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ULTRAVIOLET ASYMPTOTICS IN RENORMALIZABLE SCALAR THEORIES

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ULTRAVIOLET ASYMPTOTICS IN RENORMALIZABLE SCALAR THEORIES

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1. Introduction

Recent intensive investigations of ultraviolet behaviour of the Green functions in different quantum field theory models have led to the following two facts: 1. There exists a class of asymptotically free theories including non-abelian gauge fields [1,2] or fields with indefinite metrics [3]. 2. The analysis of subsequent logarithmic approximations may change the asymptotic behaviour drastically and, in some cases, leads to a finite value of the effective coupling constant [4]. In spite of attractiveness of

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asymptotically free theories[5], there have been found no realistic models of such a type. Moreover, the reason for presence or absence of the asymptotical freedom remains unclear.

The present paper is devoted to the investigation from this point of view of two renormalizable scalar models with the dimension of the space unequal to four: model φ^3 in six dimensions and model φ^6 in three dimensions. Though these models are not realistic, they can serve as an interesting example of the theories we spoke above.

The paper is organized as follows: In Sect. 2 we give an essential information about the dimensional regularization scheme and the notation we use. Sect. 3 is devoted to the $h\varphi^3_{(6)}$ theory. In Sect. 4 we consider 't Hooft's approach and compare the results with the usual BPHZ method. The theory $h\varphi^6_{(3)}$ is discussed in Sect. 5. The detailed calculations of multiloop diagrams in six- and three-dimensional space with the help of dimensional regularization scheme are presented in Appendices A and B. The results of calculations and some useful formulae can be found in App. C and D.

2. The dimensional regularization method

Very useful tool for our purposes proves to be the dimensional regularization scheme invented by 't Hooft and Veltman[6]. This kind of regularization is known to suit for handling the gauge theories, but it appears to be very useful in the case of scalar models as well, especially for the space-time dimensions unequal to four.

We briefly recall the scheme of the method. Let n_0 be the

space dimension of the theory we are dealing with. For the given Feynman diagram the corresponding momentum integral is convergent for the set $\{n_i\}$ of the positive integer space dimensions, including n_0 or not. Then the integral can be defined for all "dimensions" n through the analytical continuation in n from the set $\{n_i\}$ into the complex n-plane. The required formulae are given in Appendix D. The integral becomes the finite analytical function throughout the whole n-plane except for the integer positive value of n beginning from some M. At these integer points the integral has poles of the type

$$\frac{1}{(n-M-K)^N} \qquad \text{for integer } K \geqslant 0\,,\ N > 0$$

We shall regard only n near n_0, namely $n = n_0 - 2\varepsilon$.

Then the physical theory implies $\varepsilon \to 0$. Apparently this limiting process requires subtractions due to the presence of poles in ε. Only for the convergent diagram we may put ε equal to zero without any subtractions. Provided the diagram diverges or contains divergent subgraphs it has a singularity $\sim \frac{1}{\varepsilon^N}$ with $N \leq$ number of loops. Because the analytical continuation from discrete set $\{n_i\}$ is not unique, there exists a possibility to add an arbitrary polynomial to the regularized expression of the integral. So one may use different subtraction procedures. For instance, 't Hooft and Veltman[6] propose to subtract only the pole terms of the Laurent expansion in ε of the divergent integral (some consequences of this proposal will be discussed in Sect. 4). However, in the present paper we subtract the divergent integrals at the point $\kappa^2 = \lambda^2$, according to the BPHZ R-operation[7].

For the renormalization group analysis we need just ultraviolet logarithmical asymptotics, so only pole terms of the diagrams are important. Indeed the unsubtracted expression looks like

$$\frac{h^N}{(\kappa^2)^{N\varepsilon}}\left(\frac{A_N}{\varepsilon^N}+\ldots+\frac{A_1}{\varepsilon}+A_0+O(\varepsilon)\right).$$

Expanding $(\kappa^2)^{-N\varepsilon}$ in power series in ε and subtracting at $\kappa^2=\lambda^2$ we find that A_0 terms are of the type $A_0\,\varepsilon^m \ln^m\frac{\kappa^2}{\lambda^2}$ and vanish as $\varepsilon\to 0$. So they will not contribute to the logarithmical asymptotics.

We investigate renormalizable scalar models of the type $h\varphi^m_{(n_0)}$, where n_0 is a dimension of space-time. Throughout this paper we use the following notation:

$D(L,h)$ - propagator of the scalar particle with the normalization condition $D(0,h)=1$, where $L=\ln\frac{\kappa^2}{\lambda^2}$;

$\Gamma(L,h)$ - normalized proper vertex part with m external lines;

$\bar h(L,h)=h\Gamma D^{\frac{m}{2}}$ - invariant coupling constant (ICC), $\bar h(0,h)=h$.

Our metrics is $p_\mu p^\mu\equiv p^2=p_0^2-p_1^2-\ldots-p_{n_0-1}^2$.

The analysis of the ultraviolet behaviour is based on the differential Lie equations of the renormalization group

$$\frac{\partial\bar h(L,h)}{\partial L}=\varphi(\bar h(L,h)),\tag{1}$$

where $\varphi(h)$ is the Gell-Mann-Low function

$$\varphi(h)=\frac{\partial\bar h(L,h)}{\partial L}\Big|_{L=0}\tag{2}$$

3. The theory $h\varphi^3_{(6)}$

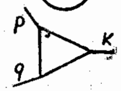
Consider the renormalizable scalar theory in six-dimensional space-time. The interaction Lagrangian looks as follows: $\mathcal{L}=\frac{(4\pi)^{3/2}h}{3!}\varphi^3_{(6)}$. The factor $(4\pi)^{3/2}$ is introduced to simplify subsequent expressions. The invariant coupling constant (ICC) is chosen in the form

$$\bar h^2(L,h)=h^2\Gamma^2 D^3.$$

The diagrams which contribute to the three-vertex Γ and propagator D are shown in Table 1. As far as we are interested in the high-energy behaviour all masses can be neglected provided that all subtractions are made far off the mass shell. Contributions to the Gell-Mann-Low function only come from the pole terms in ε (see Sect. 2) of the corresponding integrals.

In the lowest order in h the calculations are rather simple. Before making subtractions one has


$$\Rightarrow -\frac{h^2}{12\varepsilon}(\kappa^2)^{-\varepsilon}\left(1-c\varepsilon+\frac{8}{3}\varepsilon\right),$$


$$\Rightarrow \frac{h^2}{2\varepsilon}(\kappa^2)^{-\varepsilon}\left(1-c\varepsilon+\frac{\varepsilon}{2}-2f\varepsilon\right),\tag{3}$$

where c is the Euler constant arising from the expansion of Γ-function and f depends on the relation between κ^2, p^2 and q^2

$$f\left(\frac{p^2}{\kappa^2},\frac{q^2}{\kappa^2}\right)=\int_0^1 dx\int_0^1 dy\; x\ln\left\{1-y+\frac{p^2}{\kappa^2}y+x\left[\frac{q^2}{\kappa^2}y(1-y)-\frac{p^2}{\kappa^2}y-1+y\right]\right\}.$$

In the case of $p^2=q^2=\kappa^2$ we have $f\approx -0{,}468$.

In above expressions (3) we keep the finite terms in ε which are not relevant here but will be necessary for the R-operation in next-order calculations. After the subtraction at $\kappa^2=\lambda^2$ we come to the following expressions for D, Γ and ICC

$$D=1+\frac{h^2L}{12}\ ,\quad \Gamma=1-\frac{h^2L}{2}\ ,\quad \bar{h}^2=h^2\left[1-\frac{3}{4}h^2L\right]$$

and consequently the Gell-Mann-Low function

$$\varphi(h^2)=-\frac{3}{4}h^4\ .$$

Thus we have the asymptotically free theory in six dimensions. Note that this result emerges without non-abelian gauge fields and without fields with indefinite metrics.

Now we turn to the next logarithmical approximation. In App. A we consider in more detail the calculation of diagram 5, Table 1 to illustrate the use of the BPHZ R-operation in the dimensional method for the diagrams with overlapping divergences. Contributions of all diagrams are shown in Table 1. They lead to the following expressions for D, Γ and ICC:

$$D=1+\frac{h^2L}{12}-\frac{h^4L^2}{36}+\frac{h^4L}{6}\left(\frac{91}{72}+f\right)\ ,$$

$$\Gamma=1-\frac{h^2L}{2}+\frac{5h^4L^2}{16}-\frac{h^4L}{4}\left(3+f\right)\ ,$$

$$\bar{h}^2=h^2\left[1-\frac{3}{4}h^2L+\frac{9}{16}h^4L^2-\frac{125}{144}h^4L\right]\ .$$

The leading logarithmic terms form, as usual, the geometrical progression. Note that ICC is independent of f containing all the information about the relations between external momenta.

The Gell-Mann-Low function then will be

$$\varphi(h^2)=-\frac{3}{4}h^4-\frac{125}{144}h^6\ . \qquad (4)$$

Thus we can see, that the second term is of the same sign and will not change the asymptotic behaviour, just as in the selfinteraction Yang-Mills theory/12/. The theory is asymptotically free independently of the low-energy value $h=\bar{h}(0)$ up to the two-loop level.

4. 't Hooft's scheme of renormalization

It would be interesting to compare our results with those obtained by the subtraction scheme of 't Hooft and Veltman/6/. For this purpose we use the formalism recently proposed by 't Hooft/8/.

The interaction Lagrangian is $\frac{(4\pi)^{3/2}}{3!}h_B\varphi^3$. The dimension of the space-time n equals $6-2\varepsilon$, hence the bare coupling constant h_B has the mass dimension ε. We express h_B in terms of ε, "unit of mass" μ and the dimensionless renormalized coupling constant h by the following expansion

$$h_B^2=(\mu^2)^\varepsilon\left[h^2+\sum_{n=1}^{\infty}\frac{a_n(h^2)}{\varepsilon^n}\right]=(\mu^2)^\varepsilon\left[h^2+\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}a_n^{(m)}\frac{(h^2)^m}{\varepsilon^n}\right] \qquad (5)$$

Similarly the renormalization constant Z_Γ of the dimensionless Green function Γ is

$$Z_\Gamma=1+\sum_{n=1}^{\infty}\frac{c_n(h^2)}{\varepsilon^n}=1+\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}c_n^{(m)}\frac{(h^2)^m}{\varepsilon^n}\ .$$

The main fact established by 't Hooft/8/ is that we can choose $a_n^{(m)}$ and $c_n^{(m)}$ so that performing the calculations with h_B given by (5) and multiplying Γ by the corresponding Z_Γ we can put $\varepsilon=0$, i.e., all poles of the type $\frac{1}{\varepsilon^\nu}$ cancel in any given order of perturbation theory. The requirement of cancellation of the singularities allows us to calculate $a_n(h^2)$ and $c_n(h^2)$ uniquely, order by order in h^2.

The renormalized Green function is

$$\Gamma_R\left(\frac{\kappa^2}{\mu^2},h^2\right)=\lim_{\varepsilon\to 0}Z_\Gamma(h^2,\varepsilon)\cdot\Gamma_B\left(\kappa^2,h_B^2(\mu^2,h^2,\varepsilon),\varepsilon\right).$$

It satisfies the renormalization group equation of 't Hooft and Weinberg [8,9,10]

$$\left[\frac{\partial}{\partial L} - \beta(h^2)\frac{\partial}{\partial h^2} + \gamma_\Gamma(h^2)\right]\Gamma_R(L,h^2) = 0 \quad ,$$

where

$$\beta(h^2) = \left(h^2\frac{\partial}{\partial h^2} - 1\right) a_1(h^2)$$

plays the role of the Gell-Mann-Low function,

$$\gamma_\Gamma(h^2) = - h^2\frac{\partial}{\partial h^2} c_1(h^2)$$

is an anomalous dimension of Γ_R , $L = \ln\frac{\kappa^2}{\mu^2}$.

We have performed the calculations using the described method up to the two-loop level and obtained the following results:

$$h_B^2 = (\mu^2)^\varepsilon\left[h^2 - \frac{3}{4}\frac{h^4}{\varepsilon} + \frac{9}{16}\frac{h^6}{\varepsilon^2} - \frac{125}{288}\frac{h^6}{\varepsilon}\right] ,$$

$$\beta(h^2) = -\frac{3}{4}h^4 - \frac{125}{144}h^6 ,$$

$$Z_\Gamma = 1 - \frac{h^2}{2\varepsilon} + \frac{5h^4}{16\varepsilon^2} - \frac{23h^4}{96\varepsilon} ,$$

$$\gamma_\Gamma = \frac{h^2}{2} + \frac{23h^4}{48} , \qquad (6)$$

$$Z = 1 - \frac{h^2}{12\varepsilon} + \frac{5h^4}{144\varepsilon^2} - \frac{13h^4}{864\varepsilon} ,$$

$$\gamma_D = -\frac{h^2}{12} - \frac{13h^4}{432} ,$$

where Z is defined by $D_R = Z^{-1}D_B$.

One can see that $\beta(h^2)$ coincides with $\varphi(h^2)$ (4) calculated in Sect. 3. Note should be made that our results (6) are not identical with those of Macfarlane and Woo[11] who treated the same model by 't Hooft's method.

5. The theory $h\varphi^6_{(3)}$

Consider now the renormalizable scalar theory in three-dimensional space-time with the interaction Lagrangian of the form

$$\mathcal{L} = \frac{(8\pi)^2 h}{3!}\varphi^6 .$$

The ICC here is

$$\bar{h}(L,h) = h\Gamma D^3 .$$

The diagrams which contribute to the six-vertex Γ and to the propagator D are shown in Table 2. To calculate the Gell-Mann-Low function we are only interested in the pole terms in ε of the corresponding integrals. It should be noted that the diagram

does not contain such poles and will not contribute to the ICC asymptotical expression.

To illustrate the possibilities of the dimensional method in calculation of multiloop diagrams in three-dimensional space and the use of the BPHZ R-operation we consider diagram 5, Table 2 (see App. B).

The results of calculations for all the diagrams are shown in Table 2. They lead to the following expressions for D , Γ and ICC:

$$D = 1 + \frac{h^2}{90}L ,$$

$$\Gamma = 1 - \frac{10}{3}hL + \frac{100}{9}h^2L^2 - \frac{15}{4}h^2L(\pi^2+10) ,$$

$$\bar{h}(L,h) = h\left[1 - \frac{10}{3}hL + \frac{100}{9}h^2L^2 - 74.47\,h^2L\right] .$$

The Gell-Mann-Low function then looks as follows

$$\varphi(h) = -\frac{10}{3}h^2 - 74.47\,h^3 .$$

Thus in the lowest order in h for different signs of h we have the asymptotic freedom $(h>0)$ or "ghost-type" behaviour $(h<0)$. Taking into account the next terms of perturbation expansion we can see that the situation in the case of $h<0$ changes dractically just as it takes place in $\varphi^4_{(4)}$ theory/13/. Equation (1) in this approximation has the form

$$\frac{\partial \bar{h}(L,h)}{\partial L} = \varphi(\bar{h}) = -\frac{10}{3}\bar{h}^2 - 74.47\,\bar{h}^3$$

and leads to the finite asymptotic value of the ICC equal to H=-0.045. The anomalous dimension of the field will be very small: $\alpha = 2.22\cdot10^{-5}$.

So, the question of relative contribution of the next term to the function $\varphi(h)$ is very crucial. The situation thus is very similar to that in $\varphi^4_{(4)}$ theory with the coefficients of the Gell-Mann-Low function increasing even more rapidly. Therefore one can only suppose the existence of some "compensation mechanism" resulting in a finite value of $\bar{h}$.

6. Conclusion

The above considered examples allow us to conclude that asymptotical freedom is not an exceptional property of non-abelian gauge fields or of those with indefinite metrics and can arise in other quantum field models.

At the same time it should be noted that the theories with interaction of the form $h\varphi^{2n+1}$ for any h and $h\varphi^{2n}$ for h positive seem to be nonrealistic for they have no ground state from the quasi-classical point of view. But there are the

theories of such a type, namely $h\varphi^3_{(6)}$, $h\varphi^4_{(4)}$, $h\varphi^6_{(3)}$, that exhibit asymptotically free behaviour. It looks like that nature avoids asymptotical freedom.

The authors wish to express their deep gratitude to D.V.Shirkov for suggesting this subject and numerous helpful discussions and to A.V.Efremov for interest in this work.

Appendix A

Here we consider the calculation of diagram 5, Table 1. The corresponding integral looks as follows

$$J = \int \frac{d^n q \; d^n \ell}{q^2 (k-q)^2 (q-\ell)^2 \ell^2 (k-\ell)^2} \;, \qquad n = 6 - 2\varepsilon \;.$$

After the parametrization and momentum integrations according to the formula of App.D we come to the following expression

$$J = - \frac{\pi^6 \Gamma(-1+2\varepsilon)}{(k^2)^{-1+2\varepsilon}} \int_0^1 dx \, x^{1-\varepsilon} (1-x)^{-1+\varepsilon} \int_0^1 dy \int_0^1 dz \int_0^1 dt \, \frac{t\,(1-t)^{-1+\varepsilon}}{[\phi(x,y,z,t)]^{-1+2\varepsilon}} \;, \quad (A.1)$$

where $\phi(x,y,z,t) = tz(1-tz)(1-x) + y(1-t)(1-xy) - y^2(1-t)^2(1-x) - 2yzt(1-t)(1-x)$.

The integral in r.h.s. of (A.1) possesses simple pole at $\varepsilon = 0$. We rewrite it in the following way

$$J = - \frac{\pi^6 \Gamma(-1+2\varepsilon)}{(k^2)^{-1+2\varepsilon}} \left[J_1 + J_2 - J_3 - 2 J_4 \right] \;, \quad (A.2)$$

where

$$J_1 = \int_0^1 dx \, x^{1-\varepsilon} (1-x)^{\varepsilon} \int_0^1 dy \int_0^1 z \, dz \int_0^1 dt \, t^2 (1-tz)(1-t)^{-1+\varepsilon} \phi^{-2\varepsilon} \;,$$

$$J_2 = \int_0^1 dx \, x^{1-\varepsilon} (1-x)^{-1+\varepsilon} \int_0^1 dy \, y(1-xy) \int_0^1 dz \int_0^1 dt \, t\,(1-t)^{\varepsilon} \phi^{-2\varepsilon} \;,$$

$$J_3 = \int_0^1 dx \, x^{1-\varepsilon} (1-x)^{\varepsilon} \int_0^1 dy \, y^2 \int_0^1 dz \int_0^1 dt \, t\,(1-t)^{1-2\varepsilon} \phi^{-2\varepsilon} \;,$$

$$J_4 = \int_0^1 dx \, x^{1-\varepsilon} (1-x)^{\varepsilon} \int_0^1 y \, dy \int_0^1 z \, dz \int_0^1 dt \, t^2 (1-t)^{\varepsilon} \phi^{-2\varepsilon} \;.$$

It should be noted, that the integrals J_3 and J_4 are not singular in ε and hence we can put $\varepsilon = 0$. Then we immediately get

$$J_3 = \frac{1}{36} \;; \qquad J_4 = \frac{1}{24} \;. \quad (A.3)$$

As for J_1 and J_2, they are singular when integrating over t and x correspondingly. The singular integral has the form

$$T(\varepsilon) = \int_0^1 x^{-1+\varepsilon} Q(x,\varepsilon)\, dx \;.$$

Integrating by parts and expanding x^{ε} in power series in ε we find

$$T(\varepsilon) = \frac{x^{\varepsilon}}{\varepsilon} Q(x,\varepsilon) \Big|_0^1 - \frac{1}{\varepsilon} \int_0^1 \left(1 + \varepsilon \ln x + o(\varepsilon^2)\right) Q'_x(x,\varepsilon)\, dx \;.$$

Provided the remaining integral converges we may put $\varepsilon = 0$ and get

$$T(\varepsilon) = \frac{Q(0,\varepsilon)}{\varepsilon} - \int_0^1 dx \, \ln x \, Q'_x(x,0) \;.$$

Using this formula, the calculations are straightforward and give

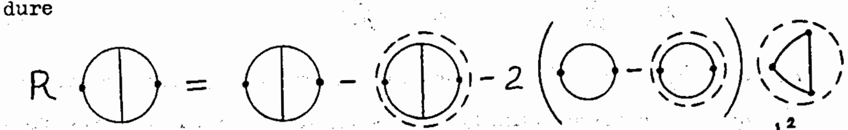
$$J_1 = \frac{\Gamma(2-\varepsilon)\Gamma(1-\varepsilon)\Gamma^2(2-2\varepsilon)}{\Gamma(3-2\varepsilon)\Gamma(4-4\varepsilon)} \frac{1}{\varepsilon} - \frac{5}{72} \;,$$

$$J_2 = \frac{\Gamma^2(2-2\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(4-4\varepsilon)\Gamma(3-\varepsilon)} \cdot \frac{1}{\varepsilon} \;. \quad (A.4)$$

Substituting (A.3) and (A.4) into (A.2) and using the Γ-function expansion we ultimately obtain

$$J(k^2) = \frac{\pi^6 k^2}{12 \varepsilon^2 (k^2)^{2\varepsilon}} \left(1 - 2c\varepsilon + 6\varepsilon\right) \;.$$

In order to remove the singularities we use the BPHZ R-operation with the subtractions at $k^2 = \lambda^2$. The diagram contains two divergent subgraphs and leads to the following R-operation procedure



where the dotted line denotes the value of the integral at λ^2.

Then for $J(k^2)$ we find:

$$R\,J(k^2) = \frac{\pi^6 k^2}{12}\left[L^2 - \left(\frac{17}{3} + 4f\right)L\right], \quad L = \ln\frac{k^2}{\lambda^2}.$$

All other diagrams can be calculated similarly.

Appendix B

Consider diagram 5, Table 2. After setting all the masses equal to zero the corresponding integral looks as follows

$$J = \int \frac{d^n q}{(p+q)^2}\,\frac{d^n \ell}{\ell^2 (k-q-\ell)^2}\,\frac{d^n r\, d^n t}{t^2 (r-t)^2 (q-r)^2}\ ; \quad n = 3-2\varepsilon . \tag{B.1}$$

Using the formulae of App. D we obtain for the internal integrations

$$\int \frac{d^n \ell}{\ell^2 (k-q-\ell)^2} = \frac{i(-\pi)^{3/2}\Gamma(\frac{1}{2}+\varepsilon)\Gamma^2(\frac{1}{2}-\varepsilon)}{\Gamma(1-2\varepsilon)\left[(k-q)^2\right]^{\frac{1}{2}+\varepsilon}}, \tag{B.2}$$

$$\int \frac{d^n r\, d^n t}{t^2 (r-t)^2 (q-r)^2} = \frac{i^2(-\pi)^3\Gamma^3(\frac{1}{2}-\varepsilon)\Gamma(2\varepsilon)}{\Gamma(\frac{3}{2}-3\varepsilon)\,(q^2)^{2\varepsilon}}. \tag{B.3}$$

Substituting (B.2) and (B.3) into (B.1) results in

$$J = \frac{i^3(-\pi)^{9/2}\Gamma(\frac{1}{2}+\varepsilon)\Gamma^5(\frac{1}{2}-\varepsilon)\Gamma(2\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(\frac{3}{2}-3\varepsilon)}\int \frac{d^n q}{\left[(k-q)^2\right]^{\frac{1}{2}+\varepsilon}(q^2)^{2\varepsilon}(p+q)^2}$$

Performing the momentum integrations and putting $k^2 = p^2 = (k+p)^2$ we come to the following expression

$$J = \frac{\pi^6\Gamma^5(\frac{1}{2}-\varepsilon)\Gamma(4\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(\frac{3}{2}-3\varepsilon)}\,\frac{1}{(k^2)^{4\varepsilon}}\int_0^1 dx\, x^{-2\varepsilon}(1-x)^{-\frac{1}{2}+\varepsilon}\int_0^1 dy\,\frac{y^{-1+2\varepsilon}}{\left[1-x(1-y(1-y))\right]^{4\varepsilon}} \tag{B.4}$$

Changing the order of integrations and using the formulae of App. D we have

$$J = \frac{\pi^6\Gamma^5(\frac{1}{2}-\varepsilon)\Gamma(\frac{1}{2}+\varepsilon)\Gamma(4\varepsilon)}{\Gamma(\frac{3}{2}-3\varepsilon)\Gamma(\frac{3}{2}-\varepsilon)}\,\frac{1}{(k^2)^{4\varepsilon}}\int_0^1 dy\; y^{-1+2\varepsilon}\,F\left(4\varepsilon, 1-2\varepsilon, \tfrac{3}{2}-\varepsilon, 1-y(1-y)\right),$$

where F is the hypergeometrical function which can be rewritten in the following way

$$F\left(4\varepsilon, 1-2\varepsilon, \tfrac{3}{2}-\varepsilon, 1-y(1-y)\right) = \frac{\Gamma\left(\frac{3}{2}-\varepsilon\right)\Gamma\left(\frac{1}{2}-3\varepsilon\right)}{\Gamma\left(\frac{3}{2}-5\varepsilon\right)\Gamma\left(\frac{1}{2}+\varepsilon\right)} F\left(4\varepsilon, 1-2\varepsilon, -\tfrac{3}{2}+3\varepsilon, y(1-y)\right) +$$
$$+\left[y(1-y)\right]^{\frac{1}{2}-3\varepsilon} \frac{\Gamma\left(\frac{3}{2}-\varepsilon\right)\Gamma\left(-\frac{1}{2}+3\varepsilon\right)}{\Gamma(4\varepsilon)\,\Gamma(1-2\varepsilon)} F\left(\tfrac{3}{2}-5\varepsilon, \tfrac{1}{2}+\varepsilon, \tfrac{3}{2}-5\varepsilon, y(1-y)\right). \tag{B.5}$$

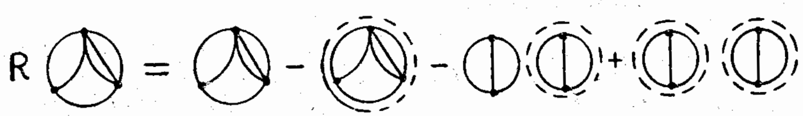
Expanding now the hypergeometrical functions in the $r.h.s.$ of (B.5) in power series in $y(1-y)$ and integrating over y, we find that the only singular term will be

$$J \Rightarrow \frac{\pi^6 \Gamma^5\left(\frac{1}{2}-\varepsilon\right)\Gamma(4\varepsilon)\Gamma\left(\frac{1}{2}-3\varepsilon\right)}{\Gamma\left(\frac{3}{2}-3\varepsilon\right)\Gamma\left(\frac{3}{2}-5\varepsilon\right)\cdot 2\varepsilon}\cdot\frac{1}{(k^2)^{4\varepsilon}} . \tag{B.6}$$

From (B.6), using the expansions of Γ-functions we immediately get

$$J = \frac{\pi^8}{2\varepsilon^2}\left(1 - 4c\,\varepsilon + 16\,\varepsilon\right)\frac{1}{(k^2)^{4\varepsilon}} .$$

For removing the singularities we use the BPHZ R-operation with the subtractions at $k^2=\lambda^2$. The diagram contains one divergent subgraph, therefore the result of the use of R-operation will be



For $J(k^2)$ this leads to

$$R\,J(k^2) = 2\pi^8\left(L^2 - 4L\right), \qquad \text{where} \quad L = \ln\frac{k^2}{\lambda^2} .$$

Appendix C

Table 1. $\mathcal{L} = \dfrac{(4\pi)^{3/2}}{3!}\,h\varphi^3$; $n = 6-2\varepsilon$; $L = \ln\dfrac{k^2}{\lambda^2}$.

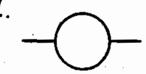
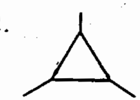
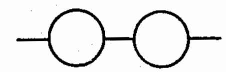
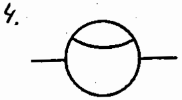
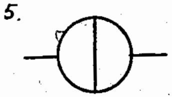
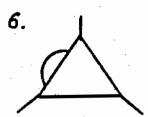
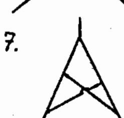
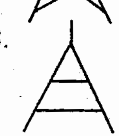
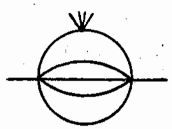
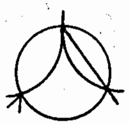
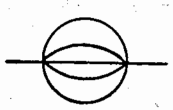
Diagram	Regularized expression	Contribution to the Green function
1. 	$-\frac{h^2}{(k^2)^{\varepsilon}}\frac{1}{12\varepsilon}\left(1-c\varepsilon+\frac{8}{3}\varepsilon\right)$	$\frac{h^2 L}{12}$
2. 	$\frac{h^2}{(k^2)^{\varepsilon}}\frac{1}{2\varepsilon}\left(1-c\varepsilon+\frac{\varepsilon}{2}-2f\varepsilon\right)$	$-\frac{h^2 L}{2}$
3. 	$\frac{h^4}{(k^2)^{2\varepsilon}}\frac{1}{144\varepsilon^2}\left(1-2c\varepsilon+\frac{16}{3}\varepsilon\right)$	$\frac{h^4 L^2}{144}$
4. 	$\frac{h^4}{(k^2)^{2\varepsilon}}\frac{1}{144\varepsilon^2}\left(1-2c\varepsilon+\frac{43}{6}\varepsilon\right)$	$\frac{h^4}{144}\left(L^2-\frac{11}{3}L\right)$
5. 	$-\frac{h^4}{(k^2)^{2\varepsilon}}\frac{1}{24\varepsilon^2}\left(1-2c\varepsilon+6\varepsilon\right)$	$-\frac{h^4}{24}\left(L^2-\frac{17}{3}L-4fL\right)$
6. 	$-\frac{h^4}{(k^2)^{2\varepsilon}}\frac{1}{16\varepsilon^2}\left(1-2c\varepsilon+\frac{13}{6}\varepsilon-4f\varepsilon\right)$	$-\frac{h^4}{16}\left(L^2+2L+4fL\right)$
7. 	$\frac{h^4}{(k^2)^{2\varepsilon}}\frac{1}{8\varepsilon}$	$-\frac{h^4 L}{4}$
8. 	$\frac{h^4}{(k^2)^{2\varepsilon}}\frac{3}{8\varepsilon^2}\left(1-2c\varepsilon+\frac{3}{2}\varepsilon-4f\varepsilon\right)$	$\frac{3}{8}h^4\left(L^2-L\right)$

Table 2. $\mathcal{L} = \frac{(8\pi)^2}{6!} h \varphi^6$; $n = 3 - 2\varepsilon$; $L = \ln \frac{\kappa^2}{\lambda^2}$.

Diagram	Regularized expression	Contribution to the Green function
1. 	$\frac{h}{(\kappa^2)^{2\varepsilon}} \frac{5}{3\varepsilon} \left(1 - 2c\varepsilon + 6\varepsilon\right)$	$-\frac{10}{3} h L$
2. 	$\frac{h^2}{(\kappa^2)^{4\varepsilon}} \frac{5}{18\varepsilon^2} \left(1 - 4c\varepsilon + 12\varepsilon\right)$	$\frac{10}{9} h^2 L^2$
3. 	$\frac{h^2}{(\kappa^2)^{4\varepsilon}} \cdot \frac{15\pi^2}{16\varepsilon}$	$-\frac{15}{4} \pi^2 h^2 L$
4. 	$-\frac{h^2}{(\kappa^2)^{4\varepsilon}} \frac{5}{8\varepsilon}$	$\frac{5}{2} h^2 L$
5. 	$\frac{h^2}{(\kappa^2)^{4\varepsilon}} \frac{5}{2\varepsilon^2} \left(1 - 4c\varepsilon + 16\varepsilon\right)$	$10 h^2 \left(L^2 - 4L\right)$
6. 	$-\frac{h^2}{(\kappa^2)^{4\varepsilon}} \frac{1}{360\varepsilon}$	$\frac{h^2}{90} L$

Appendix D

The main integration formula which holds for arbitrary n is:

$$\int \frac{d^n p}{(p^2 - 2\kappa p + m^2)^\alpha} = \frac{i(-\pi)^{\frac{n}{2}}}{(m^2 - \kappa^2)^{\alpha - \frac{n}{2}}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} .$$

With the help of this equation any scalar diagram may be evaluated if one uses the Feynman parameters:

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha - 1}(1 - x)^{\beta - 1}}{[ax + b(1 - x)]^{\alpha + \beta}} ,$$

where Γ is the Euler function.

The generalization of this equation to many factors in the denominator is

$$\frac{1}{a_1^{\alpha_1} \dots a_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 + \dots + \alpha_n)}{\Gamma(\alpha_1) \cdot \dots \cdot \Gamma(\alpha_n)} \int_0^1 dx_1 \dots \int_0^1 dx_n \frac{x_1^{\alpha_1 - 1} \dots x_n^{\alpha_n - 1} \delta(1 - x_1 - \dots - x_n)}{[a_1 x_1 + \dots + a_n x_n]^{\alpha_1 + \dots + \alpha_n}} =$$

$$= \frac{\Gamma(\alpha_1 + \dots + \alpha_n)}{\Gamma(\alpha_1) \cdot \dots \cdot \Gamma(\alpha_n)} \int_0^1 x_1^{n-2} dx_1 \int_0^1 x_2^{n-3} dx_2 \dots \int_0^1 dx_{n-1} \frac{(x_1 \cdot \dots \cdot x_{n-1})^{\alpha_1 - 1} \cdot [x_1 \cdot \dots \cdot x_{n-2}(1 - x_{n-1})]^{\alpha_2 - 1} \cdot \dots \cdot (1 - x_1)^{\alpha_n - 1}}{[a_1 x_1 \cdot \dots \cdot x_{n-1} + a_2 x_1 \cdot \dots \cdot x_{n-2}(1 - x_{n-1}) + \dots + a_n (1 - x_1)]^{\alpha_1 + \dots + \alpha_n}} .$$

For the evalution of the integrals over the Feynman parameters we use the following formulae

$$\int_0^1 dx\, x^{a-1}(1 - x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} ,$$

$$\int_0^1 dx\, x^{\lambda - 1}(1 - x)^{\mu - 1}(1 - \beta x)^{-\nu} = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda + \mu)} F(\nu, \lambda, \lambda + \mu, \beta), \ |\beta| < 1 ,$$

where F is the hypergeometrical function.

The essential properties of Γ - and F-functions are:

$$\Gamma(1 + x) = x\Gamma(x) \ ; \quad \Gamma(n) = (n - 1)! \ ; \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \ ;$$

$\Gamma(1+\varepsilon)=1-c\varepsilon+O(\varepsilon^2)$; $\Gamma(\frac{1}{2}+\varepsilon)=\Gamma(\frac{1}{2})(1-\varepsilon(c+2\ln 2)+O(\varepsilon^2))$, where C is the Euler constant; $c \approx 0.577$;

$$F(\alpha,\beta,\gamma,z)=1+\frac{\alpha\cdot\beta}{\gamma\cdot 1}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1\cdot 2}z^2+\ldots\quad;$$

$$F(\alpha,\beta,\gamma,z)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta-\gamma-1,1-z)+$$
$$+(1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z).$$

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