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CURRENTS AND FORM FACTORS IN LOBACHEVSKY MOMENTUM SPACE AND THE NUCLEON MEAN-SQUARE RADIUS

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1. In describing the interactions of nucleon with other particles, invariant form factors are introduced into the expression for the nucleon current. These form factors phenomenologically allow for the nucleon spatial distribution. However, no satisfactory description is available so far for this structure in coordinate space.

The reason for such a situation is well known: to pass over to the three-dimensional space requires a special reference frame, namely the Breit system, to be used. In this system the time component of 4-vector of momentum transfer, $q = p - k$ becomes zero (as $\vec{p} = -\vec{k}$, $p_0 = k_0$ and $q = (0, 2\vec{p})$). This makes it possible to go over to the coordinate space through the use of the three-dimensional Fourier transform for the form factor $F(-\vec{q}^{\,2})$

$$f(r) = \frac{1}{(2\pi)^3} \int F(-\vec{q}^{\,2}) e^{-i\vec{q}\vec{r}} \, d\vec{q}. \qquad (1)$$

The coordinate, therefore, is introduced as a quantity conjugate to the momentum transfer in the Breit frame. However, as for every value of $\vec{q}$ there is its own reference frame, the function $f(r)$, which describes the nucleon charge density, is not given in a definite physical reference frame.

To overcome this difficulty, the way should be found now to express the Lorentz invariants, in particular, the four momentum transfer squared, $t = q^2$, in an equivalent three-dimensional form without using a special coordinate system as that of the Breit frame type. For the spinless case, such three-dimensional form has been obtained in /1,2/ by making use of the Lobachevsky space.

The group of motions of the Lobachevsky space is the Lorentz group [3]. Therefore in [1,2] for transition to the configurational representation an expansion in unitary representations of the Lorentz group was employed instead of the Fourier transformation.

For the spin case, we are interested in, the parametrization has been found [1] for the particle current matrix elements in terms of the Lobachevsky space. The matrix elements of a scattering amplitude, constructed by means of these currents, have a form of the relativistic generalization of quantum-mechanical potentials obtained via replacing the Euclidean geometry by the Lobachevsky geometry. Therefore it is natural that the invariant form factors in the expressions for currents are to be parametrized also in terms of elements of the Lobachevsky space, and for their space description expansions over the Lorentz group representations are to be used.

Note that the Lobachevsky space was employed earlier in [5]. for describing form factors, and in [6] the expansions over the Lorentz group representations were used for studying the form factors by means of the relativistic configurational representation introduced in ref. [1]. However, in [6] the advantage of relativistic configurational representation was not exploited. In the present paper we will connect such an important nucleon characteristic as its mean-square radius with the distribution of the form factor in a new relativistic coordinate space, and demonstrate that this provides the invariant space description of the nucleon structure.

2. The way of writting the currents in terms of elements of the Lobachevsky space is most easily demonstrated by the example of a nucleon interacting with a scalar, or pseudoscalar, meson. This process is shown in Fig. 1 where solid line stands for a spinor particle, dotted line for a scalar or pseudoscalar one:

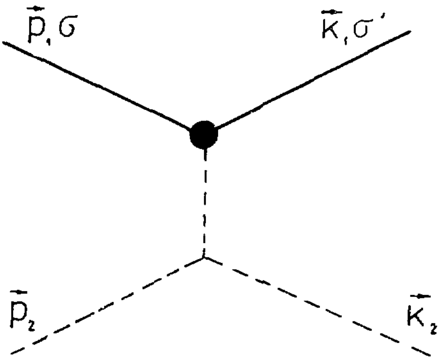

Fig. 1

Matrix elements of the hadron currents of the scattering amplitude are of the form:

$$<\vec{p},\sigma | j^{s}(0) | \vec{k},\sigma'> = j^{s}_{\sigma\sigma'}(\vec{p},\vec{k}) = \bar{u}^{\sigma}(\vec{p})\, u^{\sigma'}(\vec{k})\, F_s(q^2) \qquad (2)$$

$$<\vec{p},\sigma | j^{ps}(0) | \vec{k},\sigma'> = j^{ps}_{\sigma\sigma'}(\vec{p},\vec{k}) = \bar{u}^{\sigma}(\vec{p})\gamma^5 u^{\sigma'}(\vec{k})\, F_{ps}(q^2), \qquad (3)$$

where $F_s(q^2)$ and $F_{ps}(q^2)$ are the scalar and pseudoscalar form factors dependent on the squared 4-momentum transfer, $q=p-k$, and $u^{\sigma}(\vec{p})$ - the Dirac bispinors. The momenta p and k in (2), (3) are on the mass shell so their components are connected in the following way:

$$p_0^2 - \vec{p}^{\,2} = m^2 . \qquad (4)$$

Equation (4) defines in the Minkowski space, the three-dimensional surface of a hyperboloid on the upper sheet of which just the Lobachevsky space is realized.

Now, by using the transformation S_k corresponding to the pure Lorentz transformation (i.e., "boost") Λ_k ;

$\Lambda_k (m,0) = (k^0, \vec{k})$ in (2) and (3) we pass over to the bispinors defined in the rest frames of particles [4]:

$$j_{\sigma\sigma'}(\vec{p},\vec{k}) = \bar{u}^{\sigma}(0) S_p^{-1} S_k u^{\sigma'}(0) F_s(q^2),$$

$$j^{ps}_{\sigma\sigma'}(\vec{p},\vec{k}) = \bar{u}^{\sigma}(0) \gamma^5 S_p^{-1} S_k u^{\sigma'}(0) \cdot F_{ps}(q^2).$$

The explicit form of the transformation S_k and bispinors is given in Appendix of ref. [4].

The pure Lorentz transformations do not compose a group. Their product is not the pure Lorentz transformation but contains also the three-dimensional Wigner rotation described by the matrix $V(\Lambda p, k) \in SU(2)$

$$S_p^{-1} S_k = S_{\Lambda_p^{-1} k} \cdot D^{1/2} \{ V^{-1} (\Lambda_p, k) \}. \tag{5}$$

Let us introduce the following notation for the 4-vector $\Lambda_p^{-1} k$:

$$(k(-)p)^0 \cdot (\Lambda_p^{-1} k)^0 = \frac{k_0 p_0 - \vec{k}\vec{p}}{m} = \Delta_0 = \sqrt{m^2 + (\vec{k}(-)\vec{p})^2}, \tag{6a}$$

$$\vec{k}(-)\vec{p} \cdot (\overrightarrow{\Lambda_p^{-1} k}) = \vec{k} - \frac{\vec{p}}{m}\left(k_0 - \frac{\vec{k}\vec{p}}{p_0 + m}\right) = \vec{\Delta}. \tag{6b}$$

In spherical coordinates

$$p_0 = m\,\mathrm{ch}\chi_p \qquad k_0 = m\,\mathrm{ch}\chi_k$$

$$\vec{p} = m\,\vec{n}_p\,\mathrm{sh}\chi_p \qquad \vec{k} = m\,\vec{n}_k\,\mathrm{sh}\chi_k$$

$$\vec{n}_p^2 = \vec{n}_k^2 = 1$$

eq. (6a) takes the form

$$\mathrm{ch}\chi_\Delta = \mathrm{ch}\chi_k\,\mathrm{ch}\chi_p - \vec{n}_k\vec{n}_p\,\mathrm{sh}\chi_k\,\mathrm{sh}\chi_p, \tag{6c}$$

the three-dimensional vector $\vec{\Delta} = \vec{k}(-)\vec{p}$ is the difference of the vectors in the Lobachevsky space. In the nonrelativistic limit it converts into the usual difference of two vectors in the Euclidean space, $\vec{\Delta}_e = -\vec{q} = \vec{k} - \vec{p}$.

Consequently, it can be regarded as the three-vector of momentum transfer belonging to the Lobachevsky space. The 4-momentum transfer squared, q^2, is expressed via that 3-vector in the following way

$$q^2 = t = (p-k)^2 = 2m(m-\Lambda_0) = 2m(m - \sqrt{m^2+\vec{\Lambda}^2}). \qquad (7)$$

Therefore the form factors in (2), (3) can also be parametrized through $\vec{\Lambda}^2$, defined on the hyperboloid, i.e., again through elements of the Lobachevsky space.

With (5) taken into account, current (2) takes the form

$$j^{s}_{\sigma\sigma'}(\vec{p},\vec{k}) = \bar{u}^{\sigma}(0) S_{k(-)p} \cdot D^{1/2}\{V^{-1}(\Lambda_p,k)\} u^{\sigma'}(0) \cdot F_s(\vec{\Lambda}^2)$$

$$= \sum_{\sigma_p=-\frac{1}{2}}^{\frac{1}{2}} j^{s}_{\sigma\sigma_p}(\vec{k}(-)\vec{p}) \cdot F_s(\vec{\Lambda}^2) \cdot D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p,k)\}, \qquad (8)$$

where

$$j^{s}_{\sigma\sigma_p}(\vec{k}(-)\vec{p}) = \sqrt{2m(m+\Lambda_0)} \cdot \delta_{\sigma\sigma_p}.$$

Analogously, for (3) we have

$$j^{ps}_{\sigma\sigma'}(\vec{p},\vec{k}) = \bar{u}^{\sigma}(0)\gamma^5 S_{k(-)p}\, D^{1/2}\{V^{-1}(\Lambda_p,k)\} u^{\sigma'}(0) F_{ps}(\vec{\Lambda}^2)$$

$$= \sum_{\sigma_p=-\frac{1}{2}}^{\frac{1}{2}} j^{ps}_{\sigma\sigma_p}(\vec{k}(-)\vec{p}) \cdot F_{ps}(\vec{\Lambda}^2) \cdot D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p,k)\}, \qquad (9)$$

where

$$j^{ps}_{\sigma\sigma_p}(\vec{k}(-)\vec{p}) = \sqrt{\frac{2m}{\Lambda_0+m}}(\vec{\sigma}\vec{\Lambda})_{\sigma\sigma_p} = \sqrt{\frac{2m}{\Lambda_0+m}}\,\xi^{+\sigma}(\vec{\sigma}\vec{\Lambda})\xi^{\sigma_p}.$$

Thus, currents (2) and (3) can be represented as a product of new current $j_{\sigma\sigma_p}(\vec{k}(-)\vec{p})$, local in the Lobachevsky space, and the Wigner rotation.

The presence in (8) of the function $D^{1/2}$, containing the Wigner rotation is connected with the transformation law of state vectors under the Lorentz transformations:

$$U(\Lambda_p^{-1})|\vec{k},\sigma\rangle = \sum_{\sigma'=-s}^{s} D^{s}_{\sigma'\sigma}\{V^{-1}(\Lambda_p^{-1},k)\}|\overrightarrow{\Lambda_p^{-1}k},\sigma'\rangle .$$

The Wigner rotation has a kinematical nature. In current matrix element (8) it fulfils a removal of spin indices σ' from momentum $\vec{p}$ onto $\vec{k}$ (by the terminology of the authors of the paper [7]). Therefore the indices σ and σ_p of the current $j_{\sigma\sigma_p}(\vec{k}(-)\vec{p})$ are "sitting" on the same momentum $\vec{p}$.

Let us now examine which advantages for writting the amplitude matrix element come from the three-dimensional parametrization of currents (7), (8) as compared to four-dimensional ones (2), (3). As an example, let us consider in the Born approximation the scattering of two nucleons in the theory $\mathcal{L} = g^3 :\bar{\Psi}(x)\gamma^5\Psi(x)\phi(x):$; (without form factors). In the centre-of-mass system this process is described by the Feynmam diagram drawn in Fig. 2

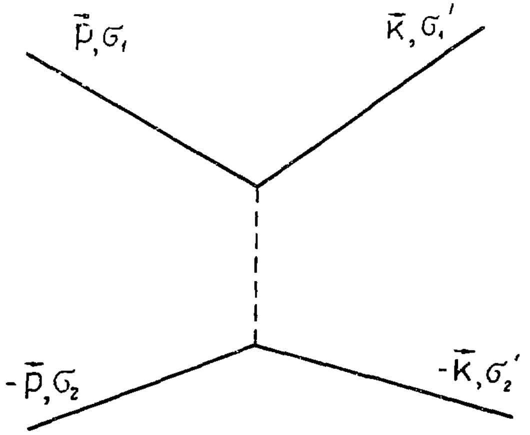

Fig. 2

and the amplitude can be written as

$$T^{(2)\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{k}) = g^2 \frac{[\bar{u}^{\sigma_1}(\vec{p})\gamma_5 u^{\sigma_1'}(\vec{k})][\bar{u}^{\sigma_2}(-\vec{p})\gamma_5 u^{\sigma_2'}(-\vec{k})]}{\mu^2-(p-k)^2} \qquad (10)$$

In virtue of (7), (8) it can be rewritten in the form

$$T^{(2)\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{k}) = \sum_{\sigma_{1p},\sigma_{2p}=-\frac{1}{2}}^{\frac{1}{2}} T^{(2)\sigma_{1p}\sigma_{2p}}_{\sigma_1\sigma_2}(\vec{k}(-)\vec{p}) \cdot$$

$$\times D^{1/2}_{\sigma_{1p}\sigma_1'}\{V^{-1}(\Lambda_p,k)\}D^{1/2}_{\sigma_{2p}\sigma_2'}\{V^{-1}(\Lambda_p,k)\},$$

where

$$T^{(2)\sigma_{1p}\sigma_{2p}}_{\sigma_1\sigma_2}(\vec{k}(-)\vec{p}) = g^2 \frac{(\vec{\sigma_1\Delta})_{\sigma_1\sigma_{1p}}(\vec{\sigma_2\Delta})_{\sigma_2\sigma_{2p}}}{\mu^2-2m^2+2m\Delta_0} \cdot \frac{2m}{m+\Delta_0}. \qquad (11)$$

Usually, when from quantum field theory information is extracted on the two-nucleon interaction one uses the potential

$$V^{(2)}(\vec{p},\vec{k}) = g^2 \frac{(\vec{\sigma_1\Delta_e})(\vec{\sigma_2\Delta_e})}{\mu^2+\vec{\Delta}_e} \qquad (12)$$

which follows from (10) in the non-relaticistic limit[8]. The vector $\vec{\Delta}_e = -\vec{q} = \vec{k}-\vec{p}$ is the difference of two vectors in the Euclidean space. Consequently, on separating the Wigner rotation, which has purely kinematic nature, we arrive at amplitude (11) which is local in the Lobachevsky space. The spin structure of (11) is direct geometric relativistic generalization of that for potential (12)* .

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* Note that usually an expansion in spin structures is performed over the vectors $\vec{k} \pm \vec{p}$ of the Euclidean space and the kinematical Wigner rotation is not separated that results in increasing number of spin structures (see, e.g., ref. [10], where the same matrix element (10) is desc[illegible] by three spin structures).

What is the factor $\frac{2m}{\Lambda_0 + m}$ in (11) can be easily understood if in (10) and (11) one puts $\mu^2 = 0$, i.e., the scalar photon exchange. Then because of $\Lambda_0^2 - \vec{\Lambda}^2 = m^2$ both expressions (11) and (12) take the same form:

$$g^2 \frac{(\vec{\sigma}_1 \vec{\Lambda})(\vec{\sigma}_2 \vec{\Lambda})}{\vec{\Lambda}^2} ,$$

and differ in the geometrical sence of quantity $\vec{\Lambda}$ only. It can be said that for $\mu = 0$ the separated in this way amplitude

$$T^{(2)\sigma_{1p}\sigma_{2p}}_{\sigma_1 \sigma_2}(\vec{p}, \vec{k}) = T^{(2)\sigma_{1p}\sigma_{2p}}_{\sigma_1 \sigma_2}(\vec{k}(-)\vec{p})$$

has absolute "geometrical" character [9].

3. In order to find local expressions in the coordinate space, analogous to those in the Lobachevsky space, transition to the configurational representation should be performed with the help of functions forming a complete system with the volume element of the Lobachevsky space,

$d\Omega_k = \frac{d\vec{k}}{k_0}$. Such fucntions realizing unitary irreducible representations of the Lorentz group (motion group in the Lobachevsky space) have been derived in [1] and have the form

$$\xi(\vec{p}, \vec{r}) = \left(\frac{p_0 - \vec{p}\vec{n}}{m}\right)^{-1-irm} \qquad (13)$$

$$p_0 = \sqrt{\vec{p}^2 + m^2} ,$$

where

$$\vec{r} = r\vec{n} \; ; \; \vec{n}^2 = 1 ; \quad 0 \leq r < \infty . \qquad (14)$$

The parameter r in (13) is connected with the eigenvalues of the Casimir operator of the Lorentz group

$$\hat{C} = -\frac{1}{4} M_{\mu\nu} M^{\mu\nu} = \vec{N}^2 - \vec{M}^2 , \qquad (15)$$

where the components of the vectors $\vec{M}$ and $\vec{N}$ are the generators of the Lorentz group [13]

$$\vec{M} = (M_{32} , M_{13} , M_{21})$$

$$\vec{N} = (M_{01} , M_{02} , M_{03}) \qquad (16)$$

via the following equation:

$$\hat{C}\, \xi (\vec{p},\vec{r}) = (\frac{1}{m^2} + r^2) \xi (\vec{p},\vec{r}) . \qquad (17)$$

To the values $0 \le r < \infty$ there correspond the so-called principal series of the irreducible unitary representations of the Lorentz group realized by functions (13). In the non-relativistic limit the functions $\xi (\vec{p},\vec{r})$ reduce to the usual plane waves [4]

$$\xi (\vec{p},\vec{r}) \to e^{i\vec{p}\vec{r}} .$$

The partial-wave expansion of (13) is as follows

$$\xi (\vec{p},\vec{r}) = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^{\ell}\, p_{\ell} (\mathrm{ch}\chi_{p} , r) P_{\ell} (\frac{\vec{p}\vec{r}}{pr}) . \qquad (18)$$

The functions

$$p_{\ell} (\mathrm{ch}\chi , r) = (-i)^{\ell} \sqrt{\frac{\pi}{2\mathrm{sh}\chi}} \frac{\Gamma (irm + \ell + 1)}{\Gamma (irm + 1)} P^{-\frac{1}{2} - \ell}_{-\frac{1}{2} + ir} (\mathrm{ch}\chi) =$$

$$= \frac{i^{\ell}\, \Gamma (-irm + 1)}{\Gamma (-irm + \ell + 1)} (\mathrm{sh}\chi)^{\ell} (\frac{d}{\mathrm{sh}\chi\, d\chi})^{\ell} p_0 (\mathrm{ch}\chi , r)$$

with

$$p_0 (\mathrm{ch}\chi , r) = \frac{\sin (rm\chi)}{rm\ \mathrm{sh}\,\chi}$$

obey the following conditions of completeness and orthogonality

$$\frac{2\,\mathrm{sh}\,\chi\cdot\mathrm{sh}\,\chi'}{\pi}\,m^3\int_0^\infty r^2 dr\cdot p_\ell^*(\mathrm{ch}\,\chi,r)\,p_\ell(\mathrm{ch}\,\chi',r)=\delta(\chi-\chi')$$

$$\frac{2rr'}{\pi}\,m^3\int_0^\infty \mathrm{sh}^2\chi\, d\chi\, p_\ell^*(\mathrm{ch}\,\chi,r)\,p_\ell(\mathrm{ch}\,\chi,r')=\delta(r-r')$$

and in the nonrelativistic limit they turn to the spherical Bessel functions

$$p_\ell(\mathrm{ch}\,\chi_p,r)\to j_\ell(pr)=\sqrt{\frac{\pi}{2pr}}\;J_{\ell+\frac{1}{2}}(pr).$$

Therefore in /1/ it was suggested to regard the functions $\xi(\vec{p},\vec{r})$ as "plane waves" in the Lobachevsky space and the parameter r as the relativistic generalization of the modulus of radius-vector.

In /1/ also the operator H_0 has been defined which stands for the free Hamiltonian

$$\hat{H}_0\xi(\vec{p},\vec{r})=p_0\xi(\vec{p},\vec{r}).$$

This operator is the finite-difference operator with a step proportional to the Compton wave length $\frac{h}{mc}$

$$\hat{H}_0=m\,\mathrm{ch}\left(\frac{i}{m}\frac{\partial}{\partial r}\right)+\frac{i}{r}\,\mathrm{sh}\left(\frac{i}{m}\frac{\partial}{\partial r}\right)-\frac{\Delta_{\theta,\phi}}{2mr^2}\,e^{\frac{i}{m}\frac{\partial}{\partial r}},\qquad(19)$$

where $\Delta_{\theta,\phi}$ is the Laplace operator on the sphere. Analogously the momentum operator /12/

$$\hat{\vec{\nabla}}_{f.dif.}=(\nabla_{x\ f.dif.};\ \nabla_{y\ f.dif.};\ \nabla_{z\ f.dif.})$$

was constructed

$$-i\hat{\nabla}_{f.dif.}\,\xi(\vec{p},\vec{r}) = \vec{p}\,\xi(\vec{p},\vec{r}) \tag{20}$$

with components

$$-i\hat{\nabla}_{x\ f.dif.} = -\sin\theta\cos\phi\left(e^{\frac{i}{m}\frac{\partial}{\partial r}} - \hat{H}_0\right) - i\left(\frac{\cos\theta\cos\phi}{r}\,\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\right)e^{\frac{i}{m}\frac{\partial}{\partial r}} \tag{21a}$$

$$-i\hat{\nabla}_{y\ f.dif.} = -\sin\theta\sin\phi\left(e^{\frac{i}{m}\frac{\partial}{\partial r}} - \hat{H}_0\right) - i\left(\frac{\cos\theta\sin\phi}{r}\,\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\cdot\sin\theta}\cdot\frac{\partial}{\partial\phi}\right)e^{\frac{i}{m}\frac{\partial}{\partial r}} \tag{21b}$$

$$-i\hat{\nabla}_{z\ f.dif.} = -\cos\theta\left(e^{\frac{i}{m}\frac{\partial}{\partial r}} - \hat{H}_0\right) + i\,\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\,e^{\frac{i}{m}\frac{\partial}{\partial r}}, \tag{21c}$$

where θ and ϕ are spherical coordinates of the vector $\vec{r} = r(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$.

Operators (21) obey the commutation relations

$$[\hat{\nabla}_x, \hat{\nabla}_y] = [\hat{\nabla}_y, \hat{\nabla}_z] = [\hat{\nabla}_z, \hat{\nabla}_x] = 0. \tag{22}$$

In the nonrelativistic limit they become the usual translation generators $-i\frac{\partial}{\partial x_k}$, written in the spherical coordinate system, which is notable for $\xi(\vec{p},\vec{r})$ because of the distinct status there of the modulus of radius-vector r.

Let us now examine what corresponds, in the new relativistic coordinate space, to amplitude (11) if it is considered as some potential. Applying the Shapiro transformation /11/ we get

$$V(\vec{r}) = g^2 \frac{1}{(2\pi)^3} \int \frac{d\vec{\Lambda}}{\Delta_0} \xi(\vec{\Delta},\vec{r}) \frac{(\vec{\sigma}_1 \vec{\Lambda})(\vec{\sigma}_2 \vec{\Delta})}{\mu^2 - 2m^2 + 2m\Delta_0} \cdot \frac{2m}{m + \Delta_0} = \tag{23}$$

$$= g^2 (\vec{\sigma}_1 \hat{\vec{r}}_{dif})(\vec{\sigma}_2 \hat{\vec{r}}_{dif}) \frac{1}{(2\pi)^3} \int \frac{d\vec{\Delta}}{\Delta_0} \frac{\xi(\vec{\Delta},\vec{r})}{\mu^2 - 2m^2 + 2m\Delta_0} \cdot \frac{2m}{m+\Delta_0} =$$

$$= g^2 (\vec{\sigma}_1 \hat{\vec{V}}_{dif})(\vec{\sigma}_2 \hat{\vec{V}}_{dif}) \cdot \frac{4m^2}{4m^2 - \mu^2} \cdot V^{\mu}_{Yuk.}(r). \tag{24}$$

To the propagator

$$\frac{1}{\mu^2 - 2m^2 + 2m\Delta_0} = \frac{1}{\mu^2 - (p-k)^2}$$

in the r -space there corresponds the relativistic Yukawa potential /1,2/

$$V^{\mu}_{Yuk.}(r) = \begin{cases} \dfrac{1}{4\pi r m} \cdot \dfrac{ch(rma_1)}{sh(rm\pi)} & \mu^2 < 4m^2, \quad a_1 = \arccos\left(\dfrac{\mu^2 - 2m^2}{2m^2}\right) \\[2ex] \dfrac{1}{4\pi r m} \cdot \dfrac{\cos(rma_2)}{sh(rm\pi)} & \mu^2 > 4m^2, \quad a_2 = Ar\,ch\left(\dfrac{\mu^2 - 2m^2}{2m^2}\right). \end{cases} \tag{25}$$

For $\mu^2 < 4m^2$ in the nonrelativistic limit it reduces to the usual Yukawa potential

$$V^{\mu}_{Yuk.}(r) \xrightarrow[\mu^2 < 4m^2]{c \to \infty} \frac{e^{-\mu r}}{4\pi r m}. \tag{26}$$

At $\mu^2 = 0$ the potential $V^{\mu}_{Yuk.}(r)$ is the relativistic generalization of the Coulomb potential

$$V_{Coulomb}(r) = \frac{1}{4\pi m r} \cdot \operatorname{cth} \pi r m \ . \tag{27}$$

Within the nonrelativistic theory potential (12) in the configurational representation has the form

$$V^{nonrel.}_{Yuk.}(\vec{r}) = g^2 \frac{1}{(2\pi)^3} \int d\vec{\Lambda}_e \, e^{i\vec{\Lambda}_e \vec{r}} \, \frac{(\vec{\sigma}_1 \vec{\Lambda}_e)(\vec{\sigma}_2 \vec{\Lambda}_e)}{\mu^2 + \vec{\Lambda}_e^2} =$$

$$= g^2 (\vec{\sigma}_1 \vec{\nabla})(\vec{\sigma}_2 \vec{\nabla}) \cdot \frac{e^{-\mu r}}{4\pi r m} \qquad \vec{\Lambda}_e = \vec{k} - \vec{p},$$

where $\vec{\nabla} = \dfrac{\partial}{\partial \vec{r}}$.

Consequently, transition to the coordinate representation with the help of the expansion over the Lorentz group representations allows one to preserve the obtained in the momentum space relativistic expressions in the form of direct relativistic generalization of quantum-mechanical potentials.

Next, we pass to the nucleon form factors. Let us define the three-dimensional distribution in relativistic configurational representation via the expression

$$F(r) = \frac{1}{(2\pi)^3} \int \xi(\vec{\Delta}, \vec{r}) F(\vec{\Delta}^2) \frac{d\vec{\Delta}}{\Delta_0}$$

$$t = (k-p)^2 = 2m(m - \Delta_0). \tag{28}$$

The transformation inverse to (28) is

$$F(\vec{\Delta}^2) = \int \xi^*(\vec{\Delta}, \vec{r}) F(r) \, d\vec{r} \tag{29}$$

and, due to the equality $t/_{\vec{\Delta}^2=0} = 0$, resulting from (7), has the property:

$$F(0) = \int F(r)dr^{*}. \tag{30}$$

In the phenomenologival description their shape is often approximated by the following expression (the Clementel-Villi formula [14]):

$$F(t) = c\,(1 - \sum_i a_i + \sum_j \frac{b_j}{1 - \frac{t}{\mu_j^2}}), \tag{31}$$

where μ_j are the vector-meson masses and a, c, a_i, b_j - constants defined from experiment.

The part of form factors (31) dependent on t is reduced to the form of relativistic propagator $\sum_j \frac{\mu_j^2 b_j}{\mu_j^2 - t}$.

In the relativistic configurational representation to this part there corresponds a set of relativistic Yukawa potentials (25).

$$F(\chi^2) \to \sum_j \mu_j^2 b_j V^{\mu}_{Yuk.}(r). \tag{32}$$

It follows from (25) that the effective range of form-factor (32) essentially depends on the mass of vector meson which defines the nucleon structure. For $\mu^2 < 4m^2$, i.e., below the threshold of nucleon-antinucleon pair production, the factor $\frac{ch(rma_1)}{sh(rm\pi)}$ diminishes the radius of the cloud of light particles with increasing their masses, i.e., when $\mu^2 \to 4m^2$. The radius of the cloud of heavy particles with masses above the production threshold of $N\bar{N}$-pair, i.e., $\mu^2 > 4m^2$, is considerably smaller and it decreases exponentially, by the $e^{-m\pi r}$ -law, with increasing r, due to the factor $\frac{\cos(rma_1^{(2)})}{sh(rm\pi)}$

Unlike the nonrelativistic Yukawa potential $\frac{e^{-\mu r}}{r}$ the dimensions of the cloud of heavy particles are defined just by the nucleon mass *) m and not by the mass of exhanged particle.

4. The relativistic configuration representation introduced above makes it possible to describe form factors in the coordinate space. In particular, the particle invariant mean-square radius $\langle r_0^2 \rangle$ defined usually by the formula

$$F(t) = F(0) + \frac{1}{6}\langle r_0^2 \rangle t + \dots \tag{33}$$

where

$$\langle r_0^2 \rangle = 6\,\frac{\left.\frac{\partial F(t)}{\partial t}\right|_{t=0}}{F(0)} \tag{34}$$

will be connected with the squared distance in relativistic configurational representation. Note that the quantity $\langle r_0^2 \rangle$, as defined by (33) may be interpreted, with the use of Fourier transform (1), as the nucleon mean-squared radius only in the Breit frame, and in all other frames of reference $\langle r_0^2 \rangle$ has no direct relation to the space distribution (if it is defined as the form factor Fourier transform). The determination of the spatial distribution of the form factor through expansion over the Lorentz group representations (28) is not connected with the Breit frame and allows us to describe the r -space distribution in an arbitrary frame of reference. To find the relation between $\langle r_0^2 \rangle$ and r^2, we consider how Casimir operator (15) operates on the form factor. The Casimir operator of the Lorentz group $\hat{C} = \vec{N}^2 - \vec{M}^2$, where (for transformation (28))

$$\vec{N} = -i\frac{\Delta_0}{m}\frac{\partial}{\partial \vec{\Delta}}; \quad \vec{M} = -\frac{i}{m}[\vec{\Delta} \times \frac{\partial}{\partial \vec{\Delta}}]$$

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*) This fact was prompted to the author by V.G.Kadyshevsky.

$$\Lambda_0^2 - \vec{\Lambda}^2 = m^2,$$

in the nonrelativistic limit reduces to the Casimir operator of the motion group of the three-dimensional Euclidean p-space $(i \frac{d}{d\vec{q}})^2$; $(\vec{q} = \vec{p} - \vec{k})$. The eigenvalues of the operator $(i \frac{d}{d\vec{q}})^2$ are given by the squared nonrelativistic coordinates r^2 which, from the group standpoint, enumerate the unitary representations of the translation group of the three-dimnesional Euclidean p-space, realized by the functions $e^{i\vec{q}\vec{r}}$. The nonrelativistic operator $(i \frac{d}{d\vec{q}})^2$ is the Laplace operator in the three-dimensional Euclidean space and in the spherical coordinates $\vec{q} = q(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ has the form

$$(i \frac{d}{d\vec{q}})^2 = -\frac{\partial^2}{\partial q^2} - \frac{2}{q}\frac{\partial}{\partial q} - \frac{\Lambda_{\theta,\phi}}{q^2} \qquad (35)$$

($\Lambda_{\theta,\phi}$ — the Laplace operator on sphere).

Analogously, in the spherical coordinates $\Lambda_0 = m\,\mathrm{ch}\,\chi$; $\vec{\Lambda} = m\,\mathrm{sh}\,\chi\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ (the hyperbolic angle χ is named "rapidity" in the Lobachevsky geometry) Casimir operator (15) has the form of the Laplace operator on hyperboloid:

$$\hat{C} = \vec{N}^2 - \vec{M}^2 = -\frac{1}{m^2}\frac{\partial^2}{\partial\chi^2} - \frac{2\,\mathrm{cth}\,\chi}{m^2}\cdot\frac{\partial}{\partial\chi} - \frac{\Lambda_{\theta,\phi}}{m^2\,\mathrm{sh}^2\chi}. \qquad (36)$$

Owing to the spherical symmetry $F(t) = F[2m^2(1-\mathrm{ch}\,\chi)]$ and even to the dependence on the variable χ on using (36), we get:

$$\frac{\{\hat{C}F[2m^2(1-\mathrm{ch}\,\chi)]\}|_{\chi=0}}{F(0)} = \frac{-\frac{6}{m^2}\frac{\partial F(t)}{\partial\chi^2}|_{\chi=0}}{F(0)} = \qquad (37)$$

$$= \frac{6\left(\frac{\mathrm{sh}\,\chi}{\chi}\right)\frac{\partial F(t)}{\partial t}|_{\chi=0}}{F(0)} = \frac{6\,\frac{\partial F(t)}{\partial t}|_{t=0}}{F(0)},$$

where we have employed the relation $\frac{1}{m^2}\cdot\frac{\partial}{\partial\chi} = -\left(\frac{\operatorname{sh}\chi}{\chi}\right)\frac{\partial}{\partial t}$

following from $t = 2m^2(1-\operatorname{ch}\chi) = -4m^2\operatorname{sh}^2\chi/2$. Expression (37) coincides with definition of mean-square radius (34) Thus, conventional definition of the mean-square radius (34) may be given the group theoretical interpretation in terms of the Casimir operator of the Lorentz group.

The eigenvalues of $\hat{C}$ are given by (17), Therefore, if the spatial distribution of from factors is determined with the help of transformation (28), the nucleon mean-square radius is expressed, using $F(r)$, as follows

$$\langle r_0^2\rangle = \frac{\hat{C}\,F(\vec{\chi}^2)\big|_{\vec{\chi}^2=0}}{F(0)} = \frac{\int\left(\frac{\hbar^2}{m^2c^2}+r^2\right)F(r)\,d\vec{r}}{\int F(r)\,d\vec{r}}$$

$$= \frac{\hbar^2}{m^2c^2} + \langle r^2\rangle . \tag{38}$$

So, the nucleon size in terms of the relativistic configurational representation is described by the

$$X^2 = \frac{\hbar^2}{m^2c^2} + r^2 , \quad 0 \le r < \infty . \tag{39}$$

It was pointed in /1/ that X^2 and r^2 have the same nonrelativistic limit, namely, the squared modulus of nonrelativistic coordinate, so both of them can pretend to be the relativistic generalization of coordinate. The lower value of X^2 is limited from below by square of the nucleon Compton wave length

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*) Unlike (17), in (38) the constants $\hbar$ and c are again introduced.

that is consistent with a result by Newton and Wigner [15] conserning impossible localization of a relativistic particle with an accuracy better than $\frac{\hbar}{mc}$. Consequantly, if one assumes that the mean-square radius is defined by (34) then, as follows from (37), (38), the coordinate r describes the nucleon structure at distances larger than the nucleon Compton wave length.

The quantity $\chi^2 = (\frac{\hbar}{mc})^2 + r^2$ – is the relativistic invariant as it is the eigenvalue of the invariant Casimir operator of the Lorentz group. Thus, the transition to the relativistic configurational representation allows one to connect invariant definition of $\langle r_0^2 \rangle$ (34) with the spatial distribution in any reference frame.

Due to the spherical symmetry of the function $F(t) = F(\vec{\Delta}^2)$, with partial expansion (18) taken into account, transformations (28) and (29) can be rewritten as

$$F(r) = \frac{1}{2\pi^2} \int \frac{\sin rm\chi}{rm\,\mathrm{sh}\,\chi} \cdot F(\vec{\Delta}^2)\,\mathrm{sh}^2\chi\, d\chi \qquad (40)$$

$$F(\vec{\Delta}^2) = 4\pi \int \frac{\sin rm\chi}{rm\,\mathrm{sh}\,\chi} F(r) r^2 dr \qquad (41)$$

where

$$\chi = \mathrm{Arch}\,\frac{\Lambda_0}{m} = \mathrm{Arch}\left(\frac{2m^2 - t}{2m^2}\right) \qquad (42)$$

For the quantity

$$\Phi(\chi) = F(\vec{\Delta}^2)\,\frac{\mathrm{sh}\,\chi}{\chi} \qquad (43)$$

eqs. (40), (41) take the form of usual transformation with the spherical zero-oder Bessel function

$$j_0(r, m\chi) = \frac{\sin rm\chi}{rm\chi}$$

$$F(r) = \frac{1}{2\pi^2} \int \frac{\sin rm\chi}{rm\chi} \Phi(\chi)\,\chi^2 d\chi \qquad (44)$$

$$\Phi(\chi) = 4\pi \int \frac{\sin rm\chi}{rm\chi} F(r) r^2 dr. \qquad (45)$$

However, in these transformations, unlike nonrelativistic Euclidean case, the relativistic coordinate is conjugate not to the modulus of vector of momentum transfer but to the "rapidity" $\chi = \mathrm{Arch}\left(\frac{2m^2 - t}{2m^2}\right)$ corresponding to the momentum transfer vector $\vec{\chi} = \vec{k}(-)\vec{p}$ of the Lobachevsky space. To a particle localized in the relativistic r-space, i.e. $F(\vec{r}) = \delta(\vec{r}) = \frac{\delta(r)}{r^2}\,\delta\left(\frac{\vec{r}}{|r|}\right)$ the form factor

$$F(t) = \frac{\chi}{\mathrm{sh}\,\chi} = \frac{2m^2 \ln\left(1 - \frac{t}{2m^2} + \frac{1}{2m^2}\sqrt{t(t-4m^2)}\right)}{\sqrt{t(t-4m^2)}} \tag{46}$$

corresponds, in accordance with (43). It decreases, for large $|t|$ by the law

$$F(t) \underset{|t| \gg 4m^2}{\sim} 2m^2 \frac{\ln\frac{|t|}{m^2}}{|t|} \tag{47}$$

The form factor $\Phi(\chi) = \frac{\mathrm{sh}\,\chi}{\chi} F(t)$ in this case equals unity, just as the constant form factor $F(t) = 1$ corresponds to the point-like particle in the usual nonrelativistic r-space, introduced by means of the Fourier transformation. Making use of (36) and (43) there is readily derived the equality

$$\frac{1}{m^2}\left\{-\frac{\partial^2}{\partial\chi^2} - \frac{2}{\chi}\frac{\partial}{\partial\chi}\right\}\Phi(\chi) = \frac{\mathrm{sh}\,\chi}{\chi}\left\{\left(\hat{C} - \frac{1}{m^2}\right)F(t)\right\} = r^2\Phi(\chi) \tag{48}$$

from which it follows that definition of the mean-square radius (34) in terms of the form factor $\Phi(\chi)$ results in the quantity

$$6\,\frac{\left.\frac{\partial\Phi(t)}{\partial t}\right|_{t=0}}{\Phi(0)} = \frac{-6\left.\frac{\partial\Phi(\chi)}{m^2\,\partial\chi^2}\right|_{\chi=0}}{\Phi(0)} = \langle r_0^2\rangle - \frac{\hbar^2}{m^2c^2} = \langle r^2\rangle . \tag{49}$$

Note also that if the standard form factor is normalized as $F(0)=1$ the same holds for $\Phi(\chi)$, i.e., $\Phi(0)=1$.

Thus, it can be thought of that just the form factor $\Phi(\chi)$ describes, in the momentum space, the nucleon structure beyond the $\frac{\hbar}{mc}$, and its distribution in the coordinate space is described, in terms of the relativistic configurational representation, by the function $F(r)$. And, as is seen from (44), (45) and (38), one has:

$$\langle r^2 \rangle = \frac{6\left.\frac{\partial \Phi(t)}{\partial t}\right|_{t=0}}{\Phi(0)} = \frac{\int r^2 F(r)\, d\vec{r}}{\int F(r)\, d\vec{r}} . \tag{50}$$

Taking various forms of the distribution $F(r)$ we shall obtain different form factors $F(t)$ which, however, have universal behaviour determined by the common factor $\frac{\chi}{\operatorname{sh}\chi}$. For instance, taking the next, after the δ-type, simple distribution, i.e., the Gaussian-type distribution,

$$F(r) = e^{-ar^2}$$

with the use of (44) we get

$$\Phi(\chi) = \frac{\pi}{a}\sqrt{\frac{\pi}{2}} \cdot e^{-\frac{\chi^2 m^2}{4a}} ,$$

that provides for $F(t)$ the expression

$$F(t) = \frac{\pi}{a}\sqrt{\frac{\pi}{2}} \cdot \frac{\chi}{\operatorname{sh}\chi} \cdot e^{-\frac{\chi^2 m^2}{4a}} ,$$

obtained earlier /6/ in analogy with the form of non-relativistic form factor. The asymptotic form for this form factor

$$F(t) \underset{|t| \gg 4m^2}{\sim} \frac{\ln \frac{t}{m^2}}{|t|}\, e^{-m^2 \ln^2 \frac{|t|}{m^2}} .$$

coincides with that for form factors introduced in /16/ .

For studying the asymptotic behaviour it is convenient to take the spectral representation for form factors. Let the spectral density of the form factor $\Phi(\chi)$ be defined through the Laplace transformation:

$$rF(r)=\frac{1}{2\pi^2}\int_0^\infty a\ \rho(a^2)e^{-ar}da. \tag{51}$$

By using (51) and (45) we obtain

$$\Phi(\chi)=\frac{1}{\pi}\int_0^\infty \frac{\rho(a^2)}{a^2+\chi^2}\,da^2, \tag{52}$$

that gives for the standard form factors the spectral representation in "rapidity" χ, conjugate to t :

$$F(t)=F(\chi)=\frac{\chi}{\operatorname{sh}\chi}\cdot\frac{1}{\pi}\int_0^\infty \frac{\rho(a^2)}{a^2+\chi^2}da^2 \tag{53}$$
$$t=-4m^2\operatorname{sh}^2\chi/2.$$

From (51) it is seen that the asymptotical behaviour of spectral density $\rho(a^2)$ and, hence, of form factor, is defined by the behaviour of $F(r)$ for small r. As far as spectral representation (52) has the same form as that in nonrelativistic theory (see, e.g., ref. [17]), with the only difference that $\vec{q}^{\,2}=(\vec{k}-\vec{p})^2$ is replaced by the "rapidity" squared χ^2 we shall simply list the final results for different cases:

I st case: Let $F(r)\underset{r\to 0}{\sim}\frac{1}{r}$. This gives for the form factors:

$$\Phi(\chi)|_{\chi^2\to\infty}\simeq\frac{\text{const}}{\chi^2};\qquad F(t)|_{\chi\to\infty}\simeq\frac{m^2}{|t|\,\ln\frac{|t|}{m^2}}. \tag{54}$$

II nd case: $F(r)|_{r=0}=\text{const}$. This corresponds to:

$$\Phi(\chi)|_{\chi^2\to\infty}\simeq\frac{\text{const}}{\chi^4};\quad F(t)|_{\chi\to\infty}\simeq\frac{m^2}{|t|\,\ln^3\frac{|t|}{m^2}} \tag{55}$$

III rd case: $rF(r)|_{r\to 0}\to\infty$ means:

$$\Phi(\chi)|_{\chi^2\to\infty}\simeq\frac{\text{const}}{(\chi^2)^\lambda};\ F(r)|_{\chi\to\infty}\simeq\frac{m^2}{|t|\,\ln^{2\lambda-1}\frac{|t|}{m^2}}, \tag{56}$$

In all the three cases $F(-\infty) = \Phi(\infty) = 0$. If $\Phi(\infty) =$ const, then on subtracting this value we arrive at the spectral representation

$$\Phi(\chi) - \Phi(\infty) = \frac{1}{\pi} \int_0^\infty \frac{\rho(a^2)\, da^2}{a^2 + \chi^2}, \tag{57}$$

from which the equation

$$r F(r) = \frac{\Phi(\infty)}{4\pi r} \delta(r) + \frac{1}{4\pi^2} \int_0^\infty \rho(a^2) e^{-ar}\, da^2 \tag{58}$$

follows. This form of the spatial distribution means that in the nucleon center, in the region defined by the Compton wave length of the nucleon itself, there is the charge $Q_0 = \Phi(\infty)$.

For the total charge (57) results in

$$Q = e F(0) = e\,\Phi(\infty) + \frac{e}{\pi} \int_0^\infty \frac{\rho(a^2)\, da^2}{a^2} \tag{59}$$

that allows (57) to be rewritten as

$$\Phi(\chi) - F(0) = -\frac{\chi^2}{\pi} \int_0^\infty \frac{\rho(a^2)\, da^2}{a^2 (a^2 + \chi^2)}. \tag{60}$$

It is easily seen from (60) that if in the nucleon center, in the region with the size of Compton wave length, there is a finite charge, then the spectral density is $\rho(a^2) \underset{a^2 \to \infty}{\sim} \frac{1}{a^{2\lambda}}$, where $\lambda > 0$. If, there is an infinite charge $Q_0 = \infty$, then $\rho(a^2) \sim$ const or $\rho(a^2) \simeq (a^2)^\lambda$. It should be emphasized that, as follows from (53), for any finite λ the standard form factor $F(t)$ remains finite, i.e., $F(\infty) = 0$.

5. To complete our consideration we formulate briefly the results we have obtained. The expressions for currents in terms of elements of the Lobachevsky space allow for the spin structur of current and scattering amplitude to be shaped into the form of the direct relativistic geometrical generalization of spin structures of analogous quantities of nonrelativistic quantum mechanics. And the transition to the relativistic configurational representation through

the expansion over the Lorentz group unitary representations permits one to preserve the three-dimensional, "nonrelativistic in form", expressions for relativistic quantities in the coordinate space, as well.

Note that the three-dimensional form of the description both in momentum and in coordinate spaces is invariant and is achieved not by the choice of some special reference frame (of the Breit type) but just through the use of the Lobachevsky space. The transition to the relativistic configuratiobal representation, in which the modulus of relative coordinate r is relativistic invariant and is related to the eigenvalue of the Casimir operator of the Lorentz group by (17), makes it possible to introduce the invariant description for form factor in the relativistic configurational representation with the help of the distribution function F(r). It is interesting to note that the usual formal definition of the nucleon mean-square raduis by formula (34) acquires the group theoretical meaning as an eigenvalue of the Casimir operator for the Lorentz group (see (37)). Of the most interest are the electromagnetic form factors of particles. These will be considered in a subsequent paper.

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