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V.I.Ogievetsky, E.Sokatchev

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Огиевецкий В.И., Сокачев Э. E2 - 8088

Примитивные представления алгебры SL(3,R)

Исследованы унитарные неприводимые представления алгебры SL(3,R), содержащие каждое неприводимое представление алгебры SO(3) не более одного раза.

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Primitive Representations of the Algebra SL(3,R)

The unitary irreducible representations of the SL(3,R) algebra are investigated, which contain every irreducible representation of its maximal compact subalgebra SO(3) not more than once.

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1. Introduction

Dothan, Gell-Mann and Ne'emann [1] introduced the algebra SL(3,R) as an algebra generating orbital excitations of hadrons. They found relations between certain representations of ASL(3,R) and the Regge sequences $k = (0,2,4,\dots)$, $k = (1,3,5,\dots)$. Later on, Biedenharn et al. [2] tried to deal analogously with the Regge series for half-integer spin. It is stated in [2] that there exist four simple ("primitive") representations within the framework of a particular realization of the algebra corresponding to the four basic Regge sequences: $\{\pi\} = (k=0,2,\dots)$, $\{\rho\} = (1,3,\dots)$, $\{N\} = (\frac{1}{2}, \frac{5}{2}, \dots)$ and $\{\Delta\} = (\frac{3}{2}, \frac{7}{2}, \dots)$.

In fact the Δ - sequence does not exist in the discussed realization, as it can be seen when a more careful analysis is carried out. Therefore it is interesting to obtain a more detailed information about the primitive representations of ASL(3,R) and to investigate whether they can reproduce the Regge classification of particles.

The representations of ASL(3,R) have also certain significance in connection with ref. [3], where the infinite-dimensional algebra of the general covariance group is shown to be a closure of the finite-dimensional algebras of the conformal and the affine [GL(4,R)] groups. One can hope to construct representations of the infinite-parameter general covariance algebra if the representations of the above-mentioned finite-dimensional algebras (particularly of ASL(3,R) as a subalgebra of GL(4,R)] are known.

Note that principal series of representations of the SL(n,R) groups have been already found by Gel'fand and Graev [4]. However, they use a functional realization of these representations while in physical applications it is more convenient to deal with a discrete basis because its elements may be directly associated with the elementary particles.

The group SL(3,R) has eight parameters. Its maximal compact subgroup is SO(3), so three of SL(3,R) generators are the generators of the space rotations J_i ($i =$ 1,2,3). The remaining five generators form a second-rank tensor with respect to SO(3). In spherical basis they can be written as T_M ($M = -2, \ldots, 2$) and satisfy the following commutation relations

$$[T_2, T_{-2}] = -4 J_3 \tag{1.a}$$

(the remaining commutators are obtainable from (1.a)). In vector basis these generators can be equivalently written as a symmetric traceless tensor T_{ij} $(i, j = 1, 2, 3)$:

$$[T_{ij}, T_{kl}] = -i(\delta_{ik}\varepsilon_{jlm} + \delta_{il}\varepsilon_{jkm} + \delta_{jk}\varepsilon_{ilm} + \delta_{jl}\varepsilon_{ikm}) J_m . \tag{1.b}$$

Consider an irreducible unitary representation (I.U.R.) of the algebra SL(3,R) realized in a Hilbert space R. Being reduced to ASD(3) this I.U.R. decomposes into the infinite sum of finite-dimensional I.U.R. of ASO(3): $R = \sum_k \oplus \varkappa_k \mathfrak{M}^k$, where $\mathfrak{M}^k$ is the $2k+1$ - dimensional space of I.U.R. of ASO(3) (k is integer or half-integer; it is called "spin" in what follows), $\varkappa_k$ is the multiplicity of the spin k representation. In general $\varkappa_k$ may be larger than unity. We shall restrict ourselves to those I.U.R. where every spin enters at most once, i.e., $\varkappa_k \leqq 1$ (following the authors of ref. 2 , we call them "primitive"). As usual, a canonical basis f^k_ν , $\nu = -k, \ldots, k$ is introduoed into the spaces $\mathfrak{M}^k$.

It is necessary to examine how the generators T_M (T_{ij}) act in the Hilbert space R in order to describe the I.U.R. of ASL(3,R)). In Section 2 the matrix elements of the generators are calculated using only the Wigner-Eckart theorem and eq. (1a) without specifying space and operators. It turns out that there exist two one-parameter sets of I.U.R. containing integer spins $k =$ 0,2, ... and $k =$ 1,3, ... resp. There is also one I.U.R. with half integer spins $k =$ ½, 5/2, ... but a sequence starting from 3/2 does not exist. Therefore the primitive I.U.R. cannot completely reproduce the Regge classification.

In Section 3 a simple realization of all primitive representations is given. The algebra of the Lorentz group is realized in the same way since all of its representations are primitive. This construction is based on the familiar creation and destruction operator formalism and is useful due to its compactness and clearness.

II. Matrix elements of the SL(3,R) generators

To evaluate the matrix elements we make use of the fact that the generators form a tensor and employ the Wigner–Eckart theorem [5]:

$$T_M f_\nu^k = \mathcal{K}_{\nu+M}^{k-2} A_k^- f_{\nu+M}^{k-2} + \mathcal{K}_{\nu+M}^{k-1} B_k^- f_{\nu+M}^{k-1} + \mathcal{K}_{\nu+M}^{k} C_k^0 f_{\nu+M}^{k}$$
$$+ \mathcal{K}_{\nu+M}^{k+1} B_k^+ f_{\nu+M}^{k+1} + \mathcal{K}_{\nu+M}^{k+2} A_k^+ f_{\nu+M}^{k+2}, \qquad (2)$$

where

$$\mathcal{K}_{\nu+M}^{k+i} \equiv C(2M, k\nu | k+i, \nu+M), \quad i = -2, \dots, 2$$

are the Clebsch–Gordan coefficients and $A_k^\pm$, $B_k^\pm$, C_k^0 are arbitrary coefficients (the reduced matrix elements) which depend on spin only. Note that by definition $A_k^- = 0$ and $B_k^- = 0$ if $k < 2$ and $k < 1$, respectively. In unitary representations the generators should satisfy the following requirements: $T_M^+ = (-1)^M T_{-M}$ or in terms of reduced matrix elements:

$$A_k^- = \overline{A_{k-2}^+}, \quad B_k^- = \overline{B_{k-1}^+}, \quad C_k^0 = \overline{C_k^0}$$

(the bar denotes complex conjugation).

The quantities A_k^-, B_k^- and C_k^0 are to be determined from the commutation relation (1.a). Consider the action of both sides of eq. (1a) on a vector f_ν^k, taking into account eq. (2). The comparison of the coefficients of vectors with equal spins leads to a set of nine finite-difference equations. We shall write down one of them as an example:

$$\nu^3[-(2k-1)B_{k-1}^+ B_k^- + (2k+3)B_{k+1}^- B_k^+ + (2k-3)A_{k-2}^+ A_k^- - (2k+5)A_{k+2}^- A_k^+ - 2C_k^{0\,2}]$$
$$+\nu[-(k^3-3k+1)B_{k-1}^+ B_k^- + (k^3+3k^2-3)B_{k+1}^- B_k^+ + (2k-3)(k^2-3k+1)A_{k-2}^+ A_k^-$$
$$-(2k+5)(k^2+5k+5)A_{k+2}^- A_k^+ + (2k^2+2k-1)C_k^{0\,2}] = -\nu \qquad (3)$$

The detailed analysis of these equations gives the following results:

a) If $B_k^\pm \equiv 0$ then the set is consistent and has the solution

$$|A_k^-|^2 = \frac{1}{4(2k-3)(2k+1)}\left[1 + \frac{c^2}{(2k-1)^2}\right], k \geqq 2, \quad C_k^0 = \frac{c}{(2k-1)(2k+3)}, \ k \geqq 0, \qquad (4)$$

where c is a real parameter (c =0 if k = 1/2 is allowed).

Since each representation contains a minimal spin k_0 the matrix element of the transition $k_0 \to k_0 - 2$ must vanish, i.e., $A_{k_0}^- = 0$. However, this is impossible when $A_{k_0}^- = 0$ according to (4). So the case $k_0 < 2$ when $A_{k_0}^- = 0$ by definition, remains only. Inthis case all the terms containing $A_{k_0}^-$ vanish in our set of equations. For instance, equation (3) for $k = k_0 < 2$ takes the form

$$-(2k_0+5)[\nu^3 + (k_0^2+5k_0+5)\nu]|A_{k_0+2}^-|^2 + [-2\nu^3 + (2k_0^2+2k_0-1)\nu]C_{k_0}^{0\,2} = -\nu. \qquad (5)$$

Inserting expression (4) for $|A_{k_0+2}^-|^2$ $(k_0 + 2 \geqq 2)$ and $C_{k_0}^0$ into (5) we arrive at the consistency condition:

$$\nu^3 + (k_0^2 - 3k_0 + 1)\nu = 0.$$

Obviously, it is satisfied for $k_0 = 0, \frac{1}{2}, 1$ but it does not hold for $k_0 = \frac{3}{2}$.

Thus, in the case $B_k^{\pm} = 0$ we see that there are three kinds of primitive I.U.R. : $k = 0, 2, 4, \dots$, c arbitrary, $k = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$, $c = 0$ and $k = 1, 3, 5, \dots$, c arbitrary.

b) An analysis of the set of equations reveals that if $B_k^{\pm}$ does not vanish for some values of k then it cannot vanish at all. But in this case the set appears to be inconsistent.

It is worthwhile to point out that representations constructed in ref. [2] are limited to the case $c = 0$. This is due to the fact that the most general expressions for the generators have not been used there.

The authors of ref. [2] have not noticed that a vector with spin less than two vanishes under the action of the lowering part of a generator. This becomes essential in the case $k_0 = \frac{3}{2}$ and it leads to the false conclusion about existence of the Δ - seuqence in [2].

III. Realization of the representations in terms of creation and destruction operators

Here we give a concrete realization for the representations discussed in the previous section. Our starting point is the well known Schwinger construction [6]. Consider two Bose operators V_α, $\alpha = 1,2$, satisfying the commutation relations

$$[V_\alpha, V_\beta] = 0, \quad [V_\alpha, V_\beta^+] = \delta_{\alpha\beta}, \quad V_\alpha|0\rangle = 0$$

(here $|0\rangle$ is the vaouum state). We can easily realize the algebra ASO(3) in terms of these operators provided the generators are taken to be

$$J_i = \frac{1}{2} V_\alpha^+ (\tau_i)_{\alpha\beta} V_\beta$$

(here τ_i are the Pauli matrices) and the basic vectors are

$$f_\nu^k = \frac{1}{\sqrt{(k+\nu)!\,(k-\nu)!}} (V_1^+)^{k+\nu} (V_2^+)^{k-\nu} |0\rangle .$$

In this scheme all scalar operators are functions of the operator $N = V_\alpha^+ V_\alpha$ whose eigenvalues are connected with the spin : $N f_\nu^k = 2k f_\nu^k$.

We can construct realizations of the algebras of various groups, containing SO(3) as their maximal compact subgroup, using the above described scheme. Take for instance the Lorentz group SO(3,1). Besides the generators J_i the group has the generators M_i, $i = 1,2,3$ forming a vector and obeying the commutation relations

$$[M_i, M_j] = -i\, \varepsilon_{ijk} J_k . \qquad (6)$$

In order to write explicitly these generators, we must find the general form of a vector operator constructed from V_α and V_α^+ and substitute it into eq . (6). It is evident that $V_\alpha (V_\alpha^+)$ are in fact some covariant (contravariant) spinors. Therefore only three kinds of vectors can be constructed from V_α and V_α^+ [7]. Namely

$$(1,0)\to u_i = \frac{1}{i} V_\alpha (\varepsilon\tau_i)_{\alpha\beta} V_\beta\,,\quad (\tfrac{1}{2},\tfrac{1}{2})\to J_i\,,\quad (0,1)\to u_i^+ = \frac{1}{i} V_\alpha^+ (\tau_i\varepsilon)_{\alpha\beta} V_\beta^+\,,$$

where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the antisymmetric 2x2 matrix.

These operators obey a number of useful identities (Appendix A) which are easily checked with the help of the completeness condition.

Multiplying these vectors by an arbitrary function of N we again obtain vector operators. So the general form of hermitian M_i is

$$M_i = A(N)\,u_i + u_i^+ A^+(N) + \varphi(N) J_i \qquad (7)$$

where $A(N)$ and $\varphi(N) = \varphi^+(N)$ are arbitrary functions (identity (A.1) requires that u, J and u^+ change the spin of the vectors f_ν^κ by -1,0,1 respectively so that these functions correspond to the Wigner-Eckart reduced matrix elements). We can now obtain a set of finite-difference equations for A(N) and $\varphi(N)$ substituting (7) into (6) and using the identities of Appendix A. The solutions of these equations provide just the same sequence of I.U.R. of ASO(3,1) as that described in ref. [8] (for a more detailed discussion see Appendix B). It is worth mentioning that besides its clarity, our realization is distinctive also because various operations over the generators are quite readily carried out due to the identities of Appendix A . (This is demonstrated in Appendix B by the example of the Casimir operators).

The case of ASL(3,R) is completely analogous. Here we must construct a tensor operator T_{ij} (it is convenient to use the vector notation for this tensor). There are exactly five constructions of tensor type:

$$(2,0)\to u_i u_j\,,\quad (\tfrac{3}{2},\tfrac{1}{2})\to u_i J_j + u_j J_i\,,\quad (1,1)\to J_i J_j + J_j J_i - \tfrac{2}{3}\delta_{ij} J^2,$$
$$(\tfrac{1}{2},\tfrac{3}{2})\to u_i^+ J_j + u_j^+ J_i\,,\quad (0,2)\to u_i^+ u_j^+ .$$

After multiplying by an arbitrary function of N and taking into account the hermiticity condition the general form of T_{ij} is

$$T_{ij} = A(N) u_i u_j + u_i^+ u_j^+ A^+(N) + B(N)(u_i J_j + u_j J_i) + (u_i^+ J_j + u_j^+ J_i) B^+(N)$$
$$+ \varphi(N)\left(J_i J_j + J_j J_i - \tfrac{2}{3}\delta_{ij} J^2\right), \qquad (8)$$

where $\varphi^+(N) = \varphi(N)$.

Substituting (8) into (1.b) we again obtain a set of finite-difference equations for the functions A(N), B(N) and $\varphi(N)$ (again identities (A) are used in obtaining the equations). This set has exactly the same features and the same solutions as set (3):

$$|A(N)|^2 = \frac{1}{16(N+1)(N+5)}\left[1 + \frac{c^2}{(N+3)^2}\right],\quad \varphi(N) = \frac{c}{(N-1)(N+3)}\,,\quad B(N)\equiv 0. \qquad (9)$$

(c is a real number parameter; $c = 0$ if $N = 1$ is permitted).

Therefore we have succeeded to realize all primitive I.U.R. of ASL(3,R) in a very simple and clear form (8) and (9).

Here the technical advantages of this type of realization (again due to the identities A) are even more obvious than in the case of the Lorentz group.

We would mention that a similar kind of realization was used in ref. [2], but there the generators were written in a spherical notation which lacks clarity and is less convenient in calculations. We point out also that there are not discussed all possible constructions of the generators.

APPENDIX A

$$[N, J_i] = 0, \quad N u_i = u_i(N-2), \quad N u_i^+ = u_i^+(N+2) \tag{A.1}$$

$$J_i u_i = 0, \quad J_i u_i^+ = 0, \quad u_i u_i = 0, \quad u_i^+ u_i^+ = 0 \tag{A.2}$$

$$J_i J_i = \tfrac{1}{4} N(N+2), \quad u_i u_i^+ = 2(N+2)(N+3), \quad u_i^+ u_i = 2N(N-1) \tag{A.3}$$

$$\left.\begin{array}{l} u_i^+ u_j - u_j^+ u_i = 4i(N-1)\varepsilon_{ijk} J_k, \quad u_i u_j^+ - u_j u_i^+ = -4i(N+3)\varepsilon_{ijk} J_k \\ u_i J_j - u_j J_i = \frac{i}{2}(N+4)\varepsilon_{ijk} u_k, \quad u_i^+ J_j - u_j^+ J_i = -\frac{i}{2}(N-2)\varepsilon_{ijk} u_k^+ \end{array}\right\} \tag{A.4}$$

$$\left.\begin{array}{l} u_i u_j^+ + u_j u_i^+ = 2(N+2)^2 \delta_{ij} - 4(J_i J_j + J_j J_i) \\ u_i^+ u_j + u_j^+ u_i = 2N^2 \delta_{ij} - 4(J_i J_j + J_j J_i) \end{array}\right\} \tag{A.5}$$

$$\left.\begin{array}{l} [J_i, J_j] = i\varepsilon_{ijk} J_k, \quad [u_i, J_j] = i\varepsilon_{ijk} u_k, \quad [u_i^+, J_j] = i\varepsilon_{ijk} u_k^+ \\ [u_i, u_j] = 0, \quad [u_i^+, u_j^+] = 0, \quad [u_i, u_j^+] = 4(N+1)\delta_{ij} - 8i\varepsilon_{ijk} J_k \end{array}\right\} \tag{A.6}$$

APPENDIX B

After inserting (7) into (6) we obtain the system of equations

$$\left|\begin{array}{l} (N+4)\varphi(N+2) - N\varphi(N) = 0 \\ (N+3)|A(N)|^2 - (N-1)|A(N-2)|^2 = \frac{1}{4} + \frac{1}{4}\varphi^2(N). \end{array}\right.$$

The solution of this set reads

$$|A(N)|^2 = \frac{1}{16}\left(1 + \frac{a}{(N+1)(N+3)} - \frac{b^2}{(N+1)(N+2)^2(N+3)}\right), \quad \varphi(N) = \frac{b}{N(N+2)},$$

where a and b are arbitrary constants (note that b is a real number since φ is hermitian; if $N=0$ is permitted then $b=0$).

Further, the existence of a minimal spin k_0 leads to the condition $A(N) u_i f_\nu^{k_0} = 0$, i.e., $A(2k_0-2) = 0$ except for the cases $k_0 = 0, \frac{1}{2}$. In the latter case $u_i f_\nu^{k_0} = 0$. This means that

$$1 + \frac{a}{(2k_0-1)(2k_0+1)} - \frac{b^2}{(2k_0-1)(2k_0)^2(2k_0+1)} = 0.$$

The existence of this relation enables us to introduce the notations (note that they correspond exactly to the Neimark notation [8]):

$$a = 1 - 4k_0^2 - 4c^2, \quad b = -4ik_0 c,$$

where c is suitably chosen constant (pure imaginary if $k_0 > 0$). The detailed analysis, shows that these notations are appropriate in the specific cases $k_0 = 0, \frac{1}{2}$ also.

Additional restrictions on the parameters a and b (that is, on k_o and c) are due to the condition $|A(N)|^2 \geqslant 0$. The final result coinciding completely with the results of ref. [8] is as follows:

a) k_o is an arbitrary non-negative integer or half-integer number, c is an arbitrary pure imaginary number. These are the conditions for the principal series of representations.

b) $k_o = 0$, $0 \leqq c < 1$ - the additional series.

At the end we give an illustration of the use of the identities (A):

$$C_1 = J^2 - M^2 = J^2 - \left[A(N)u_i + u_i^+ A^+(N) + \varphi(N) J_i\right]^2$$
$$= J^2 - |A(N)|^2 u_i u_i^+ - |A(N-2)|^2 u_i^+ u_i - \varphi^2(N) J^2 = -\frac{3}{4} - \frac{b}{4} = k_o^2 + c^2 - 1$$
$$C_2 = \vec{J} \cdot \vec{M} = J_i \left[A(N) u_i + u_i^+ A^+(N) + \varphi(N) J_i\right] = \varphi(N) J^2 = \frac{b}{4} = -i k_o c$$

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