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ON THE MINIMAL CANONICAL REALIZATIONS OF THE LIE ALGEBRA $O_c(n)$

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Summary

For the complexification of the Lie algebra of the orthogonal group in n-dimensional space it is shown that its canonical realization by means of polynomials in N pairs of canonical variables does not exist if $N < n-3$. As canonical realization by means of $N=n-2$ pairs exist, the problem of minimal canonical realization for $O_c(n)$ is, in the general case, reduced to two possibilities only. For $n < 7$ this problem is solved completely. It is further shown that, with some exceptions, the Casimir operators in canonical realization by means of $n-2$ pairs of canonical variables are realized as multiples of the identity element and that among them there is only one independent. If particularly canonical realization by means of $n-3$ pairs exists then the values of all Casimir operators are even fixed by n.

I. Introduction

In theoretical physics we often meet the Lie algebras realized through functions of pairs of canonical variables p_i, q_i or Bose creation and annihilation operators, respectively. Generally speaking, such a situation arises if we combine the assumption that observables are functions of a certain number of canonical pairs with the assumption that some of them form

Lie algebra. In this way such canonical realizations of algebras enter in the group theoretical approach to nonrelativistic quantum mechanics based,e.g.,on the spectrum generating algebras.

In a wide class of problems the realizations can help in their solution or simplification at least. If we have,e.g.,to determine matrix elements or eigenvalues of a differential operator, the solution is considerably simplified when this operator can be either embedded in realization of some Lie algebra or it is one of its Casimir operators [1, 2]. Another field where canonical realizations are used is the construction of equations invariant with respect to a given Lie algebra [3].

The canonical realizations of Lie algebras are useful also for the theory of representations. If generators of some Lie algebra G are expressed as functions of pairs p_i and q_i then, substituting p_i and q_i by their representation, we obtain the representation of G. As we deal with functions of partly noncommuting variables we have to make more exact the concept of function. The first and most simple case is to limit ourselves to the algebra of polynomials in considered number of canonical pairs. The advantage of this limitation lies in the possibility to define the space of these polynomials (the so-called Weyl algebra) purely algebraically and, consequently, to formulate algebraically also the problem of realizations.

It is known that the Weyl algebra as well as the enveloping algebra of any Lie algebra can be algebraically embedded into quotient division ring. It allows one to enlarge the functional space and to realize Lie algebras by means of rational functions of canonical variables without change of the algebraical approach.

Further extension of the functional space requires the introduction of topology.

The study of the most simple case, i.e.,the realizations in Weyl algebra is useful also for the better understanding of the more complicated situations. In this paper we deal with realizations of the complexificated Lie algebra of the orthogonal group in n-dimensional space $O_c(n)$ in the Weyl algebra W_{2N} (N-number of canonical pairs). We are interested first in the minimal number N which is necessary for the existence of realization (i.e.,isomorphism into W_{2N}) of $O_c(n)$. There is the general result of Simoni and Zaccaria [4] (see also [5]) according to which no semisimple Lie algebra of the rank r can be realized in W_{2N} if $N < r$. We prove that any realization of $O_c(n)$ does not exist even if $N < n-3$ that extends for $n > 7$ the above result. Realizations of algebras $O_c(n)$ in $W_{2(n-2)}$ exist (see,e.g.,[6]) and therefore the problem of minimal realization (i.e.,realization in W_{2N} with minimal N) reduces to two cases $N = n-3, n-2$. For $n < 7$ we can easily decide between these two possibilities. Our explicit construction of the realization of $O_c(6)$ in $W_{2,3}$ has not been, at least to our knowledge, published in the literature. The results above named are contained in theorems 1 and 2 of section 3,and section 3 itself is devoted in essential to their proof.

It was further proved in [4, 5] that Casimir operators of semisimple Lie algebra with rank r are always realized in W_{2r} by means of constant multiple of identity element. In section 4 we extend this result for $O_c(n)$ to the realizations in $W_{2(n-3)}$ and $W_{2(n-2)}$.

With some exceptions for the lowest dimensional cases (n=4,5,6) we prove moreover that realizations of all the Casimir operators in $W_{2(n-2)}$ depend on realization of the quadratic ones and in $W_{2(n-3)}$ their values depend on n only (theorem 3).

In conclusion we discuss and reformulate the results obtained for real forms of $O_{\mathbb{C}}(n)$. We introduce the involution on W_{2N} and define the skew-symmetric realization of real Lie algebra. As a special result we obtain here the existence of skew-symmetric realizations of $O(3,2)$ and $O(3,3)$ Lie algebras in W_4 and W_6 respectively. Though the skew-symmetric realizations were defined mainly with respect to the representation theory; we do not discuss here these aspects. All considerations in this paper are purely algebraical.

2. Preliminaries

A. Let H_{2N} denote the $(2N+1)$-dimensional Heiseberg Lie algebra over field of complex numbers $\mathbb{C}$, i.e., the Lie algebra with generators $\bar{p}_i, \bar{q}_i$, where

$$[\bar{p}_i, \bar{q}_j] = c\,\delta_{ij}, \quad [c, \bar{p}_i] = [c, \bar{q}_i] = 0, \quad i,j = 1,2,\dots,N.$$

Let further $\mathcal{E}(H_{2N})$ denote the enveloping algebra of H_{2N} ([7] p.173) and let $\{c-\mathbb{1}\} \subset \mathcal{E}(H_{2N})$ be two-sided ideal generated by the element $c-\mathbb{1}$. The quasienveloping algebra of H_{2N}, i.e., factoralgebra

$$W_{2N} = \mathcal{E}(H_{2N}) / \{c-\mathbb{1}\}$$

is called Weyl algebra. Equivalence classes p_i, q_i; $p_i \ni \bar{p}_i$, $q_i \ni \bar{q}_i$ generate W_{2N} and fulfill relations

$$[p_i, q_j] = \mathbb{1}\,\delta_{ij}.$$

The consequence of the Poincare-Birkhopf-Witt theorem ([7] p. 178) is that monomials

$$q^k \cdot p^l \equiv q_1^{k_1} \cdots q_N^{k_N} \cdot p_1^{l_1} \cdots p_N^{l_N}$$

form the basis of W_{2N}, i.e., that every element $w \in W_{2N}$ can be uniquely written in the form

$$w = \sum_{k,l} a_{kl}\, q^k \cdot p^l$$

($a_{kl} \equiv a_{k_1 \dots k_N l_1 \dots l_N} \in \mathbb{C}$). Similarly, as $\mathcal{E}(H_{2N})$ is the ring without nonzero dividers of zero ([7], p. 186), the same is valid for W_{2N}, i.e.,

$$w_1, w_2 \in W_{2N}, \quad w_1 \cdot w_2 = 0 \implies w_1 = 0 \ \text{ or } \ w_2 = 0 \tag{2}$$

holds.

B. The canonical realization τ of the complex (or real) Lie algebra G we shall call on isomorphism mapping of into W_{2N}:

$$\tau : G \longrightarrow W_{2N}.$$

The canonical realization of G in W_{2N} is minimal iff that in $W_{2(N-1)}$ does not exist.
The realization

$$\tau : G \longrightarrow W_{2N}$$

induces naturally the homomorphism

$$\tau' : \mathcal{E}(G) \rightarrow W_{2N}.$$

In accordance with the mentioned Poincare-Birkhopf-Witt theorem every element $g \in \mathcal{E}(G)$ can be written in the form

$$g = \sum_{a,b,\dots,c} \alpha_{ab\dots c}\, g_1^a \cdot g_2^b \cdots g_n^c$$

($\alpha_{ab\dots c} \in \mathbb{C}$; $g_1, g_2, \dots, g_n$ are equivalence classes containing generators of G). The homomorphism τ' is then defined by relation

$$\tau'(g) \equiv \sum_{a,b,\dots,c} \alpha_{ab\dots c}\, \tau(g_1)^a \cdot \tau(g_2)^b \dots \tau(g_n)^c$$

(In what follows, the homomorphism τ' will be denoted by τ).

C. The symbol $O_c(n)\ (n>2)$ denotes the comlexification of the Lie algebra of orthogonal group in the n-dimensional Euclidean space. If $L_{\mu\nu} = -L_{\nu\mu}$, $\mu, \nu = 1,2, \dots n$ denotes $\frac{1}{2} \cdot n(n-1)$ elements of basis of $O_c(n)$ then

$$[L_{\mu\nu}, L_{\rho\tau}] = \delta_{\nu\rho} L_{\mu\tau} - \delta_{\mu\rho} L_{\nu\tau} + \delta_{\nu\tau} L_{\rho\mu} - \delta_{\mu\tau} L_{\rho\nu} \qquad (3)$$

$\rho, \tau = 1,2,3 \dots , n$. Algebra $O_c(n)$ is simple (except for the case $n = 4$), its rank is $r = [\frac{n}{2}]$ and in the Cartan classification

$$O_c(2n+1) \simeq B_n \ , \quad O_c(2n) \simeq D_n \ .$$

The number of the generating Casimir operators of $O_c(n)$ equals $[\frac{n}{2}]$. All these Casimir operators can be chosen among Casimir operators

$$I_{2k} = L_{\mu_1\mu_2} L_{\mu_2\mu_3} \dots L_{\mu_{2k}\mu_1} \quad , \quad k = 1,2,\dots$$

for algebras $O_c(2n+1)$ and, adding

$$I_n' = \varepsilon_{\mu_1\nu_1\mu_2\nu_2\dots\mu_n\nu_n} L_{\mu_1\nu_1} L_{\mu_2\nu_2} \dots L_{\mu_n\nu_n} \qquad (5)$$

also for algebras $O_c(2n)$. (Here $\varepsilon_{\mu_1 \dots \nu_n}$ is completely antisymmetric Levi-Civita tensor in 2n indices and we use the summation convetion).

It is important in our further considerations that there exists the following basis of $O_c(n)$

$$L_{ij} \ , \quad P_k = L_{kn} + i L_{kn-1} \ , \quad Q_k = L_{kn} - i L_{kn-1} \ , \quad R = i L_{n-1n} \qquad (6)$$

$i,j,k = 1,2, \dots, n-2$ [x] in which commutation relations (3) have the form:

$$[L_{ij}, L_{kl}] = \delta_{jk} L_{il} - \delta_{ik} L_{jl} + \delta_{jl} L_{ki} - \delta_{il} L_{kj} \ , \qquad (7)$$

$$[L_{ij}, P_k] = \delta_{kj} P_i - \delta_{ki} P_j \ , \quad [L_{ij}, Q_k] = \delta_{kj} Q_i - \delta_{ki} Q_j \ , \qquad (8)$$

$$[L_{ij}, R] = 0 \ , \quad [R, P_k] = P_k \ , \quad [R, Q_k] = -Q_k \ , \qquad (9)$$

$$[P_i, P_j] = [Q_i, Q_j] = 0 \ , \qquad (10)$$

$$[P_i, Q_j] = -2(L_{ij} + \delta_{ij} R) . \qquad (11)$$

---

[x] Lattin indices will run always from 1 to n-2.

Note, that generators $P_1, \dots, P_{n-2}$ and $Q_1, \dots, Q_{n-2}$ form the bases of $(n-2)$-dimensional Abelian subalgebras of $O_c(n)$.

For $n-2 \geqslant 3$ we define quadratic elements of enveloping algebra $\mathcal{E}[O_c(n)]$

$$w_{i_1 \dots i_{n-5}} \equiv \frac{1}{2} \varepsilon_{i_1 \dots i_{n-5}\, ijk} L_{ij} P_k \tag{12}$$

which commute with all P_i

$$[w_{i_1 \dots i_{n-5}} , P_i] = 0 . \tag{13}$$

These elements transform under $O_c(n-2)$ generators L_{ij} as totally antisymmetric tensor and the number of its independent components equals $\binom{n-2}{3}$. Similarly we can define the quantity with the same properties with the help of generators Q_i.

It is clear that in definition of basis (6) the preference of the indices pair $(n-1, n)$ is not essential and that they can be substituted by any other pair.

3. The minimal canonical realization of $O_c(n)$.

Let us pay attention now to the problem of minimal canonical realization of $O_c(n)$. First we shall prove two simple lemmas.

Lemma 1. Let

i τ be any canonical realization of $O_c(n)$ with basis (6),

ii $p \in \mathcal{E}[O_c(n)]$ be an element, which can be written in the form

$$p = \sum_{a=0}^{A} (\alpha_a P_1 + \beta_a) \cdot P_2^{\,a}$$

$$\alpha_a , \beta_a \in \mathcal{E}[O_c(n)] , \quad [\alpha_a , L_{12}] = [\beta_a , L_{12}] = 0 ,$$

iii $\tau(p) = 0$.

Then $\tau(\alpha_a) = \tau(\beta_a) = 0$, $a = 0, \dots, A$.

Proof proceeds by contradiction. Let us assume i-iii and the existence of integer $A_1 \leqslant A$ such that

$$\tau(\alpha_A) = \tau(\beta_A) = \dots = \tau(\alpha_{A_1+1}) = \tau(\beta_{A_1+1}) = 0$$

and

$$\tau(\alpha_{A_1}) \neq 0 \quad or \quad \tau(\beta_{A_1}) \neq 0$$

therefore

$$\tau(p) = \sum_{a=0}^{A_1} [\tau(\alpha_a) \cdot \tau(P_1) + \tau(\beta_a)] \cdot \tau(P_2)^a .$$

We introduce new "variables"

$$X = \frac{1}{2}(P_2 + iP_1) , \qquad P_2 = X + Y ,$$

$$Y = \frac{1}{2}(P_2 - iP_1) , \quad \Longleftrightarrow \quad P_1 = -i(X - Y) ,$$

in which $\tau(p)$ has the form:

$$\tau(p) = \sum_{a=0}^{A_1} \{ -i\tau(\alpha_a)[\tau(X) - \tau(Y)] + \tau(\beta_a) \} \cdot \sum_{b=0}^{a} \binom{a}{b} \tau(X)^{a-b} \cdot \tau(Y)^b .$$

Further we factorize the polynomial $\tau(p)$ into the sum of polynomials $\tau(p_c)$,

$$\tau(p) = \sum_{c=-A_1-1}^{A_1+1} \tau(p_c) ,$$

where

$$\tau(p_c) = \sum_{\substack{a,b \\ a-b=c}} \tau(\gamma_{ab}) \; \tau(X)^a \cdot \tau(Y)^b .$$

The coefficients $\tau(\gamma_{ab})$ are suitable linear combinations of $\tau(\alpha_a)$ and $\tau(\beta_{\dot{b}})$. We shall write explicitly some of these polynomials:

$$\tau(p_{A_1+1}) = -i\tau(\alpha_{A_1})\cdot\tau(X)^{A_1+1}, \tag{14}$$

$$\tau(p_{A_1}) = \begin{cases} [-i\tau(\alpha_{A_1-1}) + \tau(\beta_{A_1})]\cdot\tau(X)^{A_1}, & A_1 \neq 0; \\ \tau(\beta_0), & A_1 = 0; \end{cases} \tag{15}$$

$$\tau(p_{-A_1}) = \begin{cases} [i\tau(\alpha_{A_1-1}) + \tau(\beta_{A_1})]\cdot\tau(Y)^{A_1}, & A_1 \neq 0; \\ \tau(\beta_0), & A_1 = 0. \end{cases} \tag{16}$$

From commutation relations

$$[L_{12}, X^a] = -ia X^a, \quad [L_{12}, Y^a] = +ia Y^a$$

it follows that

$$[\tau(L_{12}), \tau(p_c)] = -ic\,\tau(p_c).$$

By means of multiple commutation of $\tau(p)$ with $\tau(L_{12})$ we obtain the homogeneous system of equations for unknown $\tau(p_c)$:

$$\tau(p) = \sum_c \tau(p_c) = 0,$$
$$[\tau(L_{12}), \tau(p)] = -i\sum_c c\,\tau(p_c) = 0,$$
$$- - - - - - - - - - - - -$$
$$[\underbrace{\tau(L_{12}), \dots, [\tau(L_{12}), \tau(p)]}_{s\text{-times}} \dots] = -i\sum_c c^s \tau(p_c) = 0,$$
$$- - - - - - - - - - - - - - - - - -$$

The system has the nonzero determinant, and therefore

$$\tau(p_{A_1+1}) = \dots = \tau(p_{-A_1-1}) = 0 \tag{17}$$

holds.

Substituting $\tau(p_{A_1+1})$ from eq. (14) and using eq. (2) we obtain

$$\tau(\alpha_{A_1}) = 0$$

because the second possibility,

$$\tau(X)^{A_1+1} = 0 \implies \tau(X) = 0,$$

contradicts the isomorphism of τ.

If $A_1 = 0$ eqs. (15) and (17) give further $\tau(\beta_0) = 0$ what is the contradiction desired.

If $A_1 \neq 0$, then eqs. (15)-(16),

$$[-i\tau(\alpha_{A_1-1}) + \tau(\beta_{A_1})]\cdot\tau(X)^{A_1} = 0,$$
$$[i\tau(\alpha_{A_1-1}) + \tau(\beta_{A_1})]\cdot\tau(Y)^{A_1} = 0$$

imply, as above, the equations

$$-i\tau(\alpha_{A_1-1}) + \tau(\beta_{A_1}) = 0,$$
$$i\tau(\alpha_{A_1-1}) + \tau(\beta_{A_1}) = 0$$

from which, immediately, $\tau(\beta_{A_1}) = 0$. The proof is finished.

Lemma 2: Let

i τ be any canonical realization of $O_{\mathbb{C}}(n)$ with basis (6).

ii $p \in \mathcal{E}[O_{\mathbb{C}}(n)]$ be an element, which can be written in the form

$$p=\sum_{a_2,\dots,a_{n-2}} \beta_{a_2\dots a_{n-2}} \cdot P_2^{a_2}\dots P_{n-2}^{a_{n-2}},$$

$$\beta_{a_2\dots a_{n-2}} \in \mathcal{E}[O_c(n)],\ [\beta_{a_2\dots a_{n-2}}, L_{ij}]=0,$$

iii $\tau(p)=0$.

Then

$$\tau(\beta_{a_2\dots a_{n-2}})=0 \quad \textit{for all } a_i,\ i=2,3,\dots,n-2.$$

Proof.: For p, considered as a polynomial in "variables" P_1, P_2 all the assumptions of lemma 1 are fulfilled. As for the coefficients α_a, β_a we now have relations

$$\alpha_a=0, \quad \beta_{a_2}=\sum_{a_3\dots a_{n-2}} \beta_{a_2\dots a_{n-2}} P_3^{a_3}\dots P_{n-2}^{a_{n-2}},$$

lemma 1 asserts that

$$\tau(\beta_{a_2})=0 \text{ for all } a_2.$$

Considering β_{a_2} as a polynomial in "variables" P_1, P_3 we can again apply lemma 1, etc.

The following lemma 3 is the important assertion proved in [5]; it is formulated in the form suitable for our further use.

Lemma 3: Let

i $P_1,\dots,P_{N+1}$ be the basis of the complex $(N+1)$-dimensional Abelian Lie Algebra G,

ii τ be a canonical realization G in W_{2N}

iii $\tau(P_p)\neq \alpha_p \mathbb{1}$, $\alpha_p\in\mathbb{C}$, $p=1,2,\dots,N+1$.

Then there exists an element p:

$$0\neq p=\sum_{a_1\dots a_{N+1}} \alpha_{a_1\dots a_{N+1}} \cdot P_1^{a_1}\dots P_{N+1}^{a_{N+1}} \in \mathcal{E}(G),\ \alpha_{a_1\dots a_{N+1}}\in\mathbb{C}$$

such that $\tau(p)=0$.

Now we are in a position to prove the first our assertion concerning the canonical realizations of $O_c(n)$.

Theorem 1: If $N<n-3$ then any canonical realization of $O_c(n)$ in W_{2N} does not exist.

Proof: Assume, on the contrary, that τ is some canonical realization $O_c(n)$ in W_{2N}, $N<n-3$ and consider the commutative subalgebra of $O_c(n)$ with basis $P_2,\dots,P_{N+2}$ x). The canonical realization of none of these generators can be multiple of identity: if, say, $\tau(P_2)=\alpha\cdot\mathbb{1}$, then eq. (8) readily leads to $\tau(P_i)=0$.

According to lemma 3 there exists an element p:

$$0\neq p=\sum_{a_2\dots a_{N+2}} \alpha_{a_2\dots a_{N+2}} \cdot P_2^{a_2}\dots P_{N+2}^{a_{N+2}} \in \mathcal{E}[O_c(n)]$$

$\alpha_{a_2\dots a_{N+2}}\in\mathbb{C}$ such that $\tau(p)=0$. Lemma 2, however, asserts that then

$$\tau(\beta_{a_2\dots a_{N+2}})\equiv\tau(\alpha_{a_2\dots a_{N+2}}\cdot\mathbb{1})=\alpha_{a_2\dots a_{N+2}}\cdot\tau(\mathbb{1})=0$$

what further implies that all $\alpha_{a_2\dots a_{N+2}}=0$ and this contradicts $p\neq 0$.

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x) Note that index $N+2<n-1$, i.e.,the set $\{P_2,\dots,P_{N+2}\}\subset$ $\subset\{P_1,\dots,P_{n-2}\}$ always.

It is known (see,e.g. ,[6]) that canonical realization of $O_{\mathbb{C}}(n)$ in $W_{2(n-2)}$ exists. Therefore the consequence of theorem is that for minimal canonical realization of $O_{\mathbb{C}}(n)$ in W_{2N} only two possibilities remain open either $N=n-3$ or $N=n-2$.

For $n<7$ we are able to decide even between these two possibilities and solve the problem of minimal canonical realization therefore completely.

<u>Theorem 2:</u> The minimal canonical realization of

i $O_{\mathbb{C}}(3)$ is in W_2,

ii $O_{\mathbb{C}}(4)$ is in W_4,

iii $O_{\mathbb{C}}(5)$ is in W_4,

iv $O_{\mathbb{C}}(6)$ is in W_6.

Proof: i. As the possibility $N=n-3$ arises for $O_{\mathbb{C}}(n)$ only with $n>3$ the assertion is right.

ii. The consequence of the results contained in [4] (see also [5]) is the nonexistence of canonical realization of any semisimple Lie algebra with rank r in $W_{2(r-1)}$.
As rank of $O_{\mathbb{C}}(4)$ is 2, it cannot be realized in W_2.

iii. By the direct verification one can show that the following expressions form the canonical realization of $O_{\mathbb{C}}(5)$ in W_4 :

$$\tau(L_{12})=\frac{i}{2}(q_1p_1-q_2p_2),\ \tau(L_{13})=\frac{i}{2}(q_2p_1+q_1p_2),$$

$$\tau(L_{23})=-\frac{1}{2}(q_2p_1-q_1p_2),$$

$$\tau(P_1)=\frac{1}{2}(q_1^2+q_2^2),\ \tau(P_2)=-\frac{i}{2}(q_1^2-q_2^2),\ \tau(P_3)=-iq_1q_2,$$

$$\tau(Q_1)=\frac{1}{2}(p_1^2+p_2^2),\ \tau(Q_2)=\frac{i}{2}(p_1^2-p_2^2),\ \tau(Q_3)=ip_1p_2,$$

$$\tau(R)=\frac{1}{2}(q_1p_1+q_2p_2+1).$$

We see that all generators are realized through quadratic elements of W_4. It is generally proved [8] that all quadratic elements of W_{2N} form the Lie subalgebra isomorphic to $Sp_{\mathbb{C}}(2N)$. Our realization is the simple consequence of the isomorphism $O_{\mathbb{C}}(5)\simeq C_2\ (\simeq Sp_{\mathbb{C}}(4))$.

iv. Again by direct verification:

$$\tau(L_{12})=i(Q-q_3p_3-\alpha),\ \tau(L_{13})=\frac{i}{2}(q_1Q+p_1+q_3p_2+q_2p_3),$$

$$\tau(L_{23})=\frac{1}{2}(-q_1Q+p_1-q_3p_2+q_2p_3),\ \tau(L_{34})=i(Q-q_2p_2-\alpha),$$

$$\tau(L_{14})=-\frac{1}{2}(q_1Q-p_1-q_3p_2+q_2p_3),\ \tau(L_{24})=-\frac{i}{2}(q_1Q+p_1-q_3p_2-q_2p_3),$$

$$\tau(P_1)=q_1q_3+q_2,\quad \tau(P_2)=i(q_1q_3-q_2),$$

$$\tau(P_3)=-i(q_1q_2+q_3),\quad \tau(P_4)=q_1q_2-q_3,$$

$$\tau(Q_1)=p_1p_3+(1+Q)p_2,\ \tau(Q_2)=-i[p_1p_3-(1+Q)p_2],$$

$$\tau(Q_3)=i[p_1p_2+(1+Q)p_3],\ \tau(Q_4)=p_1p_2-(1+Q)p_3,$$

$$\tau(R)=q_2p_2+q_3p_3+1+\alpha$$

where

$$Q=-q_1p_1+q_2p_2+q_3p_3+2\alpha,\ \alpha\in\mathbb{C}.$$

As $O_{\mathbb{C}}(6)\simeq A_3\ (\simeq SU_{\mathbb{C}}(4))$ and the rank of $O_{\mathbb{C}}(6)$ equals $n-3=3$, we prove at the same time the existence of realization of the algebra A_3 by means of three pairs of canonical variables. In [4] the existence of realization of the Lie algebra A_n in quotient division ring in n canonical pairs is proved. As W_6 is properly embedded in its quotient division ring the stronger result for A_3 was obtained here.

4. Casimir operators

For the proof of the main result of this section we need two lemmas. The first of them is the slight generalization of lemma 2.

Lemma 4: Let

i τ be any canonical realization of $O_c(n)$ with basis (6),

ii $0 \neq p \in \mathcal{E}[O_c(n)]$ be an element, which can be written in the form

$$p = \sum_{a_1 \dots a_{n-2}} \beta_{a_1 \dots a_{n-2}} \cdot P_1^{a_1} \dots P_{n-2}^{a_{n-2}},$$

$$\beta_{a_1 \dots a_{n-2}} \in \mathcal{E}[O_c(n)], \quad [\beta_{a_1 \dots a_{n-2}}, L_{ij}] = 0,$$

iii $\tau(p) = 0.$

Then there exists $0 \neq p' \in \mathcal{E}[O_c(n)]$ of the form

$$p' = \sum_a \beta_a \, (P_1^2 + \dots + P_{n-2}^2)^a,$$

where coefficients β_a belong to linear envelope of the set of the coefficients $\beta_{a_1 \dots a_{n-2}}$, so that

$$\tau(p') = 0.$$

Proof: First we write the given polynomial p in the following form:

$$p = \sum_{b_1 a_2 \dots a_{n-2}} (\beta_{2b_1 a_2 \dots a_{n-2}} + \beta_{2b_1+1 a_2 \dots a_{n-2}} \cdot P_1) P_1^{2b_1} P_2^{a_2} \dots P_{n-2}^{a_{n-2}}.$$

Denoting $P_1^2 + P_2^2 + \dots + P_{n-2}^2 \equiv P^2$ we can proceed as follows:

$$p = \sum_{b_1 a_2 \dots a_{n-2}} (\beta_{2b_1 a_2 \dots a_{n-2}} + \beta_{2b_1+1 a_2 \dots a_{n-2}} \cdot P_1)[P^2 - (P_2^2 + \dots + P_{n-2}^2)]^{b_1} P_2^{a_2} \dots P_{n-2}^{a_{n-2}} =$$

$$= \sum_{b_1 a_2 \dots a_{n-2}} (\beta_{2b_1 a_2 \dots a_{n-2}} + \beta_{2b_1+1 a_2 \dots a_{n-2}} \cdot P_1) \sum_{c_1} \binom{b_1}{c_1} (P^2)^{b_1 - c_1} (P_2^2 + \dots + P_{n-2}^2)^{c_1} P_2^{a_2} \dots P_{n-2}^{a_{n-2}}$$

we see that it is possible to write p in the form

$$p = \sum_{a_1 \dots a_{n-2}} (\gamma_{a_1 \dots a_{n-2}} + \delta_{a_1 \dots a_{n-2}} \cdot P_1)(P^2)^{a_1} P_2^{a_2} \dots P_{n-2}^{a_{n-2}},$$

where coefficients $\gamma_{\dots}$ and $\delta_{\dots}$ are linear combinations (even with integer constants) of the coefficients $\beta_{\dots}$. As $p \neq 0$, at least one of polynomials

$$\sum_{a_1} \gamma_{a_1 a_2 \dots a_{n-2}} (P^2)^{a_1}, \quad \sum_{a_1} \delta_{a_1 a_2 \dots a_{n-2}} (P^2)^{a_1}$$

is nonzero. Because P^2 commutes with all L_{ij} we can apply lemma 2 asserting that the realization of all these polynomials is zero and proof is completed.

The following lemma gives two sufficient conditions for mutual dependence of the Casimir operators of $O_c(n)$ algebra in the realization τ. We use the following notation:

$$X_{(\mu\nu} Y_{\rho)\tau} \equiv X_{\mu\nu} Y_{\rho\tau} + X_{\nu\rho} Y_{\mu\tau} + X_{\rho\mu} Y_{\nu\tau}.$$

Lemma 5: Let τ be a canonical realization of $O_c(n)$, $n \geq 3$

A. If

$$\tau\left[L_{(\mu\nu} L_{\rho)\tau}\right] = \tau\left(\delta_{\tau(\mu} L_{\nu\rho)}\right) \tag{18}$$

then $\tau(I_k)$, $k \geq 3$, is a polynomial function of $\tau(I_2)$ (and $\tau(I'_m) = 0$ for even $n = 2m$) independent of τ.

B. If

$$\tau\left(L_{\mu\rho} L_{\rho\nu} + L_{\nu\rho} L_{\rho\mu}\right) = 0, \quad \mu \neq \nu \tag{19}$$

then $\tau(I_k)$, $k \geq 3$, and $\tau(I'_m)^2$ (for even $n = 2m$) are polynomial functions of $\tau(I_2)$ independent of τ.

C. If eqs. (18) and (19) hold then, moreover,

$$\tau(I_2) = -\frac{n(n-4)}{2} \mathbb{1} .$$

Proof: A. Let us introduce the abbreviation $T^{(k)}_{\mu\nu}$:

$$T^{(k)}_{\mu\nu} = \begin{cases} \delta_{\mu\nu} & , \; k = 0 , \\ L_{\mu\nu} & , \; k = 1 , \\ L_{\mu\mu_1} L_{\mu_1\mu_2} \cdots L_{\mu_{k-1}\nu} & , \; k \geq 2 . \end{cases}$$

The trace $T^{(k)}_{\mu\mu}$ coincides with the Casimir operator I_k and we define

$$I_0 = T^{(0)}_{\mu\mu} = n , \quad I_1 = T^{(1)}_{\mu\mu} = L_{\mu\mu} = 0 .$$

Further, for $k \geq 3$, we can write

$$I_k = L_{\mu\nu} L_{\nu\rho} L_{\rho\tau} T^{(k-3)}_{\tau\mu}$$

and, using commutation relations (3),

$$I_k = L_{\nu\rho} L_{\mu\nu} L_{\rho\tau} T^{(k-3)}_{\tau\mu} + (n-2)\, I_{k-1} .$$

Now we shall use for $\tau(I_k)$ relation (18) and relations (3) again, through which we obtain:

$$\tau(I_k) = -\tau(I_k) + 2(n-1)\tau(I_{k-1}) + \left[\tau(I_2) - 2(n-2)\right]\tau(I_{k-2}) - \tau(I_2)\tau(I_{k-3}) .$$

So we come to recurrent relation

$$\tau(I_k) = (n-1)\tau(I_{k-1}) + \left[\tfrac{1}{2}\tau(I_2) - n + 2\right] \cdot \tau(I_{k-2}) - \tfrac{1}{2}\tau(I_2) \cdot \tau(I_{k-3})$$

from which first assertion of part A easily follows.

The proof of the second one ($\tau(I'_m) = 0$ for $O_c(2m)$) is almost trivial; it is sufficient to substitute from (18) into definition of $\tau(I'_m)$ (eq. (5)).

B. Using commutation relations (3) and abbreviation $T^{(2)}_{\mu\nu}$ we oan rewrite eq. (19) in the form:

$$\tau(T^{(2)}_{\mu\nu}) = \frac{n-2}{2} \tau(L_{\mu\nu}) , \quad \mu \neq \nu . \tag{20}$$

From commutation relations of $\tau(L_{\mu\nu})$ with $\tau(T^{(2)}_{\mu\nu})$ we obtain:

$$\tau(T^{(2)}_{11}) = \tau(T^{(2)}_{22}) = \ldots = \tau(T^{(2)}_{nn}) ,$$

which implies

$$\frac{1}{n}\tau(I_2) \equiv \frac{1}{n}\tau(T^{(2)}_{\mu\mu}) = \tau(T^{(2)}_{11}) = \dots = \tau(T^{(2)}_{nn}). \tag{21}$$

Relations (20) and (21) can be written commonly

$$\tau(T^{(2)}_{\mu\nu}) = \frac{1}{n}\tau(I_2)\,\delta_{\mu\nu} + \frac{n-2}{2}\tau(L_{\mu\nu}).$$

As $I_k \equiv T^{(k)}_{\mu\mu} = T^{(k-2)}_{\mu\varrho}T^{(2)}_{\varrho\mu}$ we come to the recurrent relation

$$\tau(I_k) = \frac{1}{n}\tau(I_2)\,\tau(I_{k-2}) + \frac{n-2}{2}\tau(I_{k-1}).$$

The polynomial dependence of $\tau(I_k)$ on $\tau(I_2)$ is now the evident consequence.

For the proof of dependence of $\tau(I'_m)^2$ on $\tau(I_2)$ in the case of $O_{\mathbb{C}}(2m)$ algebra, we need some information concerning the centre $\mathcal{Z} \subset \mathcal{E}[O_{\mathbb{C}}(2m)]$.(see ,e.g., [9], p.565). It is known that I_{2k}, $k = 1,2,\dots,m-1$ and I'_m are the generating elements for $\mathcal{Z}$. It especially means that I_{2m} is a polynomial of these generators of $\mathcal{Z}$. As I_{2k}, I'_m are polynomials in "variables" $L_{\mu\nu}$ and for their highest degrees the relations

$$\deg I_{2k} = 2k \ , \quad \deg I'_m = m$$

are valid, then

$$I_{2m} = \alpha \cdot I'^{2}_m + I'_m\,\beta(I_2,\dots,I_{2(m-1)}) + \gamma(I_2,\dots,I_{2(m-1)}) \ , \tag{22}$$

where

$$\deg \beta(I_2,\dots,I_{2(m-1)}) \leq m \ , \quad \deg \gamma(I_2,\dots,I_{2(m-1)}) \leq 2m.$$

The polynomial β equals zero. This follows from the following considerations. The mapping $\varrho : O_c(2m) \to O_c(2m)$ defined by the relations

$$\varrho(L_{1\mu}) = -L_{1\mu} \ , \quad \varrho(L_{\mu\nu}) = L_{\mu\nu} \ , \quad \mu,\nu \neq 1$$

is the automorphism of $O_c(2m)$ (and induces naturally the automorphism of $\mathcal{E}[O_c(2m)]$ denoted by the same symbol ϱ). We see that

$$\varrho(I_{2k}) = I_{2k} \quad \text{and} \quad \varrho(I'_m) = -I'_m \ .$$

Applying ϱ to equation (22) we obtain

$$I_{2m} = \alpha I'^{2}_m - I'_m\,\beta(I_2,\dots,I_{2(m-1)}) + \gamma(I_2,\dots,I_{2(m-1)})$$

from which and eq. (22) the desired result immediately follows.

As I_{2m} does not depend on the $I_2, \dots, I_{2(m-1)}$ only, the constant $\alpha \neq 0$.

The dependence of $\tau(I'_m)^2$ on $\tau(I_2)$ is obtained immediately combining eq. (22) with the preceding result.

C. If eqs. (18) and (19) hold together, we are able to calculate value of $\tau(I_2)$. These equations, imply

$$\tau(L_{n-1n})\tau(L_{ij}) + \tau(L_{ni})\tau(L_{n-1j}) + \tau(L_{in-1})\tau(L_{nj}) = \delta_{ij}\,\tau(L_{n-1n}), \tag{18'}$$

$$\tau(L_{ij})\tau(L_{jn-1}) = \tau(L_{n-1n})\tau(L_{in}) + \frac{n-4}{2}\tau(L_{in-1}), \tag{19'}$$

$$\tau(L_{ij})\tau(L_{jn}) = -\tau(L_{n-1n})\tau(L_{in-1}) + \frac{n-4}{2}\tau(L_{in}), \tag{19''}$$

$$\tau(L_{n-1i})\tau(L_{in}) = -\tau(L_{ni})\tau(L_{in-1}) = \frac{n-2}{2}\tau(L_{n-1n}). \tag{19'''}$$

Multiplying the second and the third equation by $\tau(L_{n-1n})$ (from the left) substituting $\tau(L_{n-1n})\tau(L_{ij})$ from the first equation and using eqs. (19''') and (3), we obtain:

$$\tau(L_{in})\left[\tau(L_{n-1j})\tau(L_{jn-1}) - \tau(L^2_{n-1n}) + \frac{n-4}{2}\mathbb{1}\right] = 0,$$

$$\tau(L_{in-1})\left[\tau(L_{nj})\tau(L_{jn}) - \tau(L^2_{n-1n}) + \frac{n-4}{2}\mathbb{1}\right] = 0.$$

As $\tau(L_{in}) \neq 0$ and $\tau(L_{in-1}) \neq 0$, these equations give (see implication (2)):

$$\tau\left(L_{n-1j}L_{jn-1} + L_{nj}L_{jn}\right) = 2\tau(L^2_{n-1n}) - (n-4)\mathbb{1}.$$

It is the part of invariant $\tau(I_2)$. The other part we obtain from eq. (18') by its left multiplication by $\tau(L_{ij})$ using eqs. (19') - (19'') and (3)

$$\tau(L_{n-1n})\left[\tau(L_{ij}L_{ji}) + \tau(L_{n-1i}L_{in-1} + L_{ni}L_{in}) + \frac{(n-2)(n-4)}{2}\mathbb{1}\right] = 0,$$

from which

$$\tau(L_{ij}L_{ji}) = -\tau\left(L_{n-1i}L_{in-1} + L_{ni}L_{in}\right) - \frac{(n-2)(n-4)}{2}\mathbb{1}.$$

Substituting it into the formula

$$\tau(I_2) = -2\tau(L^2_{n-1n}) + 2\tau\left(L_{ni}L_{in} + L_{n-1i}L_{in-1}\right) + \tau(L_{ij}L_{ji})$$

the desired result is obtained.

Note: We show that eq. (18) is implied for $n \geq 5$ by relation $\tau(L_{(ij}P_{k)}) = 0$, $i \neq j \neq k \neq i$ and eq. (19) is implied by relation $\tau(P^2) \equiv \tau(P_1^2 + \dots + P_{n-2}^2) = 0$.

By means of introduced tensor $T^{(2)}_{\mu\nu}$ the equation $\tau(P^2) = 0$ can be rewritten in the from (see eq. (6)):

$$\tau\left(T^{(2)}_{n-1n} + T^{(2)}_{nn-1}\right) = i\tau\left(T^{(2)}_{nn} - T^{(2)}_{n-1n-1}\right).$$

Introducing the new tensor

$$T_{\mu\nu\rho,\tau} = L_{(\mu\nu}L_{\rho)\tau}$$

the second relation, $\tau(L_{(ij}P_{k)}) = 0$, **looks as follows:**

$$\tau\left(T_{ijk,n} + i\,T_{ijk,n-1}\right) = 0, \quad i \neq j \neq k \neq i.$$

Commuting now the first equation with $\tau(L_{jn-1})$ **and the result with** $\tau(L_{kn})$, $k \neq j$, **and the second one with** $\tau(L_{in-1})$ **we obtain:**

$$\tau\left(T^{(2)}_{kj} + T^{(2)}_{jk}\right) = 0,$$

$$\tau\left(T_{n-1jk,n}\right) = 0, \quad j \neq k.$$

With respect to tensor character of $T^{(2)}_{kj} + T^{(2)}_{jk}$ **and** $T_{n-1jk,n}$ **(commuting with suitable** $L_{\mu\nu}$**'s) we finally obtain:**

$$\tau\left(T^{(2)}_{\mu\nu} + T^{(2)}_{\nu\mu}\right) = \tau\left(L_{\mu\sigma}L_{\sigma\nu} + L_{\nu\sigma}L_{\sigma\mu}\right) = 0,$$

$$\tau\left(T_{\mu\nu\rho,\tau}\right) = \tau\left(L_{(\mu\nu}L_{\rho)\tau}\right) = 0,$$

where μ, ν, ρ, τ **are mutually different indices from the set {1,2, ... n} . The latter relation can be further written in the form**

$$\tau\left(L_{(\mu\nu}L_{\rho)\tau}\right) = \delta_{\tau(\mu}\tau(L_{\nu\rho)})$$

validing already for all $\mu, \nu, \rho, \tau = 1, 2, \dots, n$.

Now we can formulate the main theorem of this section.

Theorem 3. Let τ denote either canonical realization of $O_{\mathbb{C}}(n)$ in $W_{2(n-2)}$ when $n \neq 6$ or canonical realization of $O_{\mathbb{C}}(6)$ in W_6.

Then

i realization of all the Casimir operators equals constant multiple of identity,

ii for $n \geq 6$ realizations $\tau(I_k)$, $k = 3,4,\ldots$ and also $\tau(I'_m)^2$ (for $n = 2m$) polynomially depends on $\tau(I_2)$ in one from two possible ways.

If, especially, τ is realization of $O_{\mathbb{C}}(n)$ in $W_{2(n-3)} \subset W_{2(n-2)}$ x) and $n \neq 6$, then

iii $$\tau(I_2) = -\frac{n(n-4)}{2}\mathbb{1}.$$

and $\tau(I_k)$ are independent of τ.

Proof: For $n = 3,4$ the assertion i is a part of general result proved in [4] (see also [5]), because in these cases the rank of $O_{\mathbb{C}}(n)$ equals to n-2.

So we shall assume $n \geq 5$.

1. The proof consists of two parts. The case $n = 6$ is excluded and will be proved together with iii.

a) Consider any z from the center $\mathcal{Z}$ of $\mathcal{E}[O_{\mathbb{C}}(n)]$ and $(n-1)$-dimensional Abelian subalgebra of $\mathcal{E}[O_{\mathbb{C}}(n)]$ with basis $z, P_1, \ldots, P_{n-2}$. If we allow on the contrary to the assumption i $\tau(z) \neq \alpha\mathbb{1}$ then, according to lemma 3, there exists a complex nonzero polynomial $p \equiv p(z, P_1, \ldots, P_{n-2}) \in \mathcal{E}[O_{\mathbb{C}}(n)]$ realized as $\tau(p) = 0$.
From the lemma 4 further, the existence on nonzero polynomial

x) This possibility arises for n $\geq$5 only (see Theorem 2).

$$p' \equiv \sum_a f_a(z)(P^2)^a \in \mathcal{E}[O_{\mathbb{C}}(n)]$$

with $\tau(p') = 0$ follows, where $f_a(z)$ are polynomials in variable z. Using commutation relations

$$[R, (P^2)^a] = 2a(P^2)^a$$

by multiple commutation of $\tau(R)$ with $\tau(p')$ we oome, similarly as in the proof of lemma 1, to the homogeneous system of equations for "unknown" $\tau[f_a(z)(P^2)^a]$ solved by

$$\tau[f_a(z)(P^2)^a] = 0, \quad a = 0,1,\ldots$$

It implies further either

$$\tau[f_a(z)] = 0, \quad a = 0,1,2,\ldots$$

or

$$\tau(P^2) = 0$$

(and $\tau[f_0(z)] = 0$ if $f_0(z) \neq 0$) . As $p' \neq 0$, at least one polynomial $f_a(z) \equiv f(z) \neq 0$. Therefore either $f(z) \equiv f_0 \cdot \mathbb{1}$, $0 \neq f_0 \in \mathbb{C}$ and we obtain contradiction due to $\tau[f(z)] = f_0 \cdot \tau(\mathbb{1}) = 0$ or $\deg f(z) \geq 1$, i.e., $f(z)$ can be factorized into the product

$$f(z) = \beta \prod_b (z - \alpha_b \mathbb{1})^{n_b}, \quad \beta, \alpha_b \in \mathbb{C}.$$

Then, however, $\tau[f(z)] = 0$ implies $\tau(z) = \alpha\mathbb{1}$ (see implication (2)) which oontradicts our assumption. So, the second possibility $\tau(P^2) = 0$ remains only.

However, $\tau(P^2) = 0$ implies eq. (19) so that the assumption $\tau(z) \neq \alpha\mathbb{1}$ implies eq. (19).

b) In further investigations we have to distinguish between two cases: $n = 5$ and $n > 6$.

Case $O_{\mathbb{C}}(5)$: Let us take four-dimensional Abelian subalgebra of $\mathcal{E}[O_{\mathbb{C}}(5)]$ with basis z, w, P_1, P_2 where $w = \frac{1}{2}\epsilon_{ijk} L_{ij} P_k$ (see eq. (12)) and assume together with $\tau(z) \neq \alpha \mathbb{1}$ also $\tau(w) \neq \beta \mathbb{1}$. Then from lemmas 3 and 1 the existence of nonzero complex polynomial p,

$$0 \neq p \equiv p(z, w) \equiv \sum_a f_a(z) w^a \in \mathcal{E}[O_{\mathbb{C}}(5)]$$

with zero realization follows, because $[w, L_{ij}] = 0$.

As in the preceeding case, by multiple commutation of $\tau(p)$ with $\tau(R)$ we derive the equation

$$\tau[f_a(z) w^a] = 0$$

from which (as by our assumption $\tau(w) \neq 0$)

$$\tau[f_a(z)] = 0.$$

As we saw, this possibility leads to contradiction with the starting assumption $\tau(z) \neq \alpha \mathbb{1}$ and therefore assumption $\tau(w) \neq \beta \mathbb{1}$ has to be changed, i.e., $\tau(w) = \beta \cdot \mathbb{1}$. Commuting it with $\tau(R)$ we obtain $\tau(w) = 0$ even and we can conclude that assumption $\tau(z) \neq \alpha \mathbb{1}$ implies eq. (18). As also eq. (19) is fulfilled then according to lemma 5 C we have contradiction with $\tau(z) \neq \alpha \mathbb{1}$.

Case $O_{\mathbb{C}}(n),\ n \geq 7$. Let us introduce the following three elements from $\mathcal{E}[O_{\mathbb{C}}(n)]$:

$$w_{\pm} = (P_1 \pm i P_2) w' - (P_3 + i P_4)(w_{(234)} \mp i w_{(134)}),$$

$$w' = w_{(124)} - i w_{(123)},$$

where

$$w_{(ijk)} \equiv L_{(ij} P_{k)} \equiv L_{ij} P_k + L_{jk} P_i + L_{ki} P_j.$$

It is clear that elements $w_{(ijk)}$ differ from $w_{i_1 \dots i_{n-5}}$ (see eq. (12)) at most in sign and therefore they commute with P_i (eq. (13)). Mutual commutation relation between $w_{\pm}$ and w' looks as follows:

$$[w_{\pm}, w'] = \mp (P_3 + i P_4) w_{\pm}.$$

Let us consider now the $(n-1)$-dimensional Abelian subalgebras with bases $z, w_{\pm}, P_1, \dots, P_{n-3}$ and assume $\tau(w_{\pm}) \neq \beta_{\pm} \mathbb{1}$. Again there exists a polynomial p,

$$0 \neq p \equiv p(z, w_{\pm}, P_1, \dots, P_{n-3}) \equiv \sum_a f_a(z, P_1, \dots, P_{n-3}) w_{\pm}^a \in \mathcal{E}[O_{\mathbb{C}}(n)]$$

with $\tau(p) = 0$. As the commutation relations of w' and of the powers $w_{\pm}^a$ have simple form

$$[w_{\pm}^a, w'] = \mp a (P_3 + i P_4) w_{\pm}^a$$

we can commute $\tau(p)$ with $\tau(w')$ and we obtain again homogeneous system with nonzero determinant for "unknown" $\tau[f_a(z, P_1, \dots, P_{n-3}) w_{\pm}^a]$. Because we assume $\tau(w_{\pm}) \neq 0$ and $p \neq 0$ at least one coefficient $f_a(z, P_1, \dots, P_{n-3})$ is nonzero and

$$\tau[f_a(z, P_1, \dots, P_{n-3})] \equiv \tau\Big[\sum_{b_1 \dots b_{n-3}} \beta_{a b_1 \dots b_{n-3}}(z) \cdot P_1^{b_1} \dots P_{n-3}^{b_{n-3}}\Big] = 0.$$

Using now the lemma 2 we come to the conclusion that realization of all coefficients $\beta_{a b_1 \dots b_{n-3}}$ equals zero. We saw in part a) that it leads to contradiction with the starting assumption $\tau(z) \neq \alpha \mathbb{1}$ and therefore $\tau(z) \neq \alpha \mathbb{1}$ implies $\tau(w_{\pm}) = \beta_{\pm} \mathbb{1}$. By commuting with $\tau(R)$ we immediately obtain $\beta_{\pm} = 0$

and from the equation

$$\frac{1}{2}\tau(W_+ + W_-) = \tau\left[P_1(W_{(124)} - iW_{(123)}) - (P_3 + iP_4)W_{(234)}\right] = 0$$

by further commutation with $\tau(L_{13})$ we have:

$$\tau(P_4 W_{(124)} + P_3 W_{(123)}) = 0.$$

As we assume $n \geqslant 7$ we can repeat our consideration with other choice of indices then 1,2,3,4,e.g.,1,2,4,5 and 1,2,3,5 and we obtain also

$$\tau(P_4 W_{(124)} + P_5 W_{(125)}) = 0,$$
$$\tau(P_5 W_{(125)} + P_3 W_{(123)}) = 0$$

from which ,e.g., $\tau(W_{(123)}) = 0$. Due to the tensor character of $W_{(ijk)}$ we have

$$\tau(W_{(ijk)}) \equiv \tau(L_{(ij}P_{k)}) = 0$$

for all $i \neq j \neq k \neq i$ what implies eq. (18). We proved that assumption $\tau(\mathfrak{z}) \neq \alpha \cdot \mathbb{1}$ implies together with eq. (19) also eq. (18), which by lemma 5C, contradicts one another.

ii) In this case consider the commutative $(n-1)$-dimensional subalgebras with bases $W_\pm, P_1, \ldots, P_{n-2}$. Using lemma 3 and commutation with $\tau(W')$ as in the preceeding case, we come to the nonzero polynomial $p \equiv p(P_1, \ldots, P_{n-2})$ with zero realization $\tau(p) = 0$. From the part a) of above proof it follows

$$\tau(P^2) = 0.$$

Therefore either both $\tau(W_\pm) = 0$, i.e., eq. (18) is valid or $\tau(P^2) = 0$, i.e.,eq. (19) holds. Assertion ii now follows from lemma 5A,B.

iii) In realization of $O_c(n)$ in $W_{2(n-3)}$ (including the case $n = 6$) we take $(n-2)$-dimensional Abelian subalgebra with basis $P_1, \ldots, P_{n-2}$. Applying lemma 3 and the part a) of the proof of i we have $\tau(P^2) = 0$. From lemma 5B the assertion ii for $O_c(6)$ especially follows.

For $n \neq 6$ we can continue and take the other subalgebra with basis W, P_1, P_2 if $n = 5$ and $W_\pm, P_1, \ldots, P_{n-3}$ if $n > 7$. According to the second part of proof of assertion i we conclude:

$$\tau(W_{(ijk)}) = 0.$$

and lemma 5 C can be applied.

It remains only to prove i for $O_{\mathbb{C}}(6)$. The Abelian subalgebra in this case has the basis z, P_1, P_2, P_3 ; $z \in \mathcal{Z}$ and assumption $\tau(z) \neq \alpha \mathbb{1}$ leads, using lemmas 3 and 2, to the existence of nonzero polynomial $\gamma(z)$ with zeroness realization. It, however, contradicts $\tau(z) \neq \alpha \mathbb{1}$.
The proof of theorem is completed.

5. Concluding remarks

Up to this time we have dealt with realizations of complex Lie algebra $O_{\mathbb{C}}(n)$. As we are usually interested rather in the real Lie algebras it would be useful to apply our results to them. Due to close connection between complex Lie algebra G and its real forms the one-to-one correspondence among realizations of G and realization of any real form of G arises. If G_0 is any real form of G having basis $X_1, \dots, X_n$ and τ is a canonical realization of G_0 then the complex linear envelope of the elements $\tau(X_1), \dots, \tau(X_n)$ is realization $\tau_{\mathbb{C}}$ of G. On the contrary, if any $\tau_{\mathbb{C}}$ is given, we choose in G basis $X_1, \dots, X_n$ in which structure constants coincide with the structure constants of G_0 and real linear combinations of $\tau_{\mathbb{C}}(X_1), \dots, \tau_{\mathbb{C}}(X_n)$ define realization of G_0 . This consideration shows that all the assertions of theorems 1–3 remain valid in the case of all real forms of $O_{\mathbb{C}}(n)$.

As we mentioned in the introduction, the use of realizations in the representation theory of Lie algebras consists in simple substitution of abstract elements p_i and q_i by some representation of them, e.g., by usual Schrödinger representation. In the case of real Lie algebras we are usually interested in special realizations which leads, in the above way, to the skew-symmetric representations. To distinguish between such realizations, we have to enrich our Weyl algebra W_{2N} by involution. We define inductively antilinear mapping "$+$" of W_{2N} onto itself by relations:

$$(p_i)^+ = -p_i \, , \; (q_i)^+ = q_i \, , \; (w_1 w_2)^+ = w_2^+ w_1^+ \, ; \; w_1, w_2 \in W_{2N} \, .$$

Now we can speak, about skew-symmetric elements of W_{2N} $(w^+ = -w)$ which are, after substitution of p_i and q_i by their Schrödinger representatives, represented by skew-symmetric operators. The realization of real Lie algebra G through of skew-symmetric elements of W_{2N} only will be called by skew-symmetric realization of G . Now it is clear that if G_0 is some real form of G and $\tau_{\mathbb{C}}$ is a realization of G then corresponding realization τ of G_0 need not be skew-symmetric. Therefore the minimal skew-symmetric realization of given real form of $O_{\mathbb{C}}(n)$ needs exist neither in $W_{2(n-3)}$ nor even in $W_{2(n-2)}$ and different real forms of $O_{\mathbb{C}}(n)$ can have minimal skew-symmetric realizations in different W_{2N}, $N \geq n-3$. It can be proved that for $O(n)$ (i.e., for compact real form of $O_{\mathbb{C}}(n)$) the skew-symmetricity of realization contradicts to "constant realization" of the Casimir operators ([5]th.4.4). Together with theorem 2 it gives for $n \geqslant 3$, $n \neq 6$ the first possibility for the minimal skew-symmetric realization of $O(n)$ is in $W_{2(n-1)}$ x) and for $O(6)$ in W_8 . In the same time the skew-symmetric realization of noncompact forms $O(n-m,m)$, $0<m<n$, exist in $W_{2(n-2)}$ [6] .

For $n = 5,6$ we can derive further skew-symmetric realizations of some noncompact form of $O_{\mathbb{C}}(n)$ by means of theorem 2 even in W_4 or W_6 respectively. In the mentioned theorem the realization of $O_{\mathbb{C}}(5)$ is such that generators $iL_{12}, iL_{13}, L_{23}, iP_1, P_2, P_3, iQ_1, Q_2, Q_3$ and R are realized by skew-symmetric elements of W_4 . As by commutation of any pair of them we obtain their real linear combination (see eqs. (7) - (11)) ten generators $iL_{12}, \ldots\ldots, R$ from the basis of some real form of $O_{\mathbb{C}}(5)$ which is realized skew-symmetrically. It is not difficult to prove that this real form is just $O(3,2)$.

Similarly we prove that the realization iv contained in theorem 2 is the skew-symmetric realization of $O(3,3)$ if $Re\,\alpha = 0$.

All realizations considered until now were either minimal or the "nearest" to minimal ones. In accordance with theorem 2 this fact has two following consequences. Without exeptional cases the first consequence is the realization of Casimir operators by multiple of identity and the second one is their dependence on one of them only. (We could call realizations with the first property as the Schur realizations and the second property as the degeneration of realization). It is natural

x) As the skew-symmetric realization of $O(n,1)$ in $W_{2(n-1)}$ exists, the "subrealization" of $O(n) \subset O(n,1)$ in $W_{2(n-1)}$ is a minimal one.

to expect that enlarging the number N in W_{2N} new realizations could appear which are not Schur realizations and which are less degenerated. The question here arises whether there exist Schur realizations of $O_{\mathbb{C}}(n)$ in which degeneration is partly or fully removed, i.e., where a number of the independent Casimir operators is greater than one or even equals to $[\frac{n}{2}]$. The authors hope to give a positive answer in a subsequent paper.

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Note: When the work was finished the authors met a paper of A.Joseph, Comm.math.Phys. 36, 325 (1974) with some overlap of results (e.g., theorems 1 and 2 are contained in lemmas 3.1 and 3.2 and, on the other side, our theorem 3 generalizes, as to realization in Weyl algebra, part (5) of theorem 5.1) which were however obtained by a different methods. The assertions of our lemmas can be useful in the solution of the problem of the minimal canonical realization for $O_{\mathbb{C}}(n)$ if $n \geqslant 7$.

References:

[1.] Doebner, H.D., Pirring, B.: Spectrum-generating algebras and canonical realizations. Preprint IC/72/77.

[2.] Miller, W., Jr., On Lie algebras and some special functions of mathematical physics. AMS Mem. No.50, Providence 1964.

[3.] Palev, T.D., Nuovo Cim.,62, 585 (1969).

[4.] Simoni, A., Zaccaria, F., Nuovo Cim., 59, A, 280 (1969).

[5.] Joseph, A., J.Math.Phys., 13, 351 (1972).

[6.] Richard, J.L., Ann.Inst. H.Poincare, 8, Sec. A. No.3, 301, 1968.

[7.] Jacobson, N.: Lie algebras, Mir, Moskva, 1964 (in Russian).

[8.] Doebner, H.D., Palev, T.D. : On realizations of Lie algebras in factor spaces. Preprint IC/71/104.

[9.] Zhelobenko, D.L.:Kompaktnyje gruppy Li i ich predstavlenija, Nauka, Moskva, 1970.

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О минимальных канонических реализациях алгебры Ли $O_c(n)$

Для комплексификации алгебры Ли ортогональной группы в n-размерном пространстве Евклида показано, что не существует канонической реализации с помощью N пар канонических переменных, если $N < n - 3$.

Показано далее, что с некоторыми исключениями операторы Казимира в канонической реализации с $n - 2$ парами канонических переменных являются кратными единице и зависят только от квадратичного оператора Казимира. Если, в частности, канонические реализации с n-3 парами существуют, то значения всех операторов Казимира зависят только от n .

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On the Minimal Canonical Realizations
of the Lie Algebra $O_c(n)$

See the Summary on the reverse side of the title-page.

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