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RENORMALIZATION GROUP FOR NONRENORMALIZABLE QFT

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S U M M A R Y

Functional equations of the renormalization group (RG) are formulated for the theories with dimensional coupling constants. A special attention is paid to the theories with the coupling constants of negative-mass dimension, which are nonrenormalizable within the usual perturbation approach.

The correspondence of general solutions of the RG equation with the perturbation expansion in the ultraviolet region yields the nonanalyticity in coupling constant.

A possibility of the short distance scale invariance is discussed. An additional assumption on the finite number of the invariant charges imposes limitations on the Bogolubov R-operation parameters. As an illustration of consistency of this hypothesis an exactly soluble nonrenormalizable nonrelativistic model is considered.

1. Renormalization group in a theory with dimensional coupling constant

Renormalization group (RG) may be formulated as a group of the finite multiplicative Dyson transformations for the Green functions and "compensating" transformations of coupling constants [1]. This viewpoint makes it easy to understand that the RG invariance is conditioned neither by perturbation theory nor by the dimension of coupling constant and nor by the structure of the ultraviolet divergences. This fact was known long ago and was successfully used for the summation of the infrared divergences in quantum electrodynamics [2] (see also [1]) as well as for the problem of the nonrelativistic Coulomb screening of the electron gas [3]. To accentuate this fact, we shall present a nonstandard derivation of the RG equation which more corresponds to its fundamental nature.

Consider for simplicity an interacting scalar field theory. One of the parameters of the theory which enters into the n-particle amplitudes $T_n(p_1,\dots,p_{n-1})$ is the mass m which arises as a position of the pole of the total propagator:

$$(p^2) = \frac{d(p^2)}{p^2-m^2} \quad . \qquad (1.1)$$

In addition the amplitudes T_n also depend upon a parameter connected with normalization of one-particle state and on some parameters which characterize the strength of interaction (constants of interaction). The first parameter is fixed by a given magnitude of the function $d(p^2)$ at a point $p^2=\lambda$:

$$z_\lambda = d(p^2=\lambda). \qquad (1.2)$$

Then, obviously, the matrix elements (the observable quantities) cannot depend on the normalization of asymptotical states, therefore

$$T_n(p_i) = z_\lambda^{-n/2} T_n(p_i, m, \lambda). \qquad (1.3)$$

In particular,

$$d(p^2) = z_\lambda d(p^2, m^2, \lambda), \quad d(\lambda, m^2, \lambda) = 1. \qquad (1.4)$$

The interaction constants usually are defined as magnitudes of some matrix elements at definite values of their invariant arguments. Let us now assume that we have only one such a parameter and define it through the matrix element connected with the four-point amplitude $T_4 = \Gamma$ at the point $p_i^2 = -1/3 p_i p_j = \lambda$, i.e.,

$$\Gamma(p_i) = z_\lambda^{-2} \; \Gamma(p_i, m^2, \lambda, g_\lambda), \qquad (1.5)$$

where

$$\Gamma(\lambda, m^2, \lambda, g) = g \;. \qquad (1.6)$$

It is clear that we are free to take any point λ_1, different from λ, out of the region, where d and Γ are real functions, and other constants

$$z_{\lambda_1} = d(p^2 = \lambda_1), \quad g_{\lambda_1} = z_{\lambda_1}^{2} \; \Gamma(p_i^2 = \lambda_1).$$

The amplitudes should not depend on such a choice, i.e.,

$$z_\lambda^{-n/2} T_n(p_i, m^2, \lambda, g_\lambda) = z_{\lambda_1}^{-n/2} T_n(p_i, m^2, \lambda_1, g_{\lambda_1}). \qquad (1.7)$$

In particular,

$$z_\lambda^{-2} \; \Gamma(p_i, m^2, \lambda, g_\lambda) = z_{\lambda_1}^{-2} \; \Gamma(p_i, m^2, \lambda_1, g_{\lambda_1}), \qquad (1.8)$$

$$z_\lambda d(p^2, m^2, \lambda, g_\lambda) = z_{\lambda_1} d(p^2, m^2, \lambda_1, g_{\lambda_1}). \qquad (1.9)$$

The transformations (1.7) – (1.9) under $\lambda \to \lambda_1$ form a group with the invariant charge

$$\bar{g}(p^2, m^2, \lambda, g_\lambda) = \Gamma(p_i^2 = -1/3 p_i p_j = p^2, m^2, \lambda, g_\lambda) \cdot \qquad (1.10)$$
$$\times \, d^2(p^2, m^2, \lambda, g_\lambda),$$

obeying the well-known functional equation

$$\bar{g}(p^2, m^2, \lambda, g_\lambda) = \bar{g}(p^2, m^2, \lambda_1, \bar{g}(\lambda_1, m^2, \lambda, g_\lambda)), \qquad (1.11)$$

and the constant of interaction being determined as

$$g_\lambda = \bar{g}(\lambda, m^2, \lambda, g \;). \qquad (1.12)$$

This derivation clearly demonstrates that the RG equations are valid both for the theory with zero- and positive-mass dimension of the coupling constants (e.g., $\varphi^4_{(D)}$ when $D \le 4$) and for the theory with negative-mass dimension, nonrenormalizable from the conventional point of view (e.g., $\varphi^4_{(D)}$ when $D > 4$). The RG equations reflect the independence of a theory of a choice of definition point of the coupling constant and of a normalization of asymptotical states, and thus they are fundamental conditions in any theory. For simplicity, we only confine ourselves to a scalar field and one interaction parameter. All the arguments, however, can be generalized to other sorts of fields and several parameters, obtaining instead of eq.(1.11) the equations of multicharge RG.

The aim of this work is to analyze the consequences of the one-charge RG equation for a nonrenormalizable theory with negative-mass dimension of the coupling constant: $[g] = [m^2]^{-k}, k > 0$. We assume, of course, that such a theory does exist, i.e., the corresponding amplitudes depending on a finite number of parameters of the type of coupling constants can be determined in a self-consistent way. For simplicity we also assume that at least some of the models admit of the one-charge RG equation *.

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* As a subgroup such a group probably exists in any case.

An example of the model which satisfies the above assumptions is the soluble nonrenormalizable model of a "meson" field interacting with a fixed "nucleon" (it is considered in /4/)

$$\mathcal{L}_{int} = \frac{g}{2\pi}\delta(\vec{x})\,\varphi(\vec{x},t)\,\dot{\varphi}(\vec{x},t). \qquad (1.13)$$

The role of invariant charge (IC) plays here the meson-nucleon scattering amplitude $f(\omega)$ divided by the total meson energy ω. It has the form

$$\bar{g}(\omega) = \frac{f(\omega)}{\omega} = \frac{g}{1 - g\omega[c - \sqrt{\mu^2 - \omega^2}]} \qquad (1.14)$$

which depends on an additional parameter c. The renormalized coupling constant g in (1.14) is determined as follows:

$$g = \bar{g}(0, g).$$

Changing the normalization point we can rewrite eq.(1.14) in the form satisfying the RG equation (1.11).

Multiplying both sides of eq.(1.11) by $(p^2)^k$ and passing to the dimensionless variables one can obtain

$$\bar{\gamma}(x,y,\gamma) = \bar{\gamma}\left(\frac{x}{t},\frac{y}{t},\bar{\gamma}(t,y,\gamma)\right), \qquad (1.15)$$

where $x = p^2/\lambda$, $y = m^2/\lambda$, $t = \lambda_1/\lambda$, $\gamma = g_\lambda \lambda^k$ and $\bar{\gamma} = (p^2)^k \bar{g}$.

A remarkable feature of eq.(1.15) is its universality. It has the same form for any QFT with one coupling constant irrespective of its dimension. The general solution /5/ of eq.(1.15) in the limit $y = 0$ (i.e., in the limit $p^2, \lambda >> m^2$) has the well-known form

$$\gamma(x,\gamma) = \Phi(x^k\,\Phi^{-1}(\gamma)), \qquad (1.16)$$

where Φ is an arbitrary function.

2. RG and perturbation theory

One of the most interesting problems here is the correspondence of RG to the perturbation expansion for the nonrenormalizable theory. As is well known, the main drawback of this perturbation expansion is that the number of arbitrary subtraction constants increases with the order of coupling constant.

Consider, for simplicity, the case $k=1$. (The essential results will be formulated for the case $k\neq 1$, as well.) The perturbation expansion for IC has the form

$$\gamma(x,\gamma) = \gamma x\{1 + \gamma[Ax\ln x + c_{11}(x-1)] + \gamma^2[Bx^2\ln^2 x + Ex^2\ln x + c_{22}(x-1)^2 + c_{21}(x-1)] + \gamma^3[\cdots] + \cdots\}. \qquad (2.1)$$

Let us for the moment accept the following form of the expansion for IC

$$\bar{\gamma}(x,\gamma) = \gamma x[1 + \gamma\varphi_1(x) + \gamma^2\varphi_2(x) + \cdots] \qquad (2.2)$$

and the expansion

$$\Phi(z) = z + z^2\Phi_1 + z^3\Phi_2 \qquad (2.3)$$

for the function $\Phi(z)$. Then, substituting (2.2), (2.3) into (1.16) gives for φ_i polynomials in x which obviously contradict the perturbation series (2.1). For instance,

$$\varphi_1 = (x-1)\Phi_1. \qquad (2.4)$$

What is wrong here ? Maybe, the analyticity assumption at $\gamma=0$ (cf./6/)? If one admits a "weak" nonanalyticity in z, i.e., that Φ_i are functions of z, nonanalytic at $z=0$, then instead of (2.4) one can get

$$\varphi_1(x,\gamma) = x\,\Phi_1(x\gamma) - \Phi_1(\gamma), \qquad (2.5)$$

$$\varphi_2(x,\gamma) = \varphi_1^2(x,\gamma) + \varphi_1(x,\gamma)\frac{d\Phi_1(x\gamma)}{d\ln x\gamma} + x^2\Phi_2(x\gamma) - \Phi_2(\gamma). \qquad (2.6)$$

Comparing (2.5) and (2.6) with (2.1) shows that

$$\Phi_1(x\gamma) = a_1 \ln\frac{x\gamma}{\beta} \qquad (\text{or } \; c_{11} = a_1 \ln \gamma/\beta\;) \qquad (2.7)$$

$$\Phi_2(x\gamma) = a_2 \ln^2 x\gamma + b \ln x\gamma + d, \qquad (2.8)$$

where $a_1 = A$ and a_2, b are functions of A, B, E and β. Thus, <u>for the perturbation expansion to correspond to RG the nonanalyticity in coupling constant must appear</u>. If $k \neq 1$ it will be the branchpoint $(\gamma)^{1/k}$. For an integer $k \neq 1$ the branchpoint $\ln\gamma$ will appear, as well.

The second consequence is <u>an interdependence of the parameters c_{km}</u>. As is seen from (2.8), (2.7), (2.6), only one additional arbitrary constant d appears in the third order instead of two in (2.1). In general, using the development

$$\bar{\gamma}(x,\gamma) = \gamma + (x-1)\,\psi_1(\gamma) + (x-1)^2\,\psi_2(\gamma) + \cdots \qquad (2.9)$$

and the differential RG equation for IC

$$x\,\frac{d\bar{\gamma}(x,\gamma)}{dx} = \psi_1(\bar{\gamma}(x,\gamma)) \qquad (2.10)$$

all the functions ψ_n ($n > 1$) can easily be expressed in terms of the function ψ_1. For instance,

$$2\,\psi_2(\gamma) = \psi_1(\gamma)\left(\psi_1'(\gamma) - 1\right). \qquad (2.11)$$

Thus, we have shown that the perturbation expansion in the nonrenormalizable theory can be brought in correspondence with the RG equation. Moreover, it turns out that the whole ambiguity of IC can be expressed in terms of a first-order polynomial in x (in terms of c_{k1}), i.e., it reduces to the finite number of counterterms. This result seems to be a substantial step towards the construction of the renormalization scheme for the theories

in question. The most important problem here is the divergences and renormalization of the higher-order Green functions.

3. <u>Ultraviolet asymptotical behaviour</u>

A significant property of the expressions found in sect.2 is that the terms

$$x^n\,\Phi_n(x\gamma) - \Phi_n(x) \qquad (3.1)$$

entering into the functions $\varphi_n(x,\gamma)$ of the type (2.5), (2.6), can contain (and do contain) the senior asymptotic terms $(x\ln x\gamma)^n$. This means that the leading terms of different φ_n are independent of each other. Therefore, in particular, the differential Gell-Mann-Low equation (2.10) (in distinction with the renormalizable case) is noneffective for improving the approximating property of perturbation theory.

Moreover, it is easy to show that in this case the whole RG is of little use for this purpose. Let us imagine that we have succeeded in summing all the terms $(x\gamma\ln x\gamma)^n$ of the Feynman diagrams and have found

$$\bar{\gamma}_{P.T.}(x\gamma) = \Psi(x\gamma). \qquad (3.2)$$

Using the general solution (1.16) gives

$$\bar{\gamma}_{RG}(x,\gamma) = \Psi(x\,\Psi^{-1}(\gamma)) \qquad (3.3)$$

instead of (3.2). Due to that for small

$$\Psi^{-1}(\gamma) \simeq \gamma + O(\gamma^2)$$

the expression (3.3) is in practice the same as (3.2).

The equation (3.2) can nevertheless be useful in the region

$$x\gamma\ln x\gamma \lesssim 1, \quad \ln x\gamma \gg 1, \quad x\gamma \ll 1. \qquad (3.4)$$

For the weak interactions it allows an essential step to be made with $L_{int} = g(\bar{\psi} O \psi)(\bar{\psi} O \psi)$ (see, e.g., /11/).

To complete the paper, we would like to stress once more the fundamental character of RG and validity of its equations not only for the theories with polynomial interaction Lagrangians but also for the nonpolynomial ones, e.g., of the type of chiral invariant Lagrangian depending on a single coupling constant. A specific feature of RG in such theories is the connection of the invariant charges, constructed by means of the Green functions of various orders in the way similar to that in the Yang-Mills theory.

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