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1. Introduction

In present theoretical physics appropriate approximation schemes are of enormous importance both to systematize data coming out from the larger and larger accelerators and to search the laws, which govern the phenomena being behind the data. No doubt, one of the best guides to find successful approximations has been infinite momentum limit [1]. Most popular applications attribute to infinite momentum frame (IMF) description the following two, interrelated, implications:

1. In field theory the $P \to \infty$ limit is equivalent to quantizing the fields on a light-front [2];

2. There is a correspondence between IMF physics and two-dimensional Galilean physics [1,3].

Both of these points crystallized out from direct or indirect attempts made at interpreting the results of Weinberg's classical paper from 1966 [4]. (Though the use of IMF was initiated in current algebra [5].). The investigations of the present paper will be focused on the second point, which became a first principle in some formulation of parton model [1].

First of all we remark, that as far as kinematics is concerned IMF does not favour any subgroup of the Poincare group to the others, therefore on the kinematical level point 2 is a completely arbitrary statement. The argument for this is shortly as follows. Relativistic particle kinematics is not more, than representation theory of the Poincare group. From a group theoretical point of view infinite momentum limit is nothing else, but contraction of the Poincare group [6]. Group contraction is, however, a far not unique procedure. In the present case it turns out, that the arbitrariness manifests itself just in destroying the distinguished role of the Galilean subgroup in the infinite momentum limit. The ''favoured'' subgroup may equally be a two-dimensional Poincare group. This will be demonstrated in Sect. II.

Any distinction between the different kinematical possibilities may be of dynamical origin only. Therefore we investigate a simple interacting scalar field theory. Lagrange formalism gives us a realization of the generators of the Poincare group in terms of field operators, and we ask, how they are acting on the states of a moving Lorentz frame, especially, when the relative velocity of the frames in which the generators and states are specified tends to the velocity of light. We show, that the two kinematical cases, presented in Sect. II, can be enlarged to consistent dynamical pictures. One of them exhibits the two-dimensional Galilean structure, which is usually discussed in connection with Weinberg's infinite momentum limit of the non-covariant perturbation series. The second case shows the characteristics of a (two-dimensional)relativistic theory. This offers an alternative group-theoretical interpretation of Weinberg's result, and possibly useful new approximation schemes. After recapitulating the needed field theoretical formulas in Sect. III., these ideas will be developed in Sect. IV. Section V contains concluding remarks and ideas about applications.

Throughout the paper we will use the following terminology: The usual Poincare group in three space + one time dimensions will be called 3-Poincare group. We will deal with Galilei and Poincare groups in two space + one time dimensions, they will be called 2-Galilei group, 2-Poincare group, respectively.

II. Contraction Schemes for the 3-Poincare Group

In order to specify what in our view infinite momentum limit (IML) means on a kinematical level, we start with some elementary questions about Lorentz transformations. Consider a spin-zero, mass-m, positive timelike representation of the 3-Poincare group. Denote the basis functions of the corresponding representation space by $|p_\mu>$, $p_\mu^2 = p_0^2 - p^2 = m^2$, where the four-momenta p_μ are measured in a coordinate system 0. In this coordinate system finite Poincare transformations of the basis functions can be given with the help of the infinitesimal genetarors P_μ, $M_{\mu\nu}$ (mostly the notation $M_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$, $N_i = M^{oi}$ will be used), and an appropriate ten-dimensional parameter space. The generators satisfy the familiar commutation relations

$$[M_i, M_j] = -[N_i, N_j] = i\epsilon_{ijk} M_k,$$
$$[M_i, N_j] = i\epsilon_{ijk} N_k, \qquad \text{(II.1)}$$
$$[M^{\mu\nu}, P^\rho] = i(g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu) .$$

Now let us imagine an observer who is sitting in an 0' coordinate system. In this system we also may specify the functions of the previous representation space, on which now the Poincare transformations are generated by the operators P'_μ, $M'_{\mu\nu}$, satisfying the same algebra as (II. 1). We ask, how the observer of the 0' system will describe the functions $|P_\mu>$, given in the 0 system, if

1. the two coordinate systems, 0 and 0', are moving with a relative velocity tanh ξ, ξ being the corresponding boost angle;
2. some parameters of the ten-dimensional parameter spaces are measured on different scales in the 0 and 0' systems.

In explicit terms, from the point of view of the 0' observer the following operators are appropriate as generators of Poincare transformations on the basis functions $|p_\mu>$:

$$M_1^\xi - N_2^\xi = \lambda(\xi) U(\xi) (M'_1 - N'_2) U^{-1}(\xi) ,$$

$$M_2^\xi + N_1^\xi = \lambda(\xi)\, U(\xi)\,(M_2' + N_1')\, U^{-1}(\xi)\,,$$

$$N_1^\xi - M_2^\xi = U(\xi)\,(N_1' - M_2')\, U^{-1}(\xi)\,,$$

$$N_2^\xi + M_1^\xi = U(\xi)\,(N_2' + M_1')\, U^{-1}(\xi)\,,$$

$$M_3^\xi = U(\xi)\, M_3' U^{-1}(\xi)\,, \qquad \text{(II.2)}$$

$$N_3^\xi = U(\xi)\, N_3' U^{-1}(\xi)\,,$$

$$P_0^\xi - P_3^\xi = \lambda(\xi)\, U(\xi)\,(P_0' - P_3')\, U^{-1}(\xi)\,,$$

$$P_0^\xi + P_3^\xi = U(\xi)\,(P_0' + P_3')\, U^{-1}(\xi)\,,$$

$$P_i^\xi = U(\xi)\, P_i'\, U^{-1}(\xi)\,,\ i = 1, 2\,.$$

We denoted by $U(\xi) = e^{i\xi N_3'}$ z-boost with parameter ξ. These operators are to be used by the 0' observer, if for constructing finite Poincare transformations on the states $|p_\mu>$, the parameter space of the 0 system is applied. As is clear from (II.2), a relative $\lambda(\xi)$ scale factor is assumed between certain parameters of the 0 and o' systems. When ξ goes to infinity, the operators (II.2) either become zero (like the ones $N_1^\xi - M_2^\xi$, $N_2^\xi + M_1^\xi$, $P_0^\xi + P_3^\xi$), or can simply be related with the operators P_μ, $M_{\mu\nu}$ of the 0 system, if for the function $\lambda(\xi)$ the behaviour $\lambda(\xi)\ \underset{\xi \to \infty}{\approx}\ e^{-\xi}$ is assumed. The surviving operators give, what we call the description of the space spanned by the basis $|p_\mu>$ from the point of view of the infinite momentum frame observer. These operators read as:

$$S_1 \equiv \lim_{\xi\to\infty} (M_2^\xi + N_1^\xi) = M_2 + N_1, \quad M_3 = \lim_{\xi\to\infty} M_3^\xi, \qquad \text{(II.3)}$$

$$S_2 = \lim_{\xi \to \infty} (-M_1^{\xi} + N_2^{\xi}) = -M_1 + N_2, \qquad N_3 = \lim_{\xi \to \infty} N_3^{\xi},$$

$$\mu = \lim_{\xi \to \infty} (P_0^{\xi} - P_3^{\xi}) = P_0 - P_3, \qquad P_i = \lim_{\xi \to \infty} P_i^{\xi}, \ i = 1, 2.$$

(Later on it turns out, that in the field theoretical framework such a simple connection between the IMF operators and the 0 frame ones appears only in the free field case, see sect. IV).

It is easy to verify, that the operators S_1, S_2, M_3, P_1, P_2, μ form a two-dimensional Galilei algebra, in which μ plays the role of non-relativistic mass, M_3 generates two-dimensional rotations, P_1, P_2 translations, and S_1, S_2 are the generators of velocity transformations:

$$[S_1, S_2] = 0, \qquad [M_3, S_1] = iS_2, \quad [M_3, S_2] = -iS_1,$$

$$[P_1, P_2] = 0, \qquad [M_3, P_1] = iP_2, \quad [M_3, P_2] = -iP_1,$$

$$[P_i, \mu] = 0, \qquad [M_3, \mu] = 0, \quad [S_i, \mu] = 0,$$

$$[S_i, P_j] = i\delta_{ij}\,\mu. \qquad \text{(II.4)}$$

We may conclude, that for the IMF observer the irreducible 3-Poincare space of mass m splits up with respect to irreducible 2-Galilei representation spaces of all positive masses $\mu = p_0 - p_3$. (In what follows the same notation will be used both for the operator μ and for its eigenvalues).

There is still one more surviving operator N_3, which may be joined with the previous ones to form a closed algebra:

$$[N_3, M_3] = 0, \qquad [N_3, \mu] = i\mu,$$

$$[N_3, P_i] = 0, \qquad [N_3, S_i] = iS_i. \qquad \text{(II.5)}$$

The significance of the presence of N_3 among the IMF operators has so far not been cleared up in the literature. In our opinion it is very intimately related with the contraction procedure described above. Let us notice, that when we scaled out an infinite factor from the operators $M_1^\xi - N_2^\xi$, $M_2^\xi + N_1^\xi$ and $P_0^\xi - P_3^\xi$, we neglected a one-parameter freedom. Namely, the operators $S_i' = aS_i$, $\tilde{\tilde{\mu}} = a\mu$, P_i, M_3 would give exactly the same algebra as (II.4). On the other hand, we see from (II.5) that transformations generated by N_3 correspond just to a similar rescaling of the operators S_1, S_2, μ. This means, that from the point of view of the IMF observer the relative scale of the quantities p_1, p_2 and μ, respectively, is undetermined by construction. If, however, some fundamental) reason, like dynamics, determines this relative scale, then similarity transformation by $\exp(-i\alpha N_3)$ always enables us to redefine kinematics in order to be compatible with this natural scale.

So far we discussed IML more or less along the same lines as was done in refs. 3 and 1. Now we describe another possibility, which might also be relevant in high energy physics. Instead of (II.2) let us define the IMF generators via the following formulas:

$$M_2^\xi - N_1^\xi = a e^{-\xi} U(\xi)(M_2' + N_1')U^{-1}(\xi) + b e^{\xi} U(\xi)(N_1' - M_2')U^{-1}(\xi),$$

$$M_1^\xi + N_2^\xi = -a e^{-\xi} U(\xi)(M_1' - N_2')U^{-1}(\xi) + b e^{\xi} U(\xi)(N_2' + M_1')U^{-1}(\xi),$$

$$P_0^\xi - P_3^\xi = \frac{1}{4ab}\{a e^{-\xi} U(\xi)(P_0' - P_3')U^{-1}(\xi) +$$

$$+ b e^{\xi} U(\xi)(P_0' + P_3')U^{-1}(\xi), \qquad \text{(II.6)}$$

where a and b are arbitrary positive numbers. For the remaining seven operators we keep the definitions in (II.2). We introduce one further operator, which is not, of course, independent of the previous ten while ξ is finite, but h s important role when $\xi \to \infty$:

$$\mu'^{\xi} = a e^{-\xi} U(\xi)(P_0' - P_3') U^{-1}(\xi) - b e^{\xi} U(\xi)(P_0' + P_3') U^{-1}(\xi). \tag{II.7}$$

Now in the limit $\xi \to \infty$ the following operators survive:

$$K_1 \equiv \lim(M_2^{\xi} - N_1^{\xi}), \quad K_2 \equiv \lim(M_1^{\xi} + N_2^{\xi}), \quad M_3 \equiv \lim M_3^{\xi},$$
$$H \equiv \lim(P_0^{\xi} - P_3^{\xi}), \quad P_i \equiv \lim P_i^{\xi}, \quad \mu' \equiv \lim \mu'^{\xi} \tag{II.8}$$

and, as previously, the operator $\lim N_3^{\xi} = N_3$. It is easy to verify, that the operators enumerated in (II.8) form the Lie algebra of a Poincare group (2-Poincare group) operating in a Minkowski space of two space+ one time dimensions:

$$[K_1, K_2] = -i\,4abM_3, \qquad [M_3, K_i] = i\epsilon_{ij} K_j,$$
$$[M_3, H] = 0, \qquad [M_3, P_i] = i\epsilon_{ij} P_j, \tag{II.9}$$
$$[K_i, H] = +iP_i, \qquad [K_i, P_j] = +i\,4ab\,\delta_{ij} H.$$

All the other commutators between K_i, P_i, H, M_3, μ' are zero. It is interesting, that the two parameters a and b, introduced in the contraction procedure, appear in the algebra (II.9) only in the combination $4ab$ and the quantity $\frac{1}{\sqrt{4ab}}$ is direct analogue to the light velocity.

Since μ' commutes with all the operators K_i, M_3, P_i, H, it can be considered as number from the point of view of the algebra (II.9), and, for the IMF observer, it plays analogous role to the rest mass. (Again, the notation μ' will be used both for the operator and its eigenvalues). As a matter of fact, the Casimir operator of this algebra is $4abH^2 - P_L^2$; its eigenvalues may be prescribed as

$$4abH^2 - \vec{P}_\perp^{\,2} = m^2 + \frac{1}{4ab}\mu'^2. \qquad \text{(II.10)}$$

(The notation $\vec{P}_\perp^{\,2} = P_1^2 + P_2^2$ is introduced here).
In this case of contraction, for the IMF observer, the irreducible 3-Poincare representation space splits up to subspaces, which are irreducible with respect to a relativistic type subgroup of the 3-Poincare group, the above 2-Poincare group.

We must again discuss the role of the N_3 operator. In contrast with the previous case, now it does not form a closed algebra with the operators (II.8). It turns out, however, that its presence may again be related with an arbitrariness in the contraction prescription. Namely, the use of the numbers $ca, \frac{1}{c}b$ in (II.6,7) instead of the ones a and b leaves the algebra (II.9) unchanged. But similarity transformations of the type $e^{+i\alpha N_3}\{K, H, \mu'\} e^{-i\alpha N_3}$ induce just the change $\{a, b\} \to \{e^{\alpha} a, e^{-\alpha} b\}$. Therefore we argue, that from the point of view of the IMF observer the only use of N_3 is to restrict the values of a and b.

To conclude this section we show, that the non-relativistic contraction of the algebra (II.9) is just the one (II.4). Namely, when $b \to 0$, that is, when the "light velocity $1/\sqrt{4ab}$ goes to infinity, (II.9) reduces to (II.4).

As is obvious from (II.10), in such a limit the operator H becomes singular. For an irreducible representation this singularity can be made explicit:

$$H = H' - \frac{1}{4ab}\mu', \qquad \text{(II.11)}$$

where H is already non-singular. The algebra (II.9) becomes identical with (II.4) at $b=0$, if the following correspondence is made (at $b=0$!): $K_i \to S_i$, $\mu' \to \mu$. From (II.10) we obtain the following expression for H':

$$H' = \frac{\vec{P}_\perp^{\,2}}{2\mu} + \frac{m^2}{2\mu}, \qquad \text{(II.12)}$$

which is a non-relativistic Hamiltonian, often cited as a characteristic quantity in the infinite momentum frame.

In what follows we consider scalar field theory with ϕ^3 interaction. Standard field-theoretical machinery provides us with a realization of the 3-Poincare generators, and we can investigate in explicit terms what was outlined in the above section, with special emphasis on the interaction terms.

III. The Poincare Generators in Scalar Field Theory

In order to fix notations we cite some standard text-book material about scalar field theory /7/. In the 0' reference frame we denote the field operator by $\phi'(T,\vec{X})$, which is quantized through the standard equal-time commutators:

$$[\phi'(T,\vec{X}),\ \phi'(T,\vec{X}')] = [\frac{\partial\phi'(T,X)}{\partial T},\ \frac{\partial\phi'(T,X')}{\partial T}] = 0\ ,$$

$$[\frac{\partial\phi'(T,X)}{\partial T},\ \phi'(T,\vec{X}')] = -i\,\delta^3(X-X')\ : \qquad \text{(III.1)}$$

Our Lagrangian density will be as follows:

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi'}{\partial X^\mu}\frac{\partial\phi'}{\partial X_\mu} - m^2\phi'^2\right) + \frac{1}{2}g\,\phi'^3, \qquad \text{(III.2)}$$

This enables us to calculate the energy-momentum stress tensor:

$$\mathcal{T}^{\mu\nu} = -g^{\mu\nu}\,\mathcal{L} + \frac{\partial\phi'}{\partial X_\mu}\frac{\partial\phi'}{\partial X_\nu}\ . \qquad \text{(III.3)}$$

The subsequent investigations will be performed in the Dirac-picture, and for the field operators plane-wave expansion will be used:

$$\phi'(T,\vec{X}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3Q}{2Q_0}[A'(Q)\, e^{-iQ_\mu X^\mu} + A'^+(Q)\, e^{iQ_\mu X^\mu}], \qquad \text{(III.4)}$$

where $Q \equiv Q_\mu = (Q_0, \vec{Q})$, $Q_0 = [\vec{Q}^2 + m^2]^{1/2}$. It follows from (III.1) and (III.4), that the creation and annihilation operators commute as

$$[A'(Q), A'^+(Q')] = 2Q_0\delta^3(\vec{Q} - \vec{Q}'), \qquad \text{(III.5)}$$

Notice, that these operators are Lorentz scalars, that is, the creation and annihilation operators in the unprimed 0 reference frame,

$$A(k) = U(\xi)\, A'(Q)\, U^{-1}(\xi),$$
$$A^+(k) = U(\xi)\, A'^+(Q)\, U^{-1}(\xi), \qquad \text{(III.6)}$$

commute according to the same rules as in the 0' frame:

$$[A(k), A^+(k')] = 2k_0\delta^3(\vec{k} - \vec{k}'). \qquad \text{(III.5)}$$

The four-momenta k_μ and Q_μ are related as follows:

$$k_0 = Q_0 \operatorname{ch}\xi + Q_3 \operatorname{sh}\xi,$$
$$k_3 = Q_0 \operatorname{sh}\xi + Q_3 \operatorname{ch}\xi, \qquad \text{(III.7)}$$
$$\vec{k}_\perp = Q_\perp.$$

Now the Poincare generators, what we are going to study, in the 0' frame look like:

$$\vec{P}' = \frac{1}{2}\int \frac{d^3\vec{Q}}{2Q_0}\vec{Q}[A'(Q)A'^+(Q) + A'^+(Q)A'(Q)], \qquad \text{(III.8)}$$

$$P'_0 = \frac{1}{2}\int \frac{d^3Q}{2Q_0} Q\,[A'(Q)\,A'^{+}(Q) + A'^{+}(Q)\,A'(Q)] -$$

$$- \frac{1}{2} g \frac{1}{(2\pi)^{3/2}} \int \prod_{i=1}^{3} \left(\frac{d^3\vec{Q}^{(i)}}{2Q_0^{(i)}}\right) [A'(Q^{(1)})\,A'(Q^{(2)})\,A'(Q^{(3)}) \times$$

$$e^{-iT(Q_0^{(1)}+Q_0^{(2)}+Q_0^{(3)})}\ \delta^3(\vec{Q}^{(1)} + \vec{Q}^{(2)} + \vec{Q}^{(3)}) +$$

$$+ 3A'^{+}(Q^{(3)})\,A'(Q^{(1)})A'(Q^{(2)})\, e^{-iT(Q_0^{(1)}+Q_0^{(2)}-Q_0^{(3)})} \times \tag{III.9}$$

$$\times \delta^3(\vec{Q}^{(1)}+\vec{Q}^{(2)}-\vec{Q}^{(3)}) + \text{h.c.}\,];$$

$$\vec{M}' = \frac{i}{2}\int \frac{d^3\vec{Q}}{2Q_0}[A'(Q)\,(\vec{Q}\times\frac{\partial}{\partial\vec{Q}})\,A'^{+}(Q) - A'^{+}(Q)\,(\vec{Q}\times\frac{\partial}{\partial\vec{Q}})\,A'(Q)];$$

$$\tag{III.10}$$

$$-\vec{N}' = \frac{i}{2}\int \frac{d^3\vec{Q}}{2Q}[A'(Q)\,(Q_0\frac{\partial}{\partial\vec{Q}})\,A'^{+}(Q) - A'^{+}(Q)\,(Q_0\frac{\partial}{\partial\vec{Q}})\,A'(Q)] -$$

$$-\frac{1}{2} g \frac{1}{(2\pi)^{3/2}} \int \prod_{i=1}^{3} \left(\frac{d^3\vec{Q}^{(i)}}{2Q_0^{(i)}}\right) \times$$

$$\times [A'(Q^{(1)})\,A'(Q^{(2)})\,A'(Q^{(3)})\ e^{-i(Q_0^{(1)}+Q_0^{(2)}+Q_0^{(3)})\,T} \times$$

$$\times \frac{1}{(2\pi)^3}\int \vec{X}\, e^{-i(\vec{Q}^{(1)}+\vec{Q}^{(2)}+\vec{Q}^{(3)})\vec{X}}\, d^3\vec{X} + \tag{III.11}$$

$$+ 3A'^{+}(Q^{(3)})\,A'(Q^{(1)})\,A'(Q^{(2)})\, e^{-i(Q_0+Q_0-Q_0)\,T} \times$$

$$\times \frac{1}{(2\pi)^3}\int \vec{X}\, e^{-i(\vec{Q}^{(1)}+\vec{Q}^{(2)}-\vec{Q}^{(3)})\vec{X}}\, d^3\vec{X} + \text{h.c.}\,].$$

In the expressions (III.9) and (III.11) the interaction terms are written in normal ordered form. We mention the somewhat strange fact, that this form of the Lorentz generators $\vec{M}$, $\vec{N}$, where they are expressed via creation and annihilation operators, does not seem to be very often in the standard literature [8], although it is easy to obtain from (II.4) and the expression of P'_μ and $M'_{\mu\nu}$ by means of the tensor $\mathcal{T}^{\mu\nu}$ [7]:

$$P'_\mu = \int d^3\vec{X}\, \mathcal{T}_{0\mu}(T,\vec{X}) \,,$$
$$M'_{\mu\nu} = \int d\vec{X} [X_\nu \mathcal{T}_{0\mu}(T,\vec{X}) - X_\mu \mathcal{T}_{0\nu}(T,\vec{X})] : \qquad \text{(III.12)}$$

The only important condition in the derivation of (III.8-11) is that these operators act on the normalizable states Ψ of a Hilbert space:

$$\Psi = [c_0 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int \prod_{i=1}^{n} (\frac{d^3\vec{Q}^{(i)}}{2Q_0^{(i)}} A'^{+}(Q^{(i)}))\, c_n(Q^{(1)},\dots,Q^{(n)})]|0>,$$
$$1 = (\Psi,\Psi) = |c_0|^2 + \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} (\frac{d^3\vec{Q}^{(i)}}{2Q_0^{(i)}})\, |c_n(Q^{(1)},\dots,Q^{(n)})|^2 . \qquad \text{(III.13)}$$

Finally, we shall need the z-boosted form of the operators (III.8-11). Their calculation is straightforward by means of (III.6) and (III.7):

$$U(\xi)\vec{P}'_\perp U^{-1}(\xi) = \frac{1}{2}\int \frac{d^3\vec{k}}{2k_0} \vec{k}_\perp [A(k)A^+(k) + A^+(k)A(k)] ;$$
$$U(\xi) P'_3 U^{-1}(\xi) = \frac{1}{4}\int \frac{d^3\vec{k}}{2k_0} [-e^{\xi}(k_0-k_3) + e^{-\xi}(k_0+k_3)] \times$$
$$\times [A(k)A^+(k) + A^+(k)A(k)] ; \qquad \text{(III.14)}$$

$$U(\xi) P'_0 U^{-1}(\xi) = \frac{1}{4}\int \frac{d^3\vec{k}}{2k_0} [e^{\xi}(k_0-k_3) + e^{-\xi}(k_0+k_3)] \times$$
$$\times [A(k)A^+(k) + A^+(k)A(k)] - \frac{1}{2} g \frac{1}{(2\pi)^{3/2}} \int \prod_{i=1}^{3} (\frac{d^3\vec{k}^{(i)}}{2k_0^{(i)}}) \times$$
$$\times \{ A(k^{(1)}) A(k^{(2)}) A(k^{(3)})\, e^{-i\frac{T}{ch\xi}(k_0^{(1)}+k_0^{(2)}+k_0^{(3)})}\, \delta^3(\vec{K}^{(1)}+\vec{K}^{(2)}+\vec{K}^{(3)}) + \qquad \text{(III.15)}$$
$$+ 3A^+(k^{(3)}) A(k^{(1)}) A(k^{(2)})\, e^{-i\frac{T}{ch\xi}(k_0^{(1)}+k_0^{(2)}-k_0^{(3)})}\, \delta^3(\vec{K}^{(1)}+\vec{K}^{(2)}-\vec{K}^{(3)}) + h.c. \} ;$$

$$U(\xi)\vec{M}U^{-1}(\xi) = \frac{i}{2}\int\frac{d^3\vec{k}}{2k_0}[A(k)(\vec{K}\times\frac{\partial}{\partial\vec{K}})A^+(k) -$$

$$-A^+(k)(\vec{K}\times\frac{\partial}{\partial\vec{K}})A(k)]; \qquad (III.16)$$

$$-U(\xi)N_iU^{-1}(\xi) = ch\xi\frac{i}{2}\int\frac{d^3\vec{k}}{2k_0}(k_0-k_3\frac{sh\xi}{ch\xi})[A(k)\frac{\partial}{\partial k_i}A^+(k) -$$

$$-A^+(k)\frac{\partial}{\partial k_i}A(k)] - \frac{1}{ch\xi}\frac{1}{2}g\frac{1}{(2\pi)^{3/2}}\int\prod_{i=1}^{3}(\frac{d^3\vec{k}^{(i)}}{2k_0^{(i)}})\times$$

$$\times\{A(k^{(1)})A(k^{(2)})A(k^{(3)})e^{-i\frac{T}{ch\xi}(k_0^{(1)}+k_0^{(2)}+k_0^{(3)})}\times \qquad (III.17)$$

$$\times\frac{1}{(2\pi)^3}\int x^i e^{-\frac{i}{ch\xi}(\vec{K}^{(1)}+\vec{K}^{(2)}+\vec{K}^{(3)})\vec{x}} +$$

$$+3A^+(k^{(3)})A(k^{(1)})A(k^{(2)})e^{-i\frac{T}{ch\xi}(k_0^{(1)}+k_0^{(2)}-k_0^{(3)})}\times$$

$$\times\frac{1}{(2\pi)^3}\int x^i e^{-\frac{i}{ch\xi}(\vec{K}^{(1)}+\vec{K}^{(2)}-\vec{K}^{(3)})\vec{x}} + h.c.\}, \quad i=1,2,$$

$$-U(\xi)N_3U^{-1}(\xi) = \frac{i}{2}\int\frac{d^3\vec{k}}{2k_0}k_0[A(k)\frac{\partial}{\partial k_3}A^+(k) - A^+(k)\frac{\partial}{\partial k_3}A(k)] -$$

$$-\frac{1}{2}g\frac{1}{(2\pi)^{3/2}}\int\prod_{i=1}^{3}(\frac{d^3\vec{k}^{(i)}}{2k_0^{(i)}})\{A(k^{(1)})A(k^{(2)})A(k^{(3)})\times$$

$$\times e^{-i\frac{T}{ch\xi}(k_0^{(1)}+k_0^{(2)}+k_0^{(3)})}\times[T\frac{sh\xi}{ch\xi}\delta^3(\vec{K}^{(1)}+\vec{K}^{(2)}+\vec{K}^{(3)}) +$$

$$+\frac{1}{ch^2\xi}\frac{1}{(2\pi)^3}\int z e^{-\frac{i}{ch\xi}(\vec{K}^{(1)}+\vec{K}^{(2)}+\vec{K}^{(3)})\vec{x}}d^3\vec{x} + h.c.]\} - \qquad (III.18)$$

$$-\frac{1}{2}g\frac{1}{(2\pi)^{3/2}}\int\prod_{i=1}^{3}(\frac{d^3\vec{k}^{(i)}}{2k_0^{(i)}})\{3A^+(k^{(3)})A(k^{(1)})A(k^{(2)})\times$$

$$\times e^{-i\frac{T}{ch\xi}(k_0^{(1)}+k_0^{(2)}-k_0^{(3)})}\times[T\frac{sh\xi}{ch\xi}\delta^3(\vec{K}^{(1)}+\vec{K}^{(2)}-\vec{K}^{(3)}) +$$

$$+\frac{1}{ch^2\xi}\frac{1}{(2\pi)^3}\int z e^{-\frac{i}{ch\xi}(\vec{K}^{(1)}+\vec{K}^{(2)}-\vec{K}^{(3)})\vec{x}} d^3\vec{x} + h.c.]\}.$$

In these formulas the notation $\vec{K}$ stands for the three-component quantity $(k_1, k_2, -k_0 ch\xi + k_3 ch\xi)$, $A(k)$ and $A^+(k)$ are annihilation and creation operators of quanta with four-momentum $k_\mu = (k_0, k_1, k_2, k_3)$ in the reference frame 0. Notice the following properties of the operators (III.14-18): First, that owing to the presence of interaction they are not simple combinations of the 3-Poincare generators in the reference frame 0. (These latter operators would be of the same form as (III.8-11), but substituting $A'(Q)$ to $A(k)$ and the 0'-frame time T to the 0-frame time t). This is so only for the free parts of the generators. Second, that the interaction terms are of lower order with respect to powers of e^ξ, than those of the free-field theory. These properties are consequences of the fact, that in the definitions (III.12) T is fixed, and it is easy to see, that the substitution of T = fixed to any other fixed spacelike surface would not change these properties.

IV. The Infinite Momentum Limit

Similarly to our treatment of infinite momentum limit in Sect. II, we assume, that also the parameter spaces for the 3-Poincare transformations are different in the 0 and 0' frames. We perform the corresponding transformations on the boosted operators (III.14-18) and take the $\xi \to \infty$ limit.

a. The 2-Galilei Case

Following (II.2) and (II.3) in this case one obtains the operators:

$$S_i = -\frac{i}{2}\int\frac{d^3\vec{k}}{2k}(k_0-k_3)[A(k)\frac{\partial}{\partial k_i}A^+(k) - A^+(k)\frac{\partial}{\partial k_i}A(k)],$$

$$\vec{P}_\perp = \frac{1}{2}\int \frac{d^3\vec{k}}{2k_0}\vec{k}_\perp [A(k)A^+(k) + A^+(k)A(k)], \qquad \text{(IV.1)}$$

$$M_3 = \frac{i}{2}\int \frac{d^3\vec{k}}{2k_0}[A(k)(k_1\frac{\partial}{\partial k_2} - k_2\frac{\partial}{\partial k_1})A^+(k) -$$

$$- A^+(k)(k_1\frac{\partial}{\partial k_2} - k_2\frac{\partial}{\partial k_1})A(k)], \qquad \text{(IV.2)}$$

$$\mu = \frac{1}{2}\int \frac{d^3\vec{k}}{2k_0}(k_0 - k_3)[A(k)A^+(k) + A^+(k)A(k)],$$

$$\hat{N}_3 = -\frac{i}{2}\int \frac{d\vec{k}}{2k_0}(k_0 - k_3)[A(k)\frac{\partial}{\partial(k_0 - k_3)}A^+(k) -$$

$$-A^+(k)\frac{\partial}{\partial(k_0 - k_3)}A(k)]. \qquad \text{(IV.3)}$$

The first remarkable thing about these operators is that none of them contains the interaction. This is characteristic to non-relativistic theories, where none of the symmetry transformations alters time. So far we know neither that variable; which plays the role of "time" from the point of view of the IMF observer, nor the "Hamiltonian" governing the "time development" of the corresponding "non-relativistic" states. To find out the "Hamiltonian" we study the IML of a dynamical equation, most conveniently that of the Lippmann-Schwinger equation/9/.

From the viewpoint of the O' frame observer the Lippmann-Schwinger equation for systems given in the O frame reads as:

$$|\Psi> = |p_\mu> + \frac{1}{E^\xi - P^\xi_{o free} + i\epsilon} P^\xi_{oint} \quad |\Psi>, \qquad \text{(IV.4)}$$

where $|p_\mu>$ is an eigenstate of the free operator $P^\xi_{o free}$:

(IV.5)

$$P^\xi_{o free}\ |p_\mu> = E^\xi |p_\mu> \equiv [\frac{1}{2}(p_0 - p_3) + \frac{1}{2}e^{-\xi}(p_0 + p_3)]\ |p_\mu>.$$

The state $|p_\mu>$ develops into $|\Psi>$, which is an eigenstate of $P_0^\xi = P_{0free}^\xi + P_{0int}^\xi$ with the same eigenvalue E^ξ:

$$P_0^\xi |\Psi> = E^\xi |\Psi> . \qquad \text{(IV.6)}$$

The operators P_0^ξ, P_{0free}^ξ, P_{0int}^ξ are to be calculated from eqs. (II.2), (III.14,15), taking $T = 0$ in the interaction terms. We have already seen, that

$$\lim_{\xi\to\infty} P_0^\xi = \lim_{\xi\to\infty} P_{0free}^\xi = \frac{1}{2}\mu .$$

Therefore, in the limit $\xi\to\infty$ eqs. (IV.4,5) have non-trivial solution, $|p_\mu> \neq |\Psi>$, only, if

$$\lim_{\xi\to\infty} [\mu, V^\xi] = 0 , \qquad \text{(IV.7)}$$

$$\lim_{\xi\to\infty} [H_0 , V^\xi] \neq 0 . \qquad \text{(IV.8)}$$

We introduced the following notations:

$$\frac{1}{2} V^\xi = e^\xi P_{0int}^\xi , \quad \frac{1}{2} H_0 = e^\xi (P_{0free}^\xi - \frac{1}{2}\mu) . \qquad \text{(IV.9)}$$

All these operators are explicitly known, and one may check, that the conditions (IV.7,8) fulfill. This enables us to write down the "Lippmann-Schwinger equation" in the infinite momentum frame:

$$|\Psi> = |p_\mu> + \frac{1}{h(p) - H_0 + i\epsilon} V |\Psi> , \qquad \text{(IV.10)}$$

where

$$H_0 = \frac{1}{2} \int \frac{d^3 \vec{k}}{2k_0} h(k) [A(k) A^+(k) + A^+(k) A(k)] , \qquad \text{(IV.11)}$$

$$V = -3g\frac{1}{(2\pi)^{3/2}}\int \prod_{i=1}^{3}\left(\frac{d^3\vec{k}^{(i)}}{2k_0^{(i)}}\right)\times$$

$$\times\{A^+(k^{(3)})A(k^{(1)})A(k^{(2)})\,\delta(\vec{k}_\perp^{(1)}+\vec{k}_\perp^{(2)}-\vec{k}_\perp^{(3)})\,\delta(\mu^{(1)}+\mu^{(2)}-\mu^{(3)})+$$

$$+h.c.\}. \qquad \text{(IV.12)}$$

The notations $\mu^{(i)}=k_0^{(i)}-k_3^{(i)}$, $h=k_0+k_3$ are introduced here, and terms containing $\delta(\mu^{(i)}+\mu^{(i)}+\mu^{(i)})$ are left out, being identically zero. Notice, that the S -matrix element $\langle p'|\Psi\rangle$, when iterated by means of (IV.10), takes just the shape of Weinberg's infinite momemtum limit of the non-covariant perturbation series

Going on with our programme, we may calculate the commutators of H_0 and V with the operators (IV.1.2). Since all the commutators give zero, H_0 can be considered as free Hamiltonian, while V is the interaction Hamiltonian. Finally, calculating the commutator of H_0 with $\phi(t,\vec{x})$ which is the field operator in the 0 frame (and has analogous expansion to (III.4)), one obtains:

$$[H_0,\phi(t,\vec{x})] = -i\,2\frac{\partial}{\partial(t+z)}\phi(t,\vec{x}) \,. \qquad \text{(IV.13)}$$

That is, from the point of view of the IMF observer the "time" coordinate is $\frac{1}{2}(t+z)$

b. The 2-Poincare Case

In this case, when calculating the infinite momentum limit, $\xi\to\infty$, we must use eqs. (II.6-8). The operators M_3, $\hat{N}_3$, $\vec{P}_\perp$ are the same as in (IV.1, 3). Moreover, we obtain:

$$\vec{K} = -\frac{i}{2}\int\frac{d^3\vec{k}}{2k_0}\{A(k)\,(h'\frac{\partial}{\partial\vec{k}_\perp}+2b\vec{k}_\perp\frac{\partial}{\partial k_-})\,A^+(k) -$$

$$-A^+(k)\,(h'\frac{\partial}{\partial\vec{k}_\perp}+2b\vec{k}_\perp\frac{\partial}{\partial k_-})\,A(k)\}+$$

$$+ 3bg \frac{1}{(2\pi)^{3/2}} \int \prod_{i=1}^{3} \left(\frac{d^3 \vec{k}^{(i)}}{2k_0^{(i)}}\right) \delta(\vec{k}_-^{(1)} + \vec{k}_-^{(2)} - \vec{k}_-^{(3)}) \times$$

$$\times \{ A^+ (k^{(3)}) A(k^{(1)}) A(k^{(2)}) \cdot \frac{1}{(2\pi)^2} \int \vec{x}_\perp e^{-i(\vec{k}_\perp^{(1)} + \vec{k}_\perp^{(2)} - \vec{k}_\perp^{(3)}) \vec{x}} d\vec{x} +$$

$$+ h.c. \} ; \qquad \text{(IV.14)}$$

$$H = \frac{1}{4ab} \frac{1}{2} \int \frac{d^3\vec{k}}{2k_0} h' [A(k) A^+(k) + A^+(k) A(k)] -$$

$$- \frac{1}{4a} g \frac{1}{(2\pi)^{3/2}} \int \prod^{3} \left(\frac{d^3 \vec{k}}{2k_0^{(i)}}\right) \delta(k_-^{(1)} + k_-^{(2)} - k_-^{(3)}) \times$$

$$\times \{ 3A^+(k^{(3)}) A(k^{(1)}) A(k^{(2)}) \, \delta^2 (\vec{k}_\perp^{(1)} + \vec{k}_\perp^{(2)} - \vec{k}_\perp^{(3)}) + h.c. \} ;$$

$$\mu' = \frac{1}{2} \int \frac{d^3 \vec{k}}{2k_0} \mu'_0 [A(k) A^+(k) + A^+(k) A(k)] +$$

$$+ bg \frac{1}{(2\pi)^{4/2}} \int_{i=1}^{3} \left(\frac{d^3\vec{k}^{(i)}}{2k_0^{(i)}}\right) \delta(k_-^{(1)} + k_-^{(2)} - k_-^{(3)}) \times \qquad \text{(IV.16)}$$

$$\times \{ 3A^+(k^{(3)}) A(k^{(1)}) A(k^{(2)}) \delta^2 (\vec{k}_\perp + \vec{k}_\perp - \vec{k}_\perp) + h.c. \} .$$

In the above formulas the following notations are used: $\vec{K} = (+K_1, K_2)$, $h' = a(k_0 - k_3) + b(k_0 + k_3)$, $\mu'_0 = a(k_0 - k_3) - b(k_0 + k_3)$, **and** $2 \cdot k_- = (\mu'_0 + h')$. **In contrast with the 2-Galilean case, now *some of the* IMF generators contain the interaction. Lengthy, but straightforward calculation shows that the operators** K_1, K_2, M_3, P_1, P_2, H, μ', **including the interaction terms as well, fulfill the algebra (II.9).**

It is natural to interpret these results in the following manner. The IMF observer describes the 0-frame world by a *field theory*, the symmetry group of which is the 2-Poincare group introduced in Sect. II. This 2-Poincare group transforms the transverse space coordinates, $x = (x, y)$, **and the "time coordinate"**

$$\mathcal{T} = [(a + b) t + (a - b) z] . \qquad \text{(IV.17)}$$

It is easy to verify, that the free part H_0', of the operator H generates translations just in this variable $\mathcal{T}$:

$$[H_0', \phi(t,\vec{x})] = -i \frac{\partial}{\partial \tau} \phi(t,\vec{x}) . \qquad \text{(IV.18)}$$

In possession of eqs. (IV.17,18) we may construct the "time-dependent interaction Hamiltonian". First we split the operator H to two parts, $H = H_0' + V'$, H_0' and V' being the free part and interaction term, respectively. Then. we define

$$V'(\tau) = e^{i\tau H_0'} \, V' \, e^{-i\tau H_0'} , \qquad \text{(IV.19)}$$

and easily obtain the explicit form of $V'(\tau)$:

$$-V'(\tau) = \frac{g}{4a} \frac{1}{(2\pi)^{3/2}} \int \prod_{i=1}^{3} \left(\frac{d^3\vec{k}^{(i)}}{2k_0^{(i)}} \right) \delta(k_-^{(1)} + k_-^{(2)} - k_-^{(3)}) \times$$
$$\times \{ 3A^+(k^{(3)}) \, A(k^{(1)}) \, A(k^{(2)}) \, e^{-i\frac{\tau}{4ab}(h'^{(1)} + h'^{(2)} - h'^{(3)})} \delta^2 (\vec{k}_\perp^{(1)} + \vec{k}_\perp^{(2)} - \vec{k}_\perp^{(3)}) + h.c. \} .$$
$$\text{(IV 20)}$$

For obtaining dymanical calculational schemes we have now two possibilities:

1. Either proceeding similarly to the discussion in the 2-Galilean case, or referring to formal resemblance with usual 3-dimensional field theory, we may describ. an analogue of the Lippmann-Schwinger equation as follows:

$$| \Psi > = | p_\mu > + \frac{1}{\frac{1}{4ab} h' - H_0' + i\epsilon} V' | \Psi > , \qquad \text{(IV.21)}$$

$$\text{(IV.22)}$$
$$H_0' \, | p_\mu > = \frac{1}{4ab} h' \, | p_\mu > = \frac{1}{4ab} [\, a(p_0 - p_3) + b(p_0 + p_3) \,) \, | p_\mu > .$$

It is easy to see that eq. (IV.21) is exactly the same as the one (IV.10), the Lippmann-Schwinger equation in the 2-Galilean case. Indeed, a simple comparison of the explicit form of the operators V, V' and H_0, H_0' shows, that $V = +4aV'$, $4abH_0' = a\mu + bH_0$. These relations, combined with the commutator (IV.7), say that eq. (IV.21) does not offer more, than a reinterpretation of Weinberg's infinite momentum limit of the non-covariant perturbation series in terms of our 2-Poincare group instead of the 2-Galilei group. Namely, it appears to be the "non-covariant perturbation series" in a field theory, the notion of covariance being related with the 2-Poincare group. It will be suggested in the next section, that this reinterpretation may be of interest.

2. Based upon the "time-dependent interaction Hamiltonian" $V'(\tau)$ one may to try to work out "covariant perturbation theory". The starting point of such a theory should obviously be the " S -matrix":

$$"S" = \sum_{n=0}^{\infty} \frac{-i\sqrt{4ab}}{n!} \int_{-\infty}^{\infty} d\tau_1 \ldots \int_{-\infty}^{\infty} d\tau_n P_\tau \{V'(\tau_1) \ldots\ldots V'(\tau_n)\}, \quad \text{(IV.23)}$$

where P_τ denotes the operator of τ -ordering. The invariance properties of this operator from the point of view of the 2- and 3-Poincare groups, the relation between "S" and S -matrix of the initial scalar field theory, the elaboration of a "2-Poincare covariant" perturbation theory are problems, which we are going to discuss in a forthcoming paper.

V. Summary and Outlooks

In the previous sections we presented a systematic description of infinite momentum limit of scalar field theory with mass m and interaction ϕ^3. We demonstrated how this 3-dimensional field theory reduces to a 2-dimensional scalar field theory. However, in the new theory of lower dimensionality a continuous mass spectrum appears. It is shown, that the 2-dimensional symmetry group may be either a Galilean, or a Poincare subgroup of the original Poincare symmetry group of the 3-dimensional theory.

In that case, when IML results in a Galilean theory, we rederived Weinberg's result concerning the limit of the non-covariant perturbation series /4/. Among the Poincare generators, which survive after the limiting procedure this case, one discovers just that six operators, which do not contain the interaction and appear also in the 3-dimensional theory, if it is quantized on a light-front /10/. However, in our case the seventh surviving operator, $\hat{N}_3$, the remainder of the generator of z-boosts, does not contain the interaction, either. Therefore it is obviously different from the N_3 operator of light-front quantized field theory /10/. This fact does not seem to be noticed in investigations of the connection between IMF field theory and light-front quantized field theory /1/.

In that case, when IML results in a 2-dimensional relativistic theory, we proved, that the perturbation theory of the previous, Galilean case plays just the role of "non-covariant perturbation theory" from the point of view of this relativistic field theory. We suggested the possibility for elaborating a "covariant perturbation theory", which may offer a new calculational scheme for high energy processes.

Models based upon the relativistic 2-dimensional picture may be more advantageous than the Galilean framework for the description of processes, in which large transverse momenta appear, that is, when the 2-dimensional objects move "relativistically". It is tempting to believe, that, when also transverse momentum gets very large, the 2-dimensional relativistic dynamics somehow imitates the 3-dimensional one, and, for example, a new scaling region appears. (Such phenomenon has already been reported /11/.). For the description of this region a second "infinite momentum limit" may be useful. This "infinite momentum limit" should, of course, be understood from the point of view of the 2-Poincare group, and should, in practice, be performed with the help of eq. (IV.21), which, in the case of the relativistic 2-dimensional theory, plays the role of the "non-covariant perturbation series". Obviously, this argument can once more be repeated. Thus one obtains a sequence of three-, two-, one-dimensional theories embedded into each other, and this may perhaps be brought into correspondence with processes, in which one, two or three components of momenta are large. This picture is reminiscent to the ideas described recently by Fubini and Rebbi on different grounds /12/.

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