ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА





E2 - 8097

V.N.Pervushin, M.K.Volkov



# LOW ENERGY SCATTERING OF MASSIVE PIONS. II

1974

ЛАБОРАТОРИЯ
ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

E2 - 8097

V.N.Pervushin, M.K.Volkov

# LOW ENERGY SCATTERING OF MASSIVE PIONS. II

Submitted to *ЯФ*

## 1. Introduction

In papers of Lehmann and Honerkamp et al. 1-2 the elastic scattering amplitude has been found for massless pions in the low-energy range ( $\sqrt{s} < 1$ GeV) within the chiral quantum field theory. Since at the threshold energies it is rather important to take account of the finite pion mass, the problem has naturally emerged of extension of the previous results to the case of massive pions. This problem has been solved in our paper 3 .

The expression we have obtained for the pion-pion scattering amplitude has turned out to be rather suitable for calculating scattering lengths and effective-range parameters for partial waves. In the energy region $0 \leq s \leq 4m_\pi^2$ these partial waves satisty all rigorous inequalities obtained from crossing symmetry and positivity properties due to unitarity 4 . That these general requirements on the $\pi\pi$-scattering amplitude are fulfilled in 3 indirecty, proves the fact that at low energies ( $\sqrt{s} \ll 4\pi F_\pi$, $F_\pi$ =92 MeV) the first orders of our perturbation theory expansion in powers of the parameter $1/F_\pi$ do contain the basic information on the $\pi\pi$ -scattering process.

To start our consideration we briefly recall the most important principles of chiral quantum field theory 1-3 : 1) The base of the theory is the chiral-invariant Lagrangians, therefore the theory in the Born approximation satisfies the low-energy theorems of current algebra. 2) In this theory the use is made of the perturbation theory expansion in the dimensional perameter $1/F_\pi$ at energies considerably smaller than $4\pi F_\pi \simeq 1{,}2\,\mathrm{GeV}$, where $F_\pi$ is the pion-decay constant. This energy scale arises naturally in the chiral theories 1-3,5 . 3) In calculating higher orders of the perturbation series expansion the superpropagator (SP) method is used for regularization of the divergent diagrams. By this method, for removing the ultraviolet divergences of the diagrams it suffices to exploit the nonlinearity of chiral Lagrangian. The SP regularization is achieved through considering the whole set of two-particle diagrams 6 . 4) In constructing the superpropagator a chiral-invariant method is employed for summing all the two-vertex diagrams. The method is founded on allowing for all possible reductions of matrix elements corresponding to contractions of some lines on diagrams and decrease of the number of vertices. This occurs due to the presence of derivatives in the interaction Lagrangian. Taking into account of all possible reductions allows one to obtain the results independent of choice of a form of the chiral Lagrangian (in the limit $m_\pi=0$)*).

---

*) A detailed analysis of this technique can be found in papers by D.V.Volkov 7 , J.Honerkamp 8 and V.N.Pervushin 9 .



## 2. The $\pi\pi$ -scattering amplitude

Here the necessary results of paper 3 are listed briefly. The scattering amplitude is defined in the standard way

$$\langle i_1 i_2 | S | i_3 i_4 \rangle = I + i(2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)\times$$
$$\times[\delta_{i_1 i_2}\delta_{i_3 i_4}A(s,t,u)+\delta_{i_1 i_3}\delta_{i_2 i_4}A(t,s,u)+\delta_{i_1 i_4}\delta_{i_2 i_3}A(u,t,s)], \tag{1}$$

where $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, $u=(p_1-p_4)^2$; I the unit matrix, $\delta_{ij}$ - the Kronecker symbol.

In the one-loop approximation (i.e., all the baryon and pion loops are considered to the $1/F_\pi^4$ order of perturbation series expansion) the function $A(s,t,u)$ has the form 3 :

$$\frac{A(s,t,u)}{32\pi} = \frac{\pi}{2}\alpha_0(3\bar{s}-1) + \frac{\pi}{2}\alpha_0^2\,\Pi(\bar{s},\bar{t},\bar{u}). \tag{2}$$

$$\Pi(\bar{s},\bar{t},\bar{u}) = A\bar{s}^2 + B(\bar{t}^2+\bar{u}^2) + C\bar{s} + D - J(\bar{s})(3\bar{s}-1)^2 - \tag{3}$$
$$-J(\bar{u})[3(\bar{u}-1)(\bar{u}-\bar{t})+3\bar{u}-1] - J(\bar{t})[3(\bar{t}-1)(\bar{t}-\bar{u})+3\bar{t}-1],$$

where

$$\bar{\xi} = \xi/4m_\pi^2 \quad (\xi = s,t,u)$$
$$A = 0{,}63\ ;\ B = 20{,}5\ ;\ C = 3\ ;\ D = -1{,}5\ . \tag{4}$$
$$J(x) = 1 - \frac{1}{2}\sum_1^\infty (4x)^n \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+2)} =$$
$$= \begin{cases} y\,\mathrm{arctg}\,y^{-1} & 0<x<1\ ;\ y=(1/x-1)^{1/2} \\ \frac{\bar{y}}{2}\left[-i\pi+\ln\frac{1+\bar{y}}{1-\bar{y}}\right] & x>1 \\ \frac{\bar{y}}{2}\ln\frac{\bar{y}+1}{\bar{y}-1} & x<0 \end{cases} \quad ;\ \bar{y}=(1-1/x)^{1/2} \tag{5}$$

(Relation (5) has been derived in Appendix of paper 10 ).

The first term in eq. (2) is the Born term 11 ,the second one is the loop-diagram contribution. At energies much smaller than $4\pi F_\pi$ formula (2) represents good expansion of the $\pi\pi$ elastic scattering amplitude in the small parameter $\alpha_0$:

$$\alpha_0 = \frac{1}{3}\left(\frac{m_\pi}{2\pi F_\pi}\right)^2 = \frac{2}{103} \; . \tag{6}$$

## 3. Scattering lengths

Following paper 12 let us introduce the notation

$$\alpha_\ell^{I(n)} = \lim_{\bar{s}\to 1} \frac{1}{4^n}\frac{\partial^n}{\partial \bar{s}^n} h_\ell^I(\bar{s}) , \tag{7}$$

$$\alpha_\ell^{I(0)} = a_\ell^I \; ; \; \alpha_\ell^{I(1)} = b_\ell^I \; ; \; \alpha_\ell^{I(2)} = c_\ell^I \; , \tag{8}$$

where

$$h_\ell^I(\bar{s}) = \frac{1}{2(\bar{s}-1)^\ell}\int_{-1}^{1} dx\, P_\ell(x)\, T^I(s,x) . \tag{9}$$

Here $a_\ell^I$ are scattering lengths, $b_\ell^I$ and $c_\ell^I$ effective-range parameters, $T^I$ the amplitude in the channel with isospin I, $P_\ell(x)$ the Legendre polynomials:

$$T^I = \frac{\pi}{2}\alpha_0\, T_B^I + \frac{\pi}{2}\alpha_0^2\, \Pi^I . \tag{10}$$

$$T_B^0 = 2(3\bar{s}-1) \; ; \; T_B^1 = 3(\bar{s}-1)x \; ; \; T_B^2 = -(3\bar{s}-1) . \tag{11}$$

$$\begin{aligned} \Pi^0 &= (2B+3A)\bar{s}^2 + \frac{(4B+A)}{2}(\bar{s}-1)^2(1+x^2) + 2C\bar{s} + C + 5D - 4(3\bar{s}-1)^2 J(\bar{s}) - 12\,G^{(0)}(x), \\ \Pi^1 &= (B-A)(\bar{s}-1)^2 x + C(\bar{s}-1)x - 3(\bar{s}-1)^2 x\, J(\bar{s}) + 2\,G^{(1)}(x), \\ \Pi^2 &= 2B\bar{s}^2 + \frac{(B+A)}{2}(\bar{s}-1)^2(1+x^2) - C\bar{s} + C + 2D - (3\bar{s}-1)J(\bar{s}) - 6\,G^{(2)}(x). \end{aligned} \tag{12}$$

$$\begin{aligned} G^{(0)}(x) &= J\left(-(\bar{s}-1)\frac{(1+x)}{2}\right)\left\{(\bar{s}-1)^2\left[5\left(\frac{1+x}{2}\right)^2 - \frac{1+x}{2}\right] + (\bar{s}-1)\left[3\left(\frac{1+x}{2}\right)^2 - 1\right]\right\}, \\ G^{(1)}(x) &= J\left(-(\bar{s}-1)\frac{(1+x)}{2}\right)\left\{(\bar{s}-1)^2\left[6\left(\frac{1+x}{2}\right)^2 - 3\,\frac{1+x}{2}\right] + 3(\bar{s}-1)\left[\frac{1+x}{2} - 1\right] - 1\right\}, \\ G^{(2)}(x) &= J\left(-(\bar{s}-1)\frac{(1+x)}{2}\right)\left\{(\bar{s}-1)^2\left[4\left(\frac{1+x}{2}\right)^2 + \frac{1+x}{2}\right] + (\bar{s}-1)\left[3\,\frac{1+x}{2} + 1\right] + 1\right\}. \end{aligned} \tag{13}$$

Making use of the formula

$$\frac{1}{2}\int_{-1}^{1} dx\, P_\ell(x)\left(\frac{x+1}{2}\right)^\ell = \frac{(\ell!)^2}{(2\ell+1)!} \tag{14}$$

we obtain the values for scattering lengths and effective-range parameters presented in Table I.

For $\ell \geq 3$ the above given formulae allow one to derive the following simple expressions for scattering lengths

$$\begin{aligned} a_\ell^0 &= (2\ell+1)(4\ell+7) Z_\ell \; , \\ a_\ell^1 &= \frac{1}{3}(4\ell^2-2\ell-1) Z_\ell \; , \qquad Z_\ell = 3\pi\alpha_0^2\, \frac{2^{2(\ell-1)}\,\Gamma^3(\ell+1)\,\Gamma(\ell-2)}{\Gamma^3(2\ell+2)} \; . \\ a_\ell^2 &= (4\ell^2+3\ell+8) Z_\ell \; , \end{aligned} \tag{15}$$

The values we have calculated for scattering lengths (given in Table I) are in rather good agreement with known experimental data 12,13 and with the results of phenomenological approach of Palou and Yndurain 12 . Their model uses the Froissart-Gribov representation. Table I also lists the calculation results corresponding to the choice of parameters of their model which gives the value $a_0^0$ =0.15 (see Table II in 12 ).

All the scattering lengths for $\ell \geq 3$ obey the inequality:

$$a_{\ell+2}^I \leq a_\ell^I \, \frac{(\ell+1)(\ell+2)}{4\,(2\ell+3)(2\ell+5)} \tag{16}$$

obtained from unitarity and analyticity of scattering amplitude in papers 4,14 .

## 4. The $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude in the interval $0 \leq \bar{S} \leq 1$

For the $\pi^0\pi^0 \to \pi^0\pi^0$ S-wave

$$f_0^{00}(\bar{s}) = \frac{1}{64\pi}\int_{-1}^{1} dx \left[A(s,t,u) + A(t,s,u) + A(u,t,s)\right] \tag{17}$$

in the interval $0 \leq \bar{S} \leq 1$ A.Martin in paper 4 has found various rigorous inequalities from unitarity and crossing symmetry. To recall, let us write down these inequalities:

$$
\begin{array}{ll}
1) & f_0^{00}(\bar{s}) < f_0^{00}(1)\,, \quad 0 \le \bar{s} \le 1 \\
2) & \dfrac{d f_0^{00}(\bar{s})}{d\bar{s}} > 0\,, \quad 0{,}5 \le \bar{s} \le 1\,, \\
3) & f_0^{00}(\bar{s}) \ge 2\int_{0,5}^{1} f_0^{00}(\bar{s})\,d\bar{s}\,, \\
4) & f_0^{00}(0) > f_0^{00}\left(\tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)\right) \simeq f_0^{00}\left(\tfrac{3{,}15}{4}\right), \\
5) & \dfrac{d f_0^{00}(\bar{s})}{d\bar{s}} < 0\,, \quad 0 \le \bar{s} \le 1{,}29/4 \\
6) & \dfrac{d f_0^{00}(\bar{s})}{d\bar{s}} > 0\,, \quad 1{,}7/4 \le \bar{s} \le 1{,}76/4 \\
7) & f_0^{00}(0{,}8) > f_0^{00}\left(\tfrac{0{,}21}{4}\right) > f_0^{00}\left(\tfrac{2{,}98}{4}\right).
\end{array}
\tag{18}
$$

The direct calculations show that amplitude (2) obeys all the above inequalities. Our subsequent consideration makes it easy to see this.

Our amplitude in the range $0 \le \bar{s} \le 1$ is well described by the approximate formula:

$$A(s,t,u) = \frac{\pi}{2}\alpha_0(3\bar{s}-1) + \frac{\pi}{2}\alpha_0^2\left[A\bar{s}^2 + B(\bar{t}^2+\bar{u}^2) + C\bar{s} + D\right]. \tag{19}$$

Hence,

$$f_0^{00}(\bar{s}) \cong \frac{\pi}{6}\alpha_0^2\left[(2B+A)(5\bar{s}^2-4\bar{s}+2) + 3D + C\right]. \tag{20}$$

It is quite easy to check that the parabolic function

$$f_0^{00}(\bar{s}) = \beta(5\bar{s}^2-4\bar{s}+2) + \gamma \tag{21}$$

obeys conditions (18) for arbitrary $\beta > 0$ and $\gamma$. Thus, to fulfil the Martin inequalities (18) it is sufficient to require that the inequality

$$2B + A > 0 \tag{22}$$

holds.

Since the condition (22) is fulfilled (see formula (4)) the Martin inequalities (18) are satisfied.

## 5. Conclusion

The performed here analysis of the $\pi\pi$ scattering amplitude calculated in 3 completely proves the validity of perturbation theory even for strong interactions in the chiral quantum field theory at energies much smaller than the energy scale $4\pi F_\pi \sim 1{,}2\,GeV$.



The information on the scattering amplitude contained in the two first orders of perturbation theory ( $1/F_\pi^2$ and $1/F_\pi^4$ ) quite correctly reflects all properties of the amplitude resulting from the very general assumptions 4 . The values for scattering lengths and effective-range parameters calculated in this approximation are also well consistent both with the known experimental data and with the results of phenomenological description of the $\pi\pi$ -scattering amplitude.

We would remind here that the value for the pion form factor we have obtained in the same approach 5a is in good agreement with experiment in the threshold energy range, as well. Besides, the pion radius

$$\sqrt{\langle r^2\rangle_\pi} \sim 0{,}65\,fm \tag{23}$$

is rather consistent with recent experimental data 15 . Correct results have been found for the pion photoproduction and their polarizabilities 5b .

In conclusion the authors thank D.I.Blokhintsev, V.A.Meshcheryakov, Yu.V.Parfenov, V.V.Serebryakov and D.V.Shirkov for fruitful discussions.

## References

1. H.Lehmann.Phys.Lett.,41B,529 (1972); Preprint DESY 72/54 (1972).
2. G.Ecker,J.Honerkamp.Nucl.Phys.,B52,211 (1973); B62, 509 (1973).
3. V.N.Pervushin,M.K.Volkov. Preprint JINR,E2-7661 (1974).
4. A.Martin.Nuovo Cimento,47,265 (1967); 58A,303 (1968).
5. M.K.Volkov,V.N.Pervushin.a)Yadernaya Fizika,I9,652 (I974);
b)Preprint JINR,E2-8I00 (I974).

6. M.K.Volkov.Ann.Phys.,49, 202 (1968); Teor.Mat.Fizika,6,21(1971).
7. D.V.Volkov. Particles and Nucleus, 4, 3 (1973).
8. J.Honerkamp. Nucl. Phys., B36, 130 (1972).
9. V.N.Pervushin. JINR preprint, P2-7644 (1973);JINR,E2-8009 (1974).
10. M.K.Volkov. TMF, 6, 21 (1971).
11. S.Weinberg. Phys.Rev. Lett., 18, 567 (1971).
12. F.P.Palou, F.J.Yndurain.Nuovo Cimento, 19A, 245 (1974).
13. J.P.Baton, G.Laurens and J.Reiguier.P.L.33B, 528, 1970.
14. F.J.Yndurain. Nuovo Cimento, 64A, 225 (1969).
15. G.Adylov, F.Aliev, D.Bardin et al. JINR, E2-8047 (1974).

Table I.

| $\alpha_{\ell}^{I(n)}$ | Experiment | Our values | Values of paper 12 |
|---|---|---|---|
| $a_0^0$ | $[0{,}10;\ 0{,}60]$ | $0{,}15$ | $0{,}15 \pm 0{,}02$ |
| $a_0^2$ | $[-0{,}10;\ -0{,}03]$ | $-0{,}042$ | $-0{,}065 \pm 0{,}025$ |
| $a_1^1$ | $[0{,}032;\ 0{,}040]$ | $0{,}031$ | $0{,}0341 \pm 0{,}0036$ |
| $b_1^1$ | | $1{,}14 \cdot 10^{-3}$ | $(1{,}07 \pm 0{,}27) \cdot 10^{-3}$ |
| $a_2^0$ | $[1{,}4 \cdot 10^{-3};\ 1{,}8 \cdot 10^{-3}]$ | $1{,}85 \cdot 10^{-3}$ | $(1{,}48 \pm 0{,}08) \cdot 10^{-3}$ |
| $a_2^2$ | $[-2 \cdot 10^{-4};\ 3 \cdot 10^{-4}]$ | $2{,}6 \cdot 10^{-4}$ | $(-3 \pm 8) \cdot 10^{-5}$ |
| $b_2^0$ | | $-1{,}02 \cdot 10^{-4}$ | $(-3{,}8 \pm 1{,}1) \cdot 10^{-5}$ |
| $b_2^2$ | | $-5{,}1 \cdot 10^{-5}$ | $(-4{,}4 \pm 1{,}1) \cdot 10^{-5}$ |
| $c_2^0$ | | $2 \cdot 10^{-5}$ | $(1{,}13 \pm 0{,}36) \cdot 10^{-5}$ |
| $c_2^2$ | | $1{,}04 \cdot 10^{-5}$ | $(1{,}27 \pm 0{,}36) \cdot 10^{-5}$ |
| $a_3^1$ | | $1{,}33 \cdot 10^{-5}$ | $(3{,}8 \pm 0{,}5) \cdot 10^{-5}$ |
| $a_4^0$ | | $5 \cdot 10^{-6}$ | $(4{,}8 \pm 0{,}8) \cdot 10^{-6}$ |
| $a_4^2$ | | $2 \cdot 10^{-6}$ | $(1{,}7 \pm 0{,}8) \cdot 10^{-6}$ |

Received by Publishing Department
on July 11, 1974