigma')=\mathrm{Ran}\,P(\sigma*\sigma')$ show that $P(\sigma)$, $P(\sigma')$ do not commute too. Conversely, let σ,σ' commute. A simple calculation using relation (10) yields

$$(\tilde\gamma, P(\sigma')P(\sigma)\gamma) = \sum_{k=0}^{n'-1}\sum_{i=0}^{n-1} \Gamma_{ik}(\tilde\gamma(r'_{k+1})-\tilde\gamma(r'_k)).(\gamma(r_{i+1})-\gamma(r_i))\delta_i^{-1}(\delta'_k)^{-1}, \qquad (12)$$

where $\Gamma_{ik}=g(r_{i+1},r'_{k+1})-g(r_{i+1},r'_k)-g(r_i,r'_{k+1})+g(r_i,r'_k)$, for arbitrary $\gamma,\tilde\gamma\in\mathcal{H}$, and analogously

$$(\tilde\gamma, P(\sigma)P(\sigma')\gamma) = \sum_{i=0}^{n-1}\sum_{k=0}^{n'-1} \Gamma'_{ki}(\tilde\gamma(r_{i+1})-\tilde\gamma(r_i)).(\gamma(r'_{k+1})-\gamma(r'_k))\delta_i^{-1}(\delta'_k)^{-1}, \qquad (13)$$

where $\Gamma'_{ki}=g(r'_{k+1},r_{i+1})-g(r'_{k+1},r_i)-g(r'_k,r_{i+1})+g(r'_k,r_i)$. The interval can be decomposed due to the assumption into subintervals $\Delta^{(j)}$, $j=1,\dots,r$, such that in each of them the corresponding subpartitions fulfil either $\sigma^{(j)}\supset(\sigma')^{(j)}$ or $\sigma^{(j)}\subset(\sigma')^{(j)}$. In the first case the relation (9) implies

$$\Gamma_{ik}=\Gamma'_{ki}=\begin{cases}\delta_i & \dots\ \Delta_i\subset\Delta'_k\\ 0 & \dots\ \text{otherwise}\end{cases} \qquad (*)$$

for each $\Delta'_k\subset\Delta^{(j)}$. Similarly, if $\sigma^{(j)}\subset(\sigma')^{(j)}$, we obtain

$$\Gamma_{ik}=\Gamma'_{ki}=\begin{cases}\delta'_k & \dots\ \Delta'_k\subset\Delta_i\\ 0 & \dots\ \text{otherwise}\end{cases} \qquad (**)$$

for each $\Delta'_k\subset\Delta^{(j)}$. In particular, $\Gamma_{ik}=\Gamma'_{ki}=0$ if the intervals Δ_i and Δ'_k are disjoint; then (12) may be rewritten as follows

$$(\tilde\gamma, P(\sigma')P(\sigma)\gamma) = \sum_{j=1}^{r}(\tilde\gamma, P(\sigma')P(\sigma)\gamma)_j, \qquad (14)$$

where

$$(\tilde\gamma, P(\sigma')P(\sigma)\gamma)_j = \sum_{\Delta'_k\subset\Delta^{(j)}}\ \sum_{\Delta_i\subset\Delta'_k}\Gamma_{ik}(\tilde\gamma(r'_{k+1})-\gamma(r'_k)).(\gamma(r_{i+1})-\gamma(r_i))\delta_i^{-1}(\delta'_k)^{-1}$$

in the case that $\sigma^{(j)}\supset(\sigma')^{(j)}$. The last relation can be simplified using (*) to the form

$$(\tilde\gamma, P(\sigma')P(\sigma)\gamma)_j = \sum_{\Delta'_k\subset\Delta^{(j)}}(\tilde\gamma(r'_{k+1})-\tilde\gamma(r'_k)).(\gamma(r'_{k+1})-\gamma(r'_k))(\delta'_k)^{-1}. \qquad (15)$$

On the other hand, if $\sigma^{(j)}\subset(\sigma')^{(j)}$, then (**) yields

$$(\tilde\gamma, P(\sigma')P(\sigma)\gamma)_j = \sum_{\Delta_i\subset\Delta^{(j)}}(\tilde\gamma(r_{i+1})-\tilde\gamma(r_i)).(\gamma(r_{i+1})-\gamma(r_i))\delta_i^{-1}. \qquad (16)$$

Further $\Gamma_{ik}=\Gamma'_{ki}$ so $(\tilde\gamma, P(\sigma)P(\sigma')\gamma)$ is expressed again by the formulae (14)-(16). Consequently, $(\tilde\gamma, P(\sigma')P(\sigma)\gamma)=(\tilde\gamma, P(\sigma)P(\sigma')\gamma)$ for all $\gamma,\tilde\gamma\in\mathcal{H}$, i.e., the projections commute.

(c) If $\sigma\supset\sigma'$, then $P(\sigma)$ and $P(\sigma')$ commute according to (b). The relations (14)-(16) now read

$$(\tilde\gamma, P(\sigma')P(\sigma)\gamma) = \sum_{\Delta'_k\subset J^t}(\tilde\gamma(r'_{k+1})-\tilde\gamma(r'_k)).(\gamma(r'_{k+1})-\gamma(r'_k))(\delta'_k)^{-1} = (\tilde\gamma, P(\sigma')\gamma)$$

(cf. (11)) for arbitrary $\gamma,\tilde\gamma\in\mathcal{H}$ so that $P(\sigma')P(\sigma)=P(\sigma)P(\sigma')=P(\sigma')$, i.e., $P(\sigma)\geq P(\sigma')$. Another equivalent formulation of the last inequality is $\mathrm{Ran}\,P(\sigma)\supset\mathrm{Ran}\,P(\sigma')$. If $\sigma\not\supset\sigma'$, then there exist Δ_{i_0} and r'_{k_0} such that $r_{i_0}<r'_{k_0}<r_{i_0+1}$. Each function from $\mathrm{Ran}\,P(\sigma)$ is linear in Δ_{i_0}; it is not true for $\mathrm{Ran}\,P(\sigma')$ which contains paths having a "corner" at $r=r'_{k_0}$. Thus $P(\sigma)\not\supset\mathrm{Ran}\,P(\sigma')$ or equivalently $P(\sigma)\not\geq P(\sigma')$

(d) Let $\{e_j\}_{j=1}^{d}$ be an orthonormal basis in $\mathbb{R}^d$ and $\sigma=\{r_i\}_{i=0}^{n}\in\mathcal{P}(J^t)$. The functions

$$\gamma_{ij}:\ \gamma_{ij}(r)=e_j(g(r_{i+1},r)-g(r_i,r))\delta_i^{-1/2}, \qquad (17)$$

$i=0,\dots,n-1$, $j=1,\dots,d$, are orthonormal and span $P(\sigma)\mathcal{H}$ due to (10). ■

A sequence $\{\sigma_m\}_{m=1}^{\infty}$ of partitions is said to be <u>crumbling</u> if the lengths of all subintervals $\Delta_i(\sigma_m)$ tend to zero with $m\to\infty$, i.e., if

$$\lim_{m\to\infty}\delta(\sigma_m)=0,\qquad \delta(\sigma_m)=\max_{0\leq i\leq n(\sigma_m)-1}\delta_i(\sigma_m). \qquad (18)$$

Such sequences are of central importance for polygonal-path approximations because of the following property:

Theorem 2: Let a sequence $\{\sigma_m\}_{m=1}^{\infty} \subset \mathcal{P}(J^t)$ be crumbling, then $\text{s-}\lim_{m\to\infty} P(\sigma_m) = I$. Furthermore, the convergence is uniform in the set $\mathfrak{C}(J^t)$ of all crumbling sequences: to each $\gamma \in \mathcal{H}$, $\epsilon > 0$ there exists $\delta[\epsilon] > 0$ such that $\| P(\sigma)\gamma - \gamma \| < \epsilon$ for all $\sigma \in \mathcal{P}(J^t)$ with $\delta(\sigma) < \delta[\epsilon]$, or symbolically

$$\text{s}-\lim_{\delta(\sigma)\to 0} P(\sigma) = I. \tag{19}$$

Remark: On the other hand, it is clear that if $\text{s-}\lim_{m\to\infty} P(\sigma_m)$ exists for a non-crumbling $\{\sigma_m\}_{m=1}^{\infty}$, it cannot be equal to the unit operator. Let us take a suitable $\gamma \in \mathcal{H}$, say $\gamma(\tau) = (2t-\tau)^2 - t^2$, then

$$\|P(\sigma)\gamma - \gamma\|^2 \geq \int_{\tau_i}^{\tau_{i+1}} |2\tau - \tau_i - \tau_{i+1}|^2 \, d\tau$$

for each subinterval $\Delta_i \in \sigma$ so that $\|P(\sigma)\gamma - \gamma\| \geq (\frac{1}{3}\delta(\sigma))^{1/2}$.

Proof of the theorem: We have to show that $V \equiv \{\gamma \in AC_0[J^t; R^d] : \lim_{\delta(\sigma)\to 0} \|P(\sigma)\gamma - \gamma\| = 0\} = AC_0[J^t; R^d]$. We shall prove first that V is a closed subspace in $\mathcal{H}$. Linearity is obvious, closedness follows from the $\frac{1}{3}\epsilon$ – trick: an arbitrary Cauchy sequence $\{\gamma^{(r)}\}_{r=1}^{\infty} \subset V$ converges in the norm to some $\gamma \in \mathcal{H}$, further

$$\|P(\sigma)\gamma - \gamma\| \leq 2\|\gamma^{(r)} - \gamma\| + \|P(\sigma)\gamma^{(r)} - \gamma^{(r)}\|. \tag{$*$}$$

To any $\epsilon > 0$ there exist $r_0(\epsilon)$ and $\delta_0(r,\epsilon)$ such that

$$\|\gamma^{(r)} - \gamma\| < \frac{1}{3}\epsilon, \qquad \|P(\sigma)\gamma^{(r)} - \gamma^{(r)}\| < \frac{1}{3}\epsilon \tag{$**$}$$

for all $r > r_0(\epsilon)$ and $\sigma \in \mathcal{P}(J^t)$ with $\delta(\sigma) < \delta_0(r,\epsilon)$. For an arbitrary partition σ with $\delta(\sigma) < \delta_0(r_0(\epsilon)+1, \epsilon)$ the relations $(*)$,$(**)$ give $\|P(\sigma)\gamma - \gamma\| < \epsilon$ so that γ belongs to V, which is therefore closed.

According to Proposition 1 the elements of $\mathcal{H}$ can be expressed by trigonometric series (2). Let $\{e_j\}_{j=1}^{d}$ be some orthonormal basis in R^d. One can check easily that the functions $u_{jk} : u_{jk}(\tau) = e_j v_k(\tau)$, $j = 1, \dots, d$, $k = 1, 2, \dots$, where

$$v_1(\tau) = \tau - t, \quad v_{2N}(\tau) = \sin\left(\frac{2\pi N \tau}{t}\right), \quad v_{2N+1}(\tau) = 1 - \cos\left(\frac{2\pi N \tau}{t}\right)$$

form an orthonormal basis in $\mathcal{H}$. It is sufficient therefore to verify that $\beta v_k(.)$ is contained in V for all $\beta \in R^d$ and $k = 1, 2, \dots$. This is trivial for k=1. Assume further k=2N and $\tau \in \Delta_i^m \equiv [\tau_i^m, \tau_{i+1}^m]$. We have

$$(P(\sigma_m)\beta v_{2N})(\tau) = \beta \sin\left(\frac{2\pi N}{t}\tau_i^m\right) + \beta\left(\sin\left(\frac{2\pi N}{t}\tau_{i+1}^m\right) - \sin\left(\frac{2\pi N}{t}\tau_i^m\right)\right)(\tau - \tau_i^m)(\delta_i^m)^{-1},$$

where $\delta_i^m \equiv \delta_i(\sigma_m)$, further a simple calculation gives

$$F(\tau) \equiv -\frac{d}{d\tau}(P(\sigma_m)\beta v_{2N})(\tau) + \beta v_{2N}(\tau) =$$

$$= -2\beta \cos\left(\frac{\pi N}{t}(2\tau_i^m + \delta_i^m)\right) \sin\left(\frac{\pi N}{t}\delta_i^m\right)(\delta_i^m)^{-1} + \frac{2\pi N \beta}{t}\cos\left(\frac{2\pi N \tau}{t}\right) =$$

$$= -\beta \cos\left(\frac{2\pi N}{t}\tau_i^m\right) \sin\left(\frac{2\pi N}{t}\delta_i^m\right)(\delta_i^m)^{-1} +$$

$$+ 2\beta \sin\left(\frac{2\pi N}{t}\tau_i^m\right) \sin^2\left(\frac{\pi N}{t}\delta_i^m\right)(\delta_i^m)^{-1} + \frac{2\pi N \beta}{t}\cos\left(\frac{2\pi N \tau}{t}\right) =$$

$$= \frac{2\pi N \beta}{t}\left\{\left[\cos\left(\frac{2\pi N}{t}\tau\right) - \cos\left(\frac{2\pi N}{t}\tau_i^m\right)\right] + \right.$$

$$\left. + \cos\left(\frac{2\pi N}{t}\tau_i^m\right)\left[1 - \frac{\sin\left(\frac{2\pi N}{t}\delta_i^m\right)}{\frac{2\pi N}{t}\delta_i^m}\right] + \sin\left(\frac{2\pi N}{t}\tau_i^m\right)\frac{\sin^2\left(\frac{\pi N}{t}\delta_i^m\right)}{\frac{\pi N}{t}\delta_i^m}\right\}.$$

The last expression can be estimated by means of the inequalities

$$|\cos x - \cos y| \leq |x - y|, \qquad \left|\frac{\sin^2 x}{x}\right| \leq |x|, \qquad \left|1 - \frac{\sin x}{x}\right| \leq \frac{1}{6}x^2;$$

we obtain in this way

$$|F(\tau)| \leq \frac{2\pi N|\beta|}{t}\left\{\frac{2\pi N}{t}(\tau - \tau_i^m) + \frac{1}{6}\left(\frac{2\pi N}{t}\delta_i^m\right)^2 + \frac{\pi N}{t}\delta_i^m\right\}.$$

Further the estimates $\frac{2\pi N}{t}\delta_i^m \leq 2\pi N$, $\frac{\pi N}{3} + \frac{1}{2} < \frac{2\pi N}{3} < \pi N$ give

$$|F(\tau)| < \left(\frac{2\pi N}{t}\right)^2 |\beta| \left((\tau - \tau_i^m) + \pi N \delta_i^m\right)$$

so

$$\int_{\tau_i^m}^{\tau_{i+1}^m} |F(\tau)|^2 \, d\tau < \left(\frac{2\pi N}{t}\right)^4 \beta^2 \left(\frac{1}{3} + \pi N + \pi^2 N^2\right)(\delta_i^m)^3 <$$

$$< \left(\frac{2\pi N}{t}\right)^4 \beta^2 (\pi N + 1)^2 (\delta_i^m)^3.$$

It holds $\delta_i^m \leq \delta(\sigma_m)$ according to (18), thus we finally obtain

$$\|P(\sigma_m)\beta v_{2N} - \beta v_{2N}\|^2 = \sum_{i=0}^{n(\sigma_m)} \int_{\tau_i^m}^{\tau_{i+1}^m} |F(\tau)|^2 \, d\tau <$$

$$< (2\pi N)^4 t^{-3} \beta^2 (\pi N + 1)^2 (\delta(\sigma_m))^2.$$

Since the sequence $\{\sigma_m\}_{m=1}^{\infty}$ is assumed to be crumbling, we obtain $\lim_{m\to\infty} P(\sigma_m)\beta v_{2N} = \beta v_{2N}$; further the last inequality shows that this convergence is uniform in $\mathcal{C}(J^t)$, i.e., $\beta v_{2N} \in V$ for any natural N. A similar argument applies to v_{2N+1}, $N = 1,2,\ldots$. ∎

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