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COMPLETE INTEGRABILITY OF THE DIFFERENCE EVOLUTION EQUATIONS

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I. INTRODUCTION

The intensive development of the inverse scattering method (ISM)[1-6] has led to the discovery of a number of integrable Hamiltonian systems[7-12]. For many physically important nonlinear evolution equations (NLEE), such as the Korteweg-de Vries eq.[1,7], the non-linear Schrödinger eq.[2,8,9], the sine-Gordon eq.[3,10], there have been found and investigated the classes of soliton solutions, the infinite number of conservation laws, the Bäcklund transformations[4], the explicit form of the action-angle variables (see the review papers[3,5,12]).

The study of a class of NLEE related to the one-dimensional Dirac system

$$\left[i\sigma_3 \frac{d}{dx} + \begin{pmatrix} 0 & q(x) \\ r(x) & 0 \end{pmatrix} \right] \psi(x,\lambda) = \lambda \psi(x,\lambda) \qquad (1.1)$$

revealed the importance of the integro-differential operators $\Lambda_\pm$ (see refs.[3,4]) and their eigenfunction expansions[3,13,14]:

$$\Lambda_\pm = \frac{i}{2} [\sigma_3 \frac{d}{dx} + 2 \begin{pmatrix} q \\ r \end{pmatrix} (x) \int_x^{\pm\infty} dy\, (r, -q)(y)] . \qquad (1.2)$$

The spectral theory of the operators $\Lambda_\pm$ constructed in ref.[15] enables one to justify the interpretation, suggested in ref.[3], of the ISM as a generalized Fourier transform, which linearizes the NLEE[16]. Another important feature of the operators $\Lambda_\pm$ is that they generate the hierarchy of Hamiltonian structures for the NLEE[16,17,18].

There also exist, besides the NLEE, a number of physically important difference nonlinear evolution equations (DEE). An important example of a completely integrable DEE is provided by the Toda chain[11]. Some other important DEE are related to the discrete analogs of the Dirac system (1.1) (see refs.[19-22] and the review paper[5]). These papers contain the discrete analogs of the non-linear Schrödinger eq., the Korteweg-de Vries eq., the sine-Gordon eq. and other interesting DEE, and discuss Bäcklund transformations, conservation laws and soliton solutions.

The main result of the present paper is the proof of complete integrability and the construction of the hierarchy of symplectic structures for the DEE, related to the simplest discrete analog of the Dirac system,

$$\psi(n+1,z)=\begin{pmatrix} z & q(n) \\ r(n) & \frac{1}{z} \end{pmatrix}\psi(n,z), \qquad r(n),q(n)\underset{|n|\to\infty}{\longrightarrow} 0. \tag{1.3}$$

In sections 2 and 3 we give the necessary formulae from the spectral theory for the system (1.3) and for the discrete analog of the operator $\Lambda_\pm$ related to (1.3)

$$(\Lambda_\pm X)(n)=\Lambda_2^\pm \Lambda_1^\pm X, \qquad X=\begin{pmatrix} X_1(n) \\ X_2(n) \end{pmatrix},$$

$$(\Lambda_1^\pm X)(n)=\begin{pmatrix} X_1(n) \\ X_2(n-1) \end{pmatrix}\pm\begin{pmatrix} q(n) \\ -r(n-1) \end{pmatrix}\Sigma_n^\pm \frac{r(k)X_1(k)+q(k)X_2(k)}{h(k)};$$

$$(\Lambda_2^\pm X)(n)=h(n)\begin{pmatrix} X_1(n+1) \\ X_2(n) \end{pmatrix}\pm\begin{pmatrix} q(n) \\ -r(n) \end{pmatrix}\Sigma_{n+1}^\pm\ (r(k-1)X_1(k)+q(k)X_2(k)),$$

$$h(n)=1-q(n)r(n), \qquad \Sigma_n^+=\sum_{k=n}^{\infty},\ \ \Sigma_n^-=\sum_{k=-\infty}^{n-1}. \tag{1.4}$$

We obtain the expansions of the potential $w(n)=\begin{pmatrix} q(n) \\ -r(n) \end{pmatrix}$ and its variation $\sigma_3\delta w(n)=\begin{pmatrix} \delta q(n) \\ \delta r(n) \end{pmatrix}$ in the eigenfunctions of the operators $\Lambda_\pm$, which allow us to obtain the description of the DEE in a somewhat more general form (section 4). In section 5 we prove the complete integrability of the DEE related to the system (1.3); we construct the hierarchy of symplectic structures and calculate the action-angle variables explicitly. We also briefly discuss the possibilities of the quantization of these systems with the help of the quantum ISM /23/. Finally, in section 6 we consider two important particular cases of DEE and discuss the transition to the continuous limit.

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2. Let us start with some known facts (see refs.[19,20,22]) from the direct and inverse scattering problem for the system (1.3) provided the potentials $q(n)$, $r(n)$ tend to zero fast enough when $n \to \pm\infty$, and that

$$0 < \prod_{n=-\infty}^{\infty} |1 - q(n) r(n)| < \infty . \tag{2.1}$$

Then the Jost solutions $\psi^{\pm}$, $\phi^{\pm}$ of the problem (1.3) are uniquely determined by their asymptotic behaviour for $n \to \pm\infty$:

$$\lim_{n\to\infty} z^{n} \psi^{+}(n,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lim_{n\to\infty} z^{-n} \psi^{-}(n,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\lim_{n\to-\infty} z^{-n} \phi^{+}(n,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lim_{n\to-\infty} z^{n} \phi^{-}(n,z) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{2.2}$$

Both pairs of Jost solutions $\psi^{\pm}$ and $\phi^{\pm}$ form fundamental systems of solutions of the problem (1.3); they are linearly related to each other:

$$\phi^{\pm}(n,z) = \pm a^{\pm}(z) \psi^{\mp}(n,z) + b^{\pm}(z) \psi^{\pm}(n,z), \quad |z| = 1 . \tag{2.3}$$

The coefficients $a^{\pm}(z)$, $b^{\pm}(z)$, may be expressed through the Wronskians of the Jost solutions

$$a^{\pm}(z) = W[\phi^{\pm}, \psi^{\pm}], \quad b^{\pm}(z) = \pm W[\psi^{\mp}, \phi^{\pm}] ,$$
$$W[\phi, \psi] = V(n)(\phi_1 \psi_2 - \phi_2 \psi_1)(n,z), \tag{2.4}$$
$$V(n) = \prod_{k=n}^{\infty} h(k), \quad h(k) = 1 - q(k) r(k).$$

They satisfy the "unitarity" condition for $|z| = 1$

$$a^{+} a^{-}(z) + b^{+} b^{-}(z) = V, \quad V = \lim_{n\to-\infty} V(n). \tag{2.5}$$

The continuous spectrum of the problem (1.3) has multiplicity two and fills up the unit circle $S^1 (|z| = 1)$. The discrete spectrum $\Delta = \Delta^{+} \cup \Delta^{-}$ is located at the zeroes of $a^{\pm}(z)$:

$$\Delta^{\pm} = \{ z_j^{\pm} : \; a^{\pm}(z_j^{\pm}) = a^{\pm}(-z_j^{\pm}) = 0, \quad |z_j^{\pm}| \gtrless 1, \quad j = 1, \ldots, N^{\pm} \}. \tag{2.6}$$

Here for simplicity we assume that there is only a finite number of eigenvalues $z_j^{\pm}$, and that $N^{+} = N^{-} = N$. Note also, that if

(2.1) holds then $\psi^{+}(n,z)z^{n}$, $\phi^{+}(n,z)z^{-n}$, $a^{+}(z)$ and $\psi^{-}(n,z)z^{-n}$, $\phi^{-}(n,z)z^{n}$, $a^{-}(z)$ are analytic functions of z for $|z|>1$ and $|z|<1$, respectively. These analytic properties allow us to derive the following dispersion relation for $a^{\pm}(z)$:

$$A(z)=\frac{1}{4\pi i}\oint_{S^1}\frac{d\zeta}{\zeta}\frac{\zeta^2+z^2}{\zeta^2-z^2}\ln(1+\rho^+\rho^-)+\frac{1}{2}\sum_{j=1}^{N}\ln\frac{(z^2-z_j^{+2})z_j^-}{(z^2-z_j^{-2})z_j^+},\quad |z|=1,$$

$$A(z)=\frac{1}{2\pi i}\oint_{S^1}\frac{d\zeta}{\zeta}\frac{\zeta^2}{\zeta^2-z^2}\ln(1+\rho^+\rho^-)+\frac{1}{2}\sum_{j=1}^{N}\ln\frac{z^2-z_j^{+2}}{z^2-z_j^{-2}},\quad |z|>1,$$

$$A(z)=\frac{1}{2\pi i}\oint_{S^1}\frac{d\zeta}{\zeta}\frac{z^2}{\zeta^2-z^2}\ln(1+\rho^+\rho^-)+\frac{1}{2}\sum_{j=1}^{N}\ln\frac{(z^2-z_j^{+2})z_j^{-2}}{(z^2-z_j^{-2})z_j^{+2}},\quad |z|<1,$$

(2.7)

$$A(z)=\pm\ln a^{\pm}(z),\quad |z|\gtrless 1;\qquad A(z)=\frac{1}{2}\ln(a^+/a^-(z)),\quad |z|=1,$$

where $\rho^{\pm}(\zeta)=b^{\pm}/a^{\pm}(\zeta)$ are the reflection coefficients for the system (1.3). Note that for $|z|=1$ the integral in the r.h.s. of (2.7) should be understood in the sence of principle value.

We shall not discuss the solution of the inverse scattering problem in detail. Note only that the set of independent scattering data $T=T^+\cup T^-$

$$T^{\pm}\equiv\{\rho^{\pm}(z),\ |z|=1,\ c_j^{\pm},\ z_j^{\pm},\ |z_j^{\pm}|\gtrless 1,\ j=1,\dots,N\}\qquad(2.8)$$

enables one to reconstruct the corresponding potential $q(n)$, $r(n)$ of the problem (1.3) uniquely (see refs. /19,20,22/ where the Gel'fand-Levitan-Marchenko equation for the system (1.3) is derived). The coefficients $c_j^{\pm}$ in (2.8) are given by the ratios $c_j^{\pm}=b_j^{\pm}/\dot{a}_j^{\pm}$, where $\dot{a}_j^{\pm}=\frac{da^{\pm}}{dz}\Big|_{z=z_j^{\pm}}$ and $b_j^{\pm}$ are defined by

$$\phi^{\pm}(n,z_j^{\pm})=b_j^{\pm}\psi^{\pm}(n,z_j^{\pm}).\qquad(2.9)$$

It is easy to check, that the set T (2.8) and the dispersion relation (2.7) allow one to reconstruct uniquely the functions $a^{\pm}(z)$, $|z|\gtrless 1$ and $b^{\pm}(z)$, $|z|=1$.

3. Now let us formulate the necessary results, from the spectral theory of the operators $\Lambda_{\pm}$ (1.4) and their inverse $\hat{\Lambda}_{\pm}$:

$$(\hat{\Lambda}^{\pm} X)(n) = (\hat{\Lambda}_1^{\pm} \hat{\Lambda}_2^{\pm} X)(n), \qquad X = \begin{pmatrix} X_1(n) \\ X_2(n) \end{pmatrix},$$

$$(\hat{\Lambda}_1^{\pm} X)(n) = h(n) \begin{pmatrix} X_1(n) \\ \\ X_2(n+1) \end{pmatrix} \mp \begin{pmatrix} q(n) \\ \\ -r(n) \end{pmatrix} \Sigma_{n+1}^{\pm} (r(k) X_1(k) + q(k-1) X_2(k)), \tag{3.1}$$

$$(\hat{\Lambda}_2^{\pm} X)(n) = \begin{pmatrix} X_1(n-1) \\ \\ X_2(n) \end{pmatrix} \mp \begin{pmatrix} q(n-1) \\ \\ -r(n) \end{pmatrix} \Sigma_n^{\pm} \frac{r(k) X_1(k) + q(k) X_2(k)}{h(k)}.$$

The operators $\Lambda_{\pm}, \hat{\Lambda}_{\pm}$ are defined in the space $\ell^2(\mathbf{Z}, \mathbf{C}^2)$ of square summable vector-valued sequences $X(n) = \begin{pmatrix} X_1(n) \\ X_2(n) \end{pmatrix} \in \ell^2(\mathbf{Z}, \mathbf{C}^2)$. In $\ell^2(\mathbf{Z}, \mathbf{C}^2)$, besides the ordinary scalar product $(X, Y) = \sum_{k=-\infty}^{\infty} X^T(k) Y(k)$, we introduce the skew-scalar product:

$$[X, Y]_h = \sum_{n=-\infty}^{\infty} \frac{X^T(n) B Y(n)}{h(n)}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

It readily follows from (1.4) and (3.1) that for $X(n) \in \ell^2(\mathbf{Z}, \mathbf{C}^2)$ we have $(\Lambda_i^{\pm} \hat{\Lambda}_i^{\pm}) X = (\hat{\Lambda}_i^{\pm} \Lambda_i^{\pm}) X = X, i = 1,2$. Note also that the operators $\Lambda_-, \hat{\Lambda}_-$ are the adjoint operators of $\Lambda_+, \hat{\Lambda}_+$ with respect to the skew-scalar product (3.2), i.e., for each $X, Y \in \ell^2(\mathbf{Z}, \mathbf{C}^2)$

$$[\Lambda_- X, Y]_h = [X, \Lambda_+ Y]_h; \; [\hat{\Lambda}_- X, Y]_h = [X, \hat{\Lambda}_+ Y]_h. \tag{3.3}$$

Using (1.3) we can verify, that the eigenfunctions and the adjoint functions of the operators Λ_+ and Λ_- defined by

$$\begin{aligned} &(\Lambda_+ - z^2) \Psi^{\pm}(n,z) = 0, \; z \in S^1 \cup \Delta; \; (\Lambda_+ - z_j^{\pm 2}) \dot{\Psi}_j^{\pm}(n) = 2 z_j^{\pm} \Psi_j^{\pm}(n), \\ &(\Lambda_- - z^2) \Phi^{\pm}(n,z) = 0, \; z \in S^1 \cup \Delta; \; (\Lambda_- - z_j^{\pm 2}) \dot{\Phi}_j^{\pm}(n) = 2 z_j^{\pm} \Phi_j^{\pm}(n), \end{aligned} \tag{3.4}$$

are related to the Jost solutions (2.2) by:

$$\Psi^{\pm}(n,z) = (\psi^{\pm} \circ \psi^{\pm})(n,z), \quad \Phi^{\pm}(n,z) = (\phi^{\pm} \circ \phi^{\pm})(n,z), \quad z \in S^1 \cup \Delta,$$

$$\dot{\Psi}_j^{\pm}(n) = \frac{\partial \Psi^{\pm}(n,z)}{\partial z}\Big|_{z = z_j^{\pm}}, \quad \dot{\Phi}_j^{\pm}(n) = \frac{\partial \Phi^{\pm}(n,z)}{\partial z}\Big|_{z = z_j^{\pm}}, \tag{3.5}$$

where

$$(\psi\circ\phi)(n,z)=\frac{1}{2}V(n)\begin{pmatrix}\psi_1(n)\phi_1(n+1)+\psi_1(n+1)\phi_1(n)\\ \psi_2(n)\phi_2(n+1)+\psi_2(n+1)\phi_2(n)\end{pmatrix}.$$

Furthermore, applying the contour integration method to the integral

$$J(n,m)=\frac{1}{2\pi i}\oint_{\gamma_+}\frac{dz}{z}G^+(n,m,z)-\frac{1}{2\pi i}\oint_{\gamma_-}\frac{dz}{z}G^-(n,m,z), \tag{3.6}$$

where

$$G^{\pm}(n,m,z)=\frac{1}{(a^{\pm})}\{\theta(n-m)\Psi^{\pm}(n,z)\tilde{\Phi}^{\pm}(m,z)+$$

$$+\theta(m-n-1)[2(\phi^{\pm}\circ\psi^{\pm})(n,z)\widetilde{(\psi^{\pm}\circ\phi^{\pm})}(m,z)-\Phi^{\pm}(n,z)\tilde{\Psi}^{\pm}(m,z)]\},$$

$$\theta(n-m)=\begin{cases}1, & n\geq m,\\ 0, & n<m,\end{cases} \tag{3.7}$$

with $\tilde{\Phi}=\Phi^T B$, we derive the completeness relation for the eigenfunctions and the adjoint functions of the operators $\Lambda_{\pm}$. In (3.6) the contours are $\gamma_+=S^1\cup\bar{S}^{(\infty)}$, $\gamma_-=S^1\cup\bar{S}^{(0)}$, where S^1 is the positively oriented unit circle, and $\bar{S}^{(\infty)}$, $\bar{S}^{(0)}$ - are the negatively oriented circles of infinitely large and infinitely small radius, respectively. Omitting the calculational details we write down the result

$$-h(n)\delta(n-m)=\frac{1}{2\pi i}\oint_{S^1}\frac{dz}{z}\left[\frac{\Psi^+(n,z)\tilde{\Phi}^+(m,z)}{(a^+(z))^2}-\frac{\Psi^-(n,z)\tilde{\Phi}^-(m,z)}{(a^-(z))^2}\right]+$$

$$+\sum_{j=1}^{N}(X_j^+ + X_j^-)(n,m), \tag{3.8}$$

$$X_j^{\pm}(n,m)=\frac{1}{z_j^{\pm}\dot{a}_j^{\pm}}\{-(\frac{1}{z_j^{\pm}}+\frac{\ddot{a}_j^{\pm}}{\dot{a}_j^{\pm}})\Psi_j^{\pm}(n)\tilde{\Phi}_j^{\pm}(m)+\Psi_j^{\pm}(n)\dot{\tilde{\Phi}}_j^{\pm}(m)+\dot{\Psi}_j^{\pm}(n)\tilde{\Phi}_j^{\pm}(m).$$

Let us transpose (3.8) and make a similarity transformation with B; this will give us the adjoint to (3.8) completeness relation, which differs from (3.8) by the sign in the l.h.s. and the exchange $\Phi \to \Psi$ in the r.h.s. Subtracting (3.8) from its adjoint we get, just as in ref./15/, the symplectic completeness relation in the form:

$$h(n)\delta(n-m) = \oint_{S^1} \frac{dz}{z}[P(n,z)\tilde{Q}(m,z) - Q(n,z)\tilde{P}(m,z)] +$$
$$+ \sum_{j=1}^{N} [P_j^+(n)\tilde{Q}_j^+(m) - Q_j^+(n)\tilde{P}_j^+(m) + P_j^-(n)\tilde{Q}_j^-(m) - Q_j^-(n)\tilde{P}_j^-(m)], \quad (3.9)$$

where the symplectic basis $\{P, Q\}$ is given by

$$P(n,z) = -\frac{1}{2\pi}(\rho^+\Psi^+ + \rho^-\Psi^-) = -\frac{1}{2\pi}(\sigma^+\Phi^+ + \sigma^-\Phi^-),$$

$$Q(n,z) = \frac{iV}{2b^+b^-}(\rho^+\Psi^+ - \frac{\sigma^+}{V}\Phi^+) = \frac{iV}{2b^+b^-}(\frac{\sigma^-}{V}\Phi^- - \rho^-\Psi^-),$$

$$P_j^{\pm}(n) = \mp \frac{ic_j^{\pm}}{z_j^{\pm}}\Psi_j^{\pm}(n), \quad Q_j^{\pm}(n) = \mp \frac{i}{2}(m_j^{\pm}\dot{\Phi}_j^{\pm}(n) - c_j^{\pm}\dot{\Psi}_j^{\pm}(n)), \quad (3.10)$$

$$\sigma^{\pm} = b^{\mp}/a^{\pm}, \quad m_j^{\pm} = 1/(b_j^{\pm}\dot{a}_j^{\pm}).$$

Using these completeness relations we are able to expand any vector-valued sequence $X(n)$ which is both summable and square summable in the eigenfunctions and adjoint functions of the operators $\Lambda_{\pm}$ or in the symplectic basis (3.10). The expansion coefficients are expressed in terms of the quantities

$$[\Phi^{\pm}, X]_h, \quad [\Psi^{\pm}, X]_h,$$

where the skew-scalar product $[\ ,\ ]_h$ is incroduced in (3.2). Moreover, $X(n)=0$ if and only if all the coefficients in the expansions vanish. In two important cases: $X(n) \equiv w(n) = \begin{pmatrix} q(n) \\ -r(n) \end{pmatrix}$, and $X(n) \equiv \sigma_3 \delta w(n) = \begin{pmatrix} \delta q(n) \\ \delta r(n) \end{pmatrix}$ the coefficients (3.11) can be easily expressed in terms of the scattering data T for the problem (1.3). Indeed, it follows from (1.3) that

$$[\Phi^{\pm}, w]_h = \sum_{n=-\infty}^{\infty} \frac{\tilde{\Phi}^{\pm}(n,z)\, w(n)}{h(n)} = \frac{(\phi_1^{\pm}\ \phi_2^{\pm})(n,z)}{V(n)}\Big|_{n=-\infty}^{\infty},$$

$$[\Phi^{\pm}, \sigma_3 \delta w]_h = \frac{(\phi_2^{\pm}\, \delta\phi_1^{\pm} - \phi_1^{\pm}\, \delta\phi_2^{\pm})(n,z)}{V(n)}\Big|_{n=-\infty}^{\infty}. \qquad (3.11)$$

Now it suffices to use the relations (2.2) and (2.3) in order to get the following expansions for $w(n)$ and $\sigma_3 \delta w(n)$:

$$w(n) = -\frac{1}{2\pi i} \oint_{S^1} \frac{dz}{z} [\rho^+ \Psi^+ + \rho^- \Psi^-](n,z) - \sum_{j=1}^{N} [\frac{c_j^+}{z_n^+} \Psi_j^+(n) - \frac{c_j^-}{z_j^-} \Psi_j^-(n)], \qquad (3.12a)$$

$$w(n) = -i \oint_{S^1} \frac{dz}{z} P(n,z) - i \sum_{j=1}^{N} (P_j^+(n) + P_j^-(n)), \qquad (3.12b)$$

$$\sigma_3 \delta w(n) = \frac{1}{2\pi i} \oint_{S^1} \frac{dz}{z} [\delta\rho^+ \Psi^+ - \delta\rho^- \Psi^-](n,z) + \sum_{j=1}^{N} (Y_j^+(n) + Y_j^-(n)), \qquad (3.13a)$$

$$\sigma_3 \delta w(n) = \oint_{S^1} \frac{dz}{z} [Q\,\delta\hat{p} - P\,\delta\hat{q}](n,z) + \sum_{j=1}^{N} (Z_j^+(n) + Z_j^-(n)), \qquad (3.13b)$$

where

$$Y_j^{\pm}(n) = \Psi_j^{\pm}(n)\, \delta(\frac{c_j^{\pm}}{z_j^{\pm}}) + \dot{\Psi}_j^{\pm}(n)\, c_j^{\pm}\, \delta \ln z_j^{\pm}, \qquad Z_j^{\pm}(n) = Q_j^{\pm}(n)\, \delta\hat{p}_j^{\pm} - P_j^{\pm}(n)\, \delta\hat{q}_j^{\pm},$$

$$\hat{p}(z) = -\frac{1}{2\pi} \ln(1 + \rho^+ \rho^-(z)), \qquad \hat{q}(z) = -\frac{i}{2} \ln \frac{b^+(z)}{b^-(z)}, \qquad |z| = 1,$$

$$\hat{p}_j^{\pm} = \mp i \ln z_j^{\pm}, \qquad \hat{q}_j^{\pm} = \mp i \ln \frac{b_j^{\pm}}{\sqrt{V}}, \qquad (3.14)$$

$$\delta\hat{p}(z) = -[P, \sigma_3 \delta w]_h, \qquad \delta\hat{q}(z) = -[Q, \sigma_3 \delta w]_h.$$

4. Now we are able to formulate and prove a theorem which describes the DEE in a somewhat more general form than in refs./21,22/. To this end we consider the system (1.3) with a potential which depends on a parameter t, $w \equiv w(n,t)$ and in (3.13) confine ourselves to the variations of the form $\delta w(n,t) = \frac{\partial w}{\partial t}\delta t$, the corresponding expansions for $\sigma_3 \frac{\partial w}{\partial t}$ differ from (3.13) in that the coefficients $\delta\rho^{\pm}(z,t),\ldots$ are replaced by $\frac{\partial \rho^{\pm}}{\partial t},\ldots$.

Theorem 1. Let $f(z)$ be a meromorphic function which has no poles in a neighbourhood of the spectrum of the system (1.3). Then $w(n,t)$ satisfy the DEE

$$\sigma_3 \frac{\partial w}{\partial t} + f(\Lambda_+)\, w(n,t) = 0, \tag{4.1}$$

if and only if the scattering data T satisfy the linear equations:

$$\frac{\partial \rho^{\pm}}{\partial t} \mp f(z^2)\rho^{\pm}(z,t) = 0,$$

$$\frac{dc_j^{\pm}}{dt} \mp f(s_j^{\pm})\, c_j^{\pm}(t) = 0, \qquad \frac{ds_j^{\pm}}{dt} = 0, \tag{4.2}$$

where $s_j^{\pm} = (z_j^{\pm})^2$.

Proof. Let us insert the expansions (3.12a) and (3.13a) in the l.h.s. of (4.1) and use the relations (3.4). This gives

$$\sigma_3 \frac{\partial w}{\partial t} + f(\Lambda_+)\, w(n,t) = \frac{1}{2\pi i}\oint_{S^1} \frac{dz}{z}[(\rho_t^{+} - f(z^2)\rho^{+})\Psi^{+}(n,z) -$$

$$-(\rho_t^{-} + f(z^2)\rho^{-})\Psi^{-}(n,z)] + \sum_{j=1}^{N}(U_j^{+}(n) + U_j^{-}(n)),$$

$$U_j^{\pm}(n) = [\frac{d}{dt}(\frac{c_j^{\pm}}{z_j^{\pm}}) - f(s_j^{\pm})\frac{c_j^{\pm}}{z_j^{\pm}}]\Psi_j^{\pm}(n) + \frac{c_j^{\pm}}{z_j^{\pm}}\frac{dz_j^{\pm}}{dt}\dot{\Psi}_j^{\pm}(n). \tag{4.3}$$

Thus DEE (4.1) holds if and only if all coefficients in the expansion (4.3) vanish, i.e., if and only if the relations (4.2) hold.

Theorem 2. The potential $w(n,t)$ satisfies the DEE (4.1) if and only if the variables $\{\hat{p}, \hat{q}\}$ satisfy the linear equations

$$\frac{\partial \hat{q}(z,t)}{\partial t} = -if(z^2), \qquad \frac{\partial \hat{p}(z)}{\partial t} = 0, \qquad |z| = 1$$
$$\frac{d\hat{q}_j^{\pm}(t)}{dt} = -if(s_j^{\pm}), \qquad \frac{d\hat{p}_j^{\pm}}{dt} = 0. \tag{4.4}$$

Besides if w satisfies the DEE (4.1), it satisfies also the DEE (4.1),

$$\sigma_3 \frac{\partial w}{\partial t} + f(\Lambda_-)w(n,t) = 0, \tag{4.5}$$

and vice versa.

The first part of theorem 2 is proved similarly to theorem 1 using the expansions (3.12b) and (3.13b) over the symplectic basis. The equivalence of the DEE (4.1) and (4.5) follows from the relations $\Lambda_+ P(n,z) = \Lambda_- P(n,z) = z^2 P(n,z)$ (compare (3.4) and (3.10)), which provide the identity

$$f(\Lambda_+)w(n,t) = f(\Lambda_-)w(n,t). \tag{4.6}$$

To get a generating functional for the conservation laws of the DEE just as in the case of the Dirac system (1.2) we take the function $A(z)$ (2.7).

Really, from (2.7) and (4.2) we immediately obtain that $\frac{dA(z,t)}{dt} = 0$ for all z. To get the conserved quantities we may take the asymptotic expansion coefficients C_n :

$$A(z) = \sum_{p=1}^{\infty} C_p z^{-2p}, \; |z| \gg 1; \qquad A(z) = -\sum_{p=1}^{\infty} C_{-p} z^{2p}, \; |z| \ll 1; \tag{4.7}$$

which may be expressed both as functionals of the potential $w(n,t)$ and as functionals of the scattering data T. In the paper[20] the recurrent relations are obtained, which allow one to express C_p as functionals of $w(n)$. Here we write down a compact expression for C_p in terms of $w(n)$ and the operator Λ_+, which is derived similarly to those in[16,18]:

$$C_p = \frac{1}{p} \sum_{n=-\infty}^{\infty} \sum_{k=n}^{\infty} \frac{\tilde{w}(k)\,\Lambda_+^p\, w(k)}{h(k)} \quad , \quad p \neq 0,$$

$$C_0 = -\ln V = -\sum_{n=-\infty}^{\infty} \ln h(n). \tag{4.8}$$

From the dispersion relation (2.7) it follows that

$$C_p = \frac{i}{2\pi} \oint_{S^1} \frac{dz}{z} z^{2p} \ln(1+\rho^+\rho^-) - \frac{1}{2p} \sum_{j=1}^{N} [(s_j^+)^p - (s_j^-)^p], \quad p \neq 0,$$

$$C_0 = -\ln V = \frac{i}{2\pi} \oint_{S^1} \frac{dz}{z} \ln(1+\rho^+\rho^-) - \frac{1}{2} \sum_{j=1}^{N} \ln(s_j^+ / s_j^-). \tag{4.9}$$

Below when discussing the Hamiltonian structure of the DEE we shall need an expression for the variations:

$$\delta C_p = \sum_{n=-\infty}^{\infty} \frac{\widetilde{\sigma_3 \delta w(n)}\, \Lambda_+^p\, w(n)}{h(n)}. \tag{4.10}$$

At the end of this section we give an explicit form of the four simplest C_p, $p = \pm 1, \pm 2$:

$$C_1 = \sum_{n=-\infty}^{\infty} q(n) r(n-1), \qquad C_{-1} = \sum_{n=-\infty}^{\infty} r(n) q(n-1),$$

$$C_2 = \sum_{n=-\infty}^{\infty} [\, q(n) r(n-2) h(n-1) - \frac{1}{2} (q(n) r(n-1))^2 \,], \tag{4.11}$$

$$C_{-2} = \sum_{n=-\infty}^{\infty} [\, r(n) q(n-2) h(n-1) - \frac{1}{2} (r(n) q(n-1))^2 \,].$$

5. In order to write down the DEE (5.1) as Hamiltonian equations of motion we must introduce a symplectic structure on the manifold of potentials $\mathcal{F} = \{q(n), r(n)\}$ and a Hamiltonian $H[q(n), r(n)]$. As a Hamiltonian we take the following linear combination of the integrals of motion:

$$H_f = -i \sum_p f_p C_p = -i \sum_{n=-\infty}^{\infty} \sum_{k=n}^{\infty} \frac{\tilde{w}(k) F(\Lambda_+) w(k)}{h(k)} + \tag{5.1}$$

$$+ i f_0 \sum_{n=-\infty}^{\infty} \ln h(n),$$

$$f(z) = \sum_p f_p z^p , \qquad F(z) = \int^z \frac{ds}{s} (f(s) - f_0).$$

The symplectic structure on $\mathcal{F}$ is given by the 2-form

$$\Omega_0 = 2i \sum_{n=-\infty}^{\infty} \frac{\delta q(n) \wedge \delta r(n)}{h(n)} \equiv$$
$$\equiv i[\sigma_3 \delta_2 w, \sigma_3 \delta_1 w]_h - i[\sigma_3 \delta_1 w, \sigma_3 \delta_2 w]_h , \qquad (5.2)$$

where $\delta_1 w$ and $\delta_2 w$ are two independent variations of the potential w of the system (1.3). It is easy to check that the 2-form Ω_0 on $\mathcal{F}$ is skew-symmetric and closed and satisfies the Jacobi identity.

Using (4.10) we can verify, that the Hamiltonian equations of motion generated by H_f (5.1) with respect to Ω_0 (5.2)

$$\Omega_0(\sigma_3 \frac{\partial w}{\partial t}, \cdot) = \delta H_f (\cdot), \qquad (5.3)$$

coinside with (4.1).

Let us express H_f and Ω_0 in terms of the scattering data T.

To do this we insert the expansion (3.13) in (5.2); the summation over n leads again to the quantities $[\sigma_3 \delta w, P]_h$ and $[\sigma_3 \delta w, Q]_h$ (see (3.14). Thus we immediately obtain Ω_0 in canonical form:

$$\Omega_0 = 2i \oint_{S^1} \frac{dz}{z} \delta \hat{p}(z) \wedge \delta \hat{q}(z) + 2i \sum_{j=1}^{N} (\delta \hat{p}_j^+ \wedge \delta \hat{q}_j^+ + \delta \hat{p}_j^- \wedge \delta \hat{q}_j^-), \qquad (5.4)$$

where $\{\hat{p}, \hat{q}\}$ are given in (3.14). From (4.9) and (5.1) we easily get:

$$H_f = -\oint_{S^1} \frac{dz}{z} f(z^2) \hat{p}(z) + \frac{i}{2} \sum_{j=1}^{N} (F_1(s_j^+) - F_1(s_j^-)),$$
$$F_1(z) = \int^z \frac{ds}{s} f(s). \qquad (5.5)$$

Formulae (5.4) and (5.5) lead us to the conclusion, that the new variables $\{\hat{p}, \hat{q}\}$ (3.14) are action-angle variables. Thus we have proved the complete integrability of the whole class of DEE (4.1). It is also easy to check that the Hamiltonian equations of motion (5.3) for H_f (5.5) and Ω_0 (5.4) lead exactly to the linear system of equations (4.4) for $\{\hat{p}, \hat{q}\}$.

The symplectic structure chosen above is not unique. We can introduce a one-parameter family (a hierarchy) of 2-forms Ω_m, which are related to Ω_0 (5.2) through the operator Λ_+:

$$\Omega_m = i[\sigma_3 \delta_2 w, \Lambda_+^m \sigma_3 \delta_1 w]_h - i[\sigma_3 \delta_1 w, \Lambda_+^m \sigma_3 \delta_2 w]. \tag{5.6}$$

To prove that Ω_m are symplectic one can proceed as in /17/ and express Ω_m im terms of the scattering data. Using again the expansion (3.13) we obtain:

$$\Omega_m = 2i \oint_{S^1} \frac{dz}{z} z^{2m} \delta \hat{p}(z) \wedge \delta \hat{q}(z) + + 2i \sum_{j=1}^{N} [(s_j^+)^m \delta \hat{p}_j^+ \wedge \delta \hat{q}_j^+ + (s_j^-)^m \delta \hat{p}_j^- \wedge \delta \hat{q}_j^-]. \tag{5.7}$$

Note that the form Ω_m and the Hamiltonian $H_{f^{(m)}}$ (5.1) with $f^{(m)}(z) = z^m f(z)$ also generate the DEE (4.1); i.e., to each DEE (4.1) there correspond one-parameter family (a hierarchy) of Hamiltonian structures.

Note that from the consistency of the symplectic structures generated by Ω_m it follows, that the corresponding Lagrangian manifolds

$$\mathfrak{M}^{(m)} \equiv \{ g \in \mathfrak{M}^{(m)}: \Omega_m|_g = 0 \}$$

must coinside, i.e., $\mathfrak{M}^{(m)} = \mathfrak{M}^{(0)} \equiv \mathfrak{M}$ for all m. Really, let us introduce $\mathfrak{M}^{(0)}$ by (see /16/):

$$\mathfrak{M}^{(0)} \equiv \{ f \in \mathfrak{M}^{(0)} : f(n) = \oint_{S^1} \frac{dz}{z} g(z) P(n,z) + \sum_{j=1}^{N} [\tilde{g}_j^+ P_j^+(n) + + \tilde{g}_j^- P_j^-(n)] \}, \tag{5.8}$$

and show that all Ω_m vanish on $\mathfrak{M}^{(0)}$. For this it is sufficient to note that the requirement $\sigma_3 \delta w \in \mathfrak{M}^{(0)}$ is equivalent to the requirement, $\delta \hat{q}(z) = \delta \hat{q}_j^{\pm} = 0$ (compare (3.13b) and (5.8)), which in its own turn is equivalent to $\Omega_m = 0$ (see (5.7)). Let us list without proof (see refs. /15,18/) the main features of $\mathfrak{M}^0$: i) $\mathfrak{M}^0$ is the maximal Lagrangian manifold of the DEE (4.1), i.e., $\dim \mathfrak{M}^0 = \operatorname{codim} \mathfrak{M}^0$; ii) on $\mathfrak{M}^0$ the operator Λ_+ is "self-adjoint", i.e., $\Lambda_+ = \Lambda_-$; iii) $w \in \mathfrak{M}^0$, and therefore $f(\Lambda_+)w = f(\Lambda_-)w \in \mathfrak{M}^0$ (see (3.12) and (4.6)).

Resently the quantum ISM has been invented and actively developed (see the review paper /23/), which has led to a number of exactly solvable two-dimensional quantum models. A crucial role in this approach is played by the so-called Yang-Baxter relation. Its detailed study has led to a new way of calculating the classical Poisson brackets between the entries of the scattering data /24/. For this we need the so-called classical r-matrix determined from the relation:

$$\{L_n(z) \overset{\otimes}{,} L_n(\zeta)\}_0 = [r(z,\zeta),\ L_n(z) \otimes L_n(\zeta)],$$
$$L_n(z) = \begin{pmatrix} z & q(n) \\ r(n) & \frac{1}{z} \end{pmatrix}, \qquad (5.9)$$

where $L_n(z) \otimes L_n(\zeta)$ is the tensor product of the matrices $L_n(z)$ and $L_n(\zeta)$; the 4x4-matrix in the l.h.s. of (5.9) consists of the Poisson brackets between the corresponding elements of $L_n(z)$ and $L_n(\zeta)$. In our case the only non-trivial Poisson bracket equals $\{q(n), r(n)\}_0 = ih(n)$ (see (5.2)). From (5.9) we get

$$r(\tau) = -\frac{i}{2} \begin{pmatrix} \operatorname{cth} \tau & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\operatorname{sh} \tau} & 0 \\ 0 & \frac{1}{\operatorname{sh} \tau} & -1 & 0 \\ 0 & 0 & 0 & \operatorname{cth} \tau \end{pmatrix}, \quad \tau = \ln \frac{z}{\zeta}. \qquad (5.10)$$

Furthermore, from (5.9) it follows that /24/:

$$\{S(z) \overset{\otimes}{,} S(\zeta)\} = [r(\tau),\ S(z) \otimes S(\zeta)], \qquad S(z) = \prod_{n=-\infty}^{\infty} L_n(z). \qquad (5.11)$$

Thus the r-matrix (5.10) provides the Poisson brackets between the scattering data of the problem (1.3). Without going into the calculational details we note, that the Poisson brackets between the action-angle variables are

$$\{\hat{p}(z), \hat{q}(z)\}_0 = i z \delta(z-\zeta)$$

which is consistent with the canonical form of Ω_0 (5.4).

Now, using the relation

$$R(z,\zeta) L_n(z) \otimes L_n(\zeta) = L_n(\zeta) \otimes L_n(z) R(z,\zeta) \qquad (5.12)$$

we can calculate the quantum R-matrix for this system. Provided that the operators $q(n)$ and $r(n) = \epsilon q^+(n)$ satisfy the commutation relation

$$[q(n), q^+(m)] = \delta_{nm} \hbar (1 - \epsilon q^+(n) q(n)),$$

where $\hbar$ is the Plank constant, we obtain ($\hbar = \epsilon (1-e^{-2\eta})$):

$$R(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^+ & \gamma & 0 \\ 0 & \gamma & \beta^- & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} \beta^{\pm} &= \frac{e^{\pm\eta}\,\mathrm{sh}\,\tau}{\mathrm{sh}(\tau+\eta)}, \\ \gamma &= \frac{\mathrm{sh}\,\eta}{\mathrm{sh}(\tau+\eta)}, \\ \tau &= \ln\frac{z}{\zeta} \end{aligned} \qquad (5.13)$$

The commutation relations between the entries of the monodromy matrix S(z) (see (5.11)) are calculated using the R-matrix (5.13) from the relation

$$R(\tau)\, S(z) \otimes S(\zeta) = S(\zeta) \otimes S(z)\, R(\tau).$$

The further realization of the ideas of the quantum ISM for this system will be described in a separate paper.

6. In this last section we briefly discuss some particular DEE and the continuous limit case. The soliton solutions of these DEE are obtained in refs./19,22/ and we will not write them down here.

A. The difference non-linear Schrödinger (DNS) equation

$$i\,\frac{\partial q(n,t)}{\partial t} = -[\,1-\epsilon\,|q(n)|^2\,][\,q(n+1)+q(n-1)\,] + 2q(n), \quad \epsilon = \pm 1, \qquad (6.1)$$

is obtained from (4.1) with $f(z)=i(2-z-\frac{1}{z})$ provided that the involution $q(n) = \epsilon\, r^*(n)$ holds. The involution imposes the following restrictions on the scattering data:

$$a^+(z) = a^{-*}\left(\frac{1}{z^*}\right), \quad b^+(z) = -\epsilon\, b^{-*}\left(\frac{1}{z^*}\right), \quad |z| = 1,$$
$$z_j^+ = \frac{1}{(z_j^-)^*}, \quad c_j^- = \epsilon\, \frac{c^{+*}}{(z_j^{+*})^2}, \qquad (6.2)$$

which lead to the following canonical form of Ω_{DNS}(5.4):

$$\Omega_{DNS} = \epsilon\, \Omega_0 \big|_{q=\epsilon r^*} = 2i\epsilon \oint_{S^1} \frac{dz}{z}\, \delta \arg b^+(z) \wedge \delta\, \frac{\ln(1-\epsilon|\rho^+|^2)}{\pi} -$$

$$-4\epsilon \sum_{j=1}^{N} [\, \delta \alpha_j \wedge \delta \ln \Big|\frac{b_j^+}{\sqrt{V}}\Big| + \delta \ln \rho_j \wedge \delta \arg b_j^+ \,], \tag{6.3}$$

where $z_j^+ = \rho_j e^{i\alpha_j}$. The Hamiltonian for the DEE (6.1) equals

$$H_{DNS} = \epsilon(2C_0 - C_1 - C_{-1})\big|_{q=\epsilon r^*} =$$

$$= -\sum_{n=-\infty}^{\infty} \{ q^*(n)[q(n+1) + q(n-1)] + 2\epsilon \ln(1-\epsilon|q(n)|^2) \} = \tag{6.4}$$

$$= i\epsilon \oint_{S^1} \frac{dz}{z}(z^2 + \frac{1}{z^2} - 2)\hat{p}(z) + \epsilon \sum_{j=1}^{N} [(\rho_j^2 - \frac{1}{\rho_j^2})\cos 2\alpha_j - 2\ln \rho_j^2 .$$

The explicit form of the action-angle variables for all DEE, obtained from (4.1) with the involution $q(n)=\epsilon r(n)^*$ is obvious from (6.3). From (6.2) and (4.7) it follows that $C_p = C^*_{-p}$.

B. The modified difference Korteweg-de Vries (MDKDV) equation /20,21/

$$\frac{\partial q(n,t)}{\partial t} = [1-\epsilon\, q^2(n)][q(n+1) - q(n-1)], \qquad \epsilon = \pm 1, \tag{6.5}$$

is obtained from (4.1) with $f(z) = \frac{1}{z} - z$ provided that the involution $q(n) = \epsilon r(n)$ holds. This involution imposes the restrictions

$$a^+(z) = a^-(\frac{1}{z}), \qquad b^+(z) = -\epsilon\, b^-(\frac{1}{z}), \qquad |z| = 1,$$

$$z_j^+ = \frac{1}{z^-}, \qquad c_j^+ = \epsilon\, c_j^-\, z_j^{+2}. \tag{6.6}$$

Note, that on the invariant subspace $\mathcal{F}_\epsilon$ specified by the involution $q(n) = \epsilon\, r(n)$ the 2-form Ω_0 (5.4) vanishes, i.e., $\Omega_0|_{q=\epsilon r} \equiv 0$. Therefore we are to use another symplectic structure $\Omega_1|_{q=\epsilon r} \equiv -\Omega_{-1}|_{q=\epsilon r}$

$$\Omega_{MDKDV} = i\Omega_1|_{q=\epsilon r} =$$

$$= -2\epsilon \sum_{n=-\infty}^{\infty} [\, 2\delta q(n) \wedge \delta q(n+1) + \delta \ln h(n) \wedge \delta(\sum_{k=n}^{\infty} q(k) q(k-1))] = \tag{6.7}$$

$$= -\frac{2}{\pi}\int_0^{\pi} d\tau \sin 2\tau\, \delta(\ln(1-\epsilon\rho^+(e^{i\tau})\rho^+(e^{-i\tau}))\wedge\delta(-\frac{i}{2}\ln\frac{b^+(e^{i\tau})}{b^+(e^{-i\tau})}) +$$

$$+ \quad 2\sum_{j=1}^{N}\delta\, \mathrm{ch}(\zeta_j + i\omega_j)\wedge\delta(\xi_j + i\rho_j),$$

$$\ln s_j^+ = \zeta_j + i\omega_j, \quad \ln\frac{b_j^+}{\sqrt{V}} = \xi_j + i\rho_j . \tag{6.7}$$

The Hamiltonian of the DEE (6.5) is

$$H_{MDKDV} = C_0 - C_2 = -\sum_{n=-\infty}^{\infty}\{\ln(1-\epsilon(q(n))^2) +$$

$$+\epsilon q(n)q(n-2)[1-(q(n-1))^2] - \frac{1}{2}[q(n)q(n-1)]^2\} =$$

$$= -\frac{2}{\pi}\int_0^{\pi} d\tau\, \sin^2 2\tau \ln[1-\epsilon\rho^+(e^{i\tau})\rho^+(e^{-i\tau})] - \tag{6.8}$$

$$-\sum_{j=1}^{N}[\zeta_j + i\omega_j - \frac{1}{2}\mathrm{sh}\, 2(\zeta_j + i\omega_j)].$$

If, besides $q(n) = \epsilon r(n)$, we require $q(n) = q^*(n)$ then the scattering data will satisfy both (6.2) and (6.6). In this case the eigenvalues appear either in 4-tuples $(z_j^+, z_j^{+*}, -z_j^+, -z_j^{+*})$, or pairwise if among z_α^+ there occur real or purely imaginary numbers. Let us introduce

$$z_j^+ = e^{(\zeta_j + i\omega_j)/2}, \qquad b_j^+/\sqrt{V} = e^{\beta_j + i\rho_j}, \quad j = 1,\dots,N_1 ,$$

$$z_\alpha^+ = e^{\epsilon_\alpha/2}, \qquad b_\alpha^+/\sqrt{V} = e^{\gamma_\alpha}, \quad \alpha = 1,\dots,N_2 ,$$

$$z_\beta^+ = i e^{\eta_\beta/2}, \qquad b_\beta^+/\sqrt{V} = i e^{\rho_\beta}, \quad \beta = 1,\dots,N_3 .$$

In this case the 2-forms $\Omega^{(m)} = -\Omega^{(-m)} = i\Omega_m$ become real and we have

$$\Omega^{(m)} = \frac{1}{\pi}\int_0^{2\pi} d\tau\, \sin 2m\tau\, \delta\ln(1-\epsilon|\rho^+(e^{i\tau})|^2)\wedge\delta\arg b^+(e^{i\tau})$$

$$-\frac{2}{m}\sum_{j=1}^{N_1}\{\delta(\cos(m\omega_j)\mathrm{ch}(m\zeta_j))\wedge\delta\beta_j - \delta(\sin(m\omega_j)\mathrm{sh}(m\zeta_j))\wedge\delta\rho_j\} -$$

$$-\frac{2}{m}\sum_{\alpha=1}^{N_2}\delta\,\mathrm{ch}(m\epsilon_\alpha)\wedge\delta\gamma_\alpha - \frac{2}{m}\sum_{\beta=1}^{N_3}\delta\,\mathrm{ch}(m\eta_\beta)\wedge\delta\rho_\beta . \tag{6.9}$$

From (6.9) with $m=1$ we easily get the action-angle variables for the DEE (6.5) with real-valued $q(n)$. Note that in (6.5) we can change the variables $q(n)=\mathrm{tg}\,u(n)$ for $\epsilon=1$ and $q(n)=\mathrm{th}\,u(n)$ for $\epsilon=-1$. As a result the DEE (6.5) goes into

$$\frac{\partial u(n,t)}{\partial t}=\mathrm{tg}\,u(n+1)-\mathrm{tg}\,u(n-1),\quad \epsilon=1,$$
$$\frac{\partial u(n,t)}{\partial t}=\mathrm{th}\,u(n+1)-\mathrm{th}\,u(n-1),\quad \epsilon=-1. \tag{6.10}$$

The equivalence of (6.5) and (6.10) is obvious only for $\epsilon=-1$; for $\epsilon=1$ the change of variables $q(n)=\mathrm{tg}\,u(n)$ is not one-to-one.

Let us briefly discuss the transition to the continuous limit (from system (1.3) to Zakharov-Shabat system (1.1)). Let us introduce the parameter Δ such, that

$$z=e^{i\lambda\Delta},\quad q(n)=\Delta u(n\Delta),\quad r(n)=\Delta v(n\Delta),$$
$$f(n,z)=\phi(\Delta n,\lambda). \tag{6.11}$$

In the limit $\Delta\to 0$ for $\Delta n=x$ fixed, we see that (1.3) goes into (1.1), the operators $\Lambda_\pm$ (1.4) go into the corresponding operators $\Lambda_\pm$ (1.2), and (since $h(n)\underset{\Delta\to 0}{\to}1$) we obtain for the symplectic form the well known formula (see ref. /8/) multiplied by Δ. For the Hamiltonian H_{DNS} (6.4) we get

$$H_{DNS}\simeq\Delta^3\int_{-\infty}^{\infty}dx[-u_{xx}(x)u^*(x)+\epsilon|u(x)|^4]=\Delta^3 H_{NS}.$$

In order to get the nonlinear Schrödinger equation $i\frac{\partial u}{\partial t}+u_{xx}-2\epsilon|u^2|u=0$ we must also change the evolution parameter $t\to t/\Delta^2$. In our second example the continuous limit of eq. (6.5) coinsides with the modified Korteweg-de Vries eq. provided that we put in addition $x\to x-\Delta t$.

The transition matrix $S(z)$ (7) of the system (1.3) goes into the corresponding transition matrix $S(\lambda)$ of (1.1) with no additional multiplicative factors. In order to get the eigenvalues of the system (1.1) we must also rescale the eigenvalues of (1.3), e.g., $z_j^\pm=\exp(i\lambda_j^\pm\Delta)$.

The higher integrals of motion J_n for the system (1.1) can be obtained from suitable linear combinations of C_k, $k=0,\pm1,\ldots$ which is determined by the dispersion. Thus for the integral J_n of the system (1.1) which corresponds to dispersion λ^n we must consider the expression $(z^2-\frac{1}{z^2})^n$, which for $\Delta\to 0$ is of order$(\lambda\Delta)^n$. If we insert C_k instead of z^{2k} we obtain J_n in the leading term in the limit $\Delta \to 0$. Note also that in DEE we must change the evolution parameter $t \to t/\Delta^n$.

At the end we note that there exist multicomponent analogs of the DEE solvable through the ISM. The corresponding linear problem is of the form:

$$\psi(n+1, z) = \begin{pmatrix} z\, 1_s, & q(n) \\ r^T(n) & \frac{1}{z} 1_p \end{pmatrix} \psi(n,z),$$

where $q(n)$ and $r(n)$ are $s\times p$ matrices. The analogical constructions in this case lead to non-trivial complications. Here we write down only the operator, analogous to (1.4); for brevity we take the vector case ($p=1$):

$$\Lambda_\pm X(n) = \Lambda_2^\pm \Lambda_1^\pm X(n), \qquad X = \begin{pmatrix} X_1(n) \\ X_2(n) \end{pmatrix}, \; X_i(n) = \begin{pmatrix} X_i^{(1)} \\ \vdots \\ X_i^{(s)} \end{pmatrix}, \quad i=1,2$$

$$\Lambda_1^\pm X(n) = \begin{pmatrix} h(n) X_1(n) \\ h(n-1) X_2(n-1) \end{pmatrix} \pm \Sigma_n^\pm \begin{pmatrix} X_1(k) r^T(k) q(n) + q(k) X_2^T(k) q(n) \\ -r(k) r^T(n-1) X_1(k) - X_2(k) r^T(n-1) q(k) \end{pmatrix}$$

$$\Lambda_2^\pm X(n) = \begin{pmatrix} X_1(n+1) \\ X_2(n) \end{pmatrix} \pm \frac{1}{h(n)} \begin{pmatrix} q(n) \\ -r(n) \end{pmatrix} \Sigma_{n+1}^\pm (r^T(k-1) X_1(k) + q^T(k) X_2(k)),$$

$$h(n) = 1 - r^T(n) q(n).$$

Another way of generalization is to consider the Z_2-graded case. The structure of the corresponding Hamiltonian systems is much more complicated. These problems will be considered in a separate paper.

REFERENCES

1. Gardner C.S. et al. Phys.Rev.Lett., 1967, 19, p. 1095; P.Lax Commun. Pure Appl.Math., 1968, 21, p. 467.
2. Zakharov V.E., Shabat A.B. JETP (in Russian), 1971, 61, p 118.
3. Ablowitz M., et al. Studies in Appl.Math., 1974, 53, p.249.
4. Calogero F., Degasperis A. Nuovo Cimento, 1976, 32B, p.201.
5. Ablowitz M. Studies in Appl.Math., 1978, 58, p. 17.
6. Zakharov V.E. et al. "Soliton Theory. The Inverse Problem Method". (in Russian), M., "Nauka", 1980.

7. Zakharov V.E., Faddeev L.D. Funct.Anal. and Appl. (in Russian), 1971, 5, p. 18.
8. Zakharov V.E., Manakov S.V. TMF, (in Russian), 1974, 19, p. 332.
9. Gerdjikov V.S., Kulish P.P. Bulg.Journ. of Phys. (in Russian), 1978, 5, p. 337.
10. Fadeev L.D., Takhtadjan L.A. TMF (in Russian), 1974, 21, p. 160; Trudy MIAN USSR (in Russian), 1976, 142, p. 254.
11. Manakov S.V. JETP (in Russian), 1974, 67, p. 543.
12. Faddeev L.D. In:"Solitons", ed. Bullough. Springer Verlag 1979.
13. Kaup D.J. Journ. of Math.Ann. and Appl., 1976, 54, p. 849.
14. Kaup D.J., Newell A.C. Adv. in Math., 1979, 31, p. 67.
15. Gerdjikov V.S. Khristov E.Kh. Mat.Zametki (in Russian), 1980, 28, p. 501; JINR, E2-12731, Dubna, 1979.
16. Gerdjikov V.S., Khristov E.Kh. JINR, E2-12-742, Dubna, 1979; Bulgarian Journ. Phys. (in Russian), 1980, 7, p.119.
17. Kulish P.P., Reiman A.G. Zapiski Nauchnich Seminarov LOMI (in Russian), Leningrad, USSR, 1978, 77, p. 134.
18. Gerdjikov V.S., Ivanov M.I., Kulish. TMF (in Russian), 1980, 44, p. 342.
19. Ablowitz M., Ladik J.F. Journ.Math.Phys., 1975, 16, p.598.
20. Ablowitz M., Ladik J.F. Journ.Math.Phys., 1976, 17, p. 1011.
21. Chiu S.C., Ladik J.F. Journ.Math.Phys., 1977, 18, p.690.
22. Levi D., Ragnisco O. Preprint Instituto di Fisica G.Marconi Universita di Roma, 1978, No. 100.
23. Faddeev L.D. Preprint LOMI (in Russian), P2-79, 1979.
24. Sklyanin, Preprint LOMI, E-3-79, 1979.

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