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J.Hruby

A POSSIBILITY FOR OBTAINING CONSTRAINTS IN EXTENDED SUPERSYMMETRY

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The study of the two-dimensional supersymmetric models, in which fields appear in a nonlinear representation, was started in works [1].

A conserved supercurrent in models with supersymmetry solitons has the form [2]:

$$\partial_\mu [(\partial\!\!\!/\phi - V'(\phi))\gamma^\mu \psi] = 0 . \qquad (1)$$

From (1) it is possible to obtain supercharges Q_α, $\alpha=1,2$, and show that the usual anticommutation relation is invalid [3]. The true relation has the form:

$$\{Q_\alpha, Q_\beta\} = 2(\gamma . P\gamma_0)_{\alpha\beta} + 2iT(\gamma_5\gamma_0)_{\alpha\beta} , \qquad (2)$$

where the central charges exist due to the nonlinear boundary condition and $T = \int dx V'(\phi) \frac{d\phi}{dx}$.

It is interesting why in such supersymmetric models the central charges appear. It is well known that the usual superalgebra can be modified to include central charges, as it has been shown mathematically by R.Haag et al. [4], in the case of extended supersymmetry.

The examples of the nonlinear supermodels, which are invariant under O(2) extended supersymmetry, are the super-O(3) sigma model and super-CP^{n-1} models [5,6]. These sigma models and their supersymmetric generalization have deep group-geometrical nature [7].

From our point of view they are interesting in the case when O(3) sigma model and CP^1 model are equivalent [8] and moreover they are equivalent to the sine-Gordon model [9]. It was explicitly shown in [10] that these models are equivalent also in supersymmetric case.

For our purpose to show the relation between N=2 extended supersymmetry and complex numbers, we have to start at the Bose level with the CP^1 model. This is a $SU(2)/SU(1)\times U(1)$ model, where the coset space $SU(2)/U(1)$ can be identified with the one-dimensional complex projective space, CP^1 involves two complex fields $\phi_i(x)$, $i=1,2$ and $\bar{\phi}_i(x) \in SU(2)/U(1)$. The fields $\phi_i(x)$ satisfying a constraint $\bar{\phi}_i \phi^i = 1$ and two fields related by the U(1) gauge transformation

$$\phi'_i(x) = e^{i\lambda(x)}\,\phi_i(x) \qquad (3)$$

should be considered equivalent.

It can be also interpreted so that the automorphism group U(1), which preserves the norm of complex numbers, is the gauge group.

The U(1) local gauge-invariant action of the model has the form /8/:

$$S = \frac{1}{2}\int d^2x\,(\overline{D_\mu \phi_i})(D_\mu \phi^i), \qquad (4)$$

where $D_\mu = \partial_\mu + iA_\mu$ and the Abelian gauge field A_μ has the form $A_\mu = \frac{i}{2}\bar{\phi}_i \overleftrightarrow{\partial}_\mu \phi^i$ and transforms under (3) like $A'_\mu = A_\mu - \partial_\mu \lambda$.

If we want to achieve our aim, we have to construct super-CP^1 model directly in U(1)-gauge-invariant supersymmetric way using the connection with complex numbers and functions. The gauge group U(1), which preserves the norm of complex numbers, will give complex supersymmetry that is equivalent to O(2) real extended supersymmetry.

Here "real" means that the Grassmann variables θ_i, $i = 1,2$, are real anticommuting Majorana spinors and "complex" means the complex composition of two real variables in the full analogy between real and complex numbers.

By analogy with the complex function we shall write a complex superfield $C(x,\theta,\bar\theta) \equiv C(x,\theta_1 + i\theta_2, \theta_1 - i\theta_2)$. The supersymmetry transformation on the complex superspace $(x,\theta,\bar\theta)$ was first defined in two dimensions by M.Ademollo et al. /11/:

$$\delta x_\mu = -\frac{i}{2}[\epsilon\gamma_\mu\bar\theta + \bar\epsilon\gamma_\mu\theta], \quad \delta\theta = \epsilon, \quad \delta\bar\theta = \bar\epsilon,$$

and on the superfields $C(x,\theta,\bar\theta)$ acts as follows:

$$\delta C = [\epsilon Q + \bar\epsilon \bar Q]C, \qquad (5)$$

where

$$Q = \frac{\partial}{\partial\theta} - \frac{i}{2}\not\partial\bar\theta, \quad \bar Q = \frac{\partial}{\partial\bar\theta} - \frac{i}{2}\not\partial\theta, \quad \not\partial = \gamma.\partial.$$

These supercharges anticommute with the covariant derivatives:

$$D = \frac{\partial}{\partial\theta} + \frac{i}{2}\not\partial\bar\theta = \frac{1}{2}(D^1 - iD^2), \quad \bar D = \frac{\partial}{\partial\bar\theta} + \frac{i}{2}\not\partial\theta = \frac{1}{2}(D^1 + iD^2).$$

We decompose the complex superfield into the real and imaginary part: $C(x,\theta,\bar\theta) = A(x,\theta_1,\theta_2) + iB(x,\theta_1,\theta_2)$. Following /12/ by analogy with complex functions Cauchy-Riemann equations give:

$$D^1 A = D^2 B, \qquad (6a)$$

$$D^1 B = -D^2 A. \qquad (6b)$$

It means the complex superfield will be an analytical superfield [12], when (6a,b) are valid. But it means that the chirality condition

$$\bar{D}C = 0 \tag{7}$$

in [11] is the analyticity condition. There this restriction was obtained using the new shifted variables $x' = x - \frac{i}{2}\theta\bar{\theta}$ (a complex Bose variable).

The condition (7) actually plays a role of the invariant constraint:

$$C(x, \theta, \bar{\theta}) = C(x - \frac{i}{2}\bar{\theta}\gamma\theta, \theta), \tag{8}$$

what means that the graded Lie algebra in complex supersymmetry can be realized in a smaller parametric superspace with the complex Bose variable but independent of the spinor $\bar{\theta}$.

In analogy with the Laplace equation for real and imaginary part of the complex function the superfield equation of motion follows:

$$\bar{D}DC = 0. \tag{9}$$

The corresponding action has the form:

$$S = \frac{1}{8}\int d^2x\, d^2\theta\, d^2\bar{\theta}\, \bar{C}C. \tag{10}$$

If we want to have the action (10) also local U(1)-gauge-invariant, we have to use the receipt given in [13]. We have to introduce the vector superfield $V(x, \theta, \bar{\theta})$, which transforms as

$$V \to V + i(\bar{\Lambda} - \Lambda), \tag{11}$$

under the U(a) gauge transformation:

$$C \to e^{i\Lambda} C, \quad \bar{C} \to e^{-i\Lambda} \bar{C}, \tag{12}$$

where Λ is also an analytical superfield $(\bar{D}\Lambda = 0)$.

It can be shown, after [13] that the action

$$S = \frac{1}{8}\int d^2x\, d^2\theta\, d^2\bar{\theta}\, [V - \bar{C}Ce^{V}] \tag{13}$$

in supersymmetric and U(1)-gauge-invariant. The action (13) is exactly the action given in [6], but obtained in a general way using the analyticity of the superfields.

The constraints will result from the equation of motion for the vector superfield

$$\bar{C}C = e^{-V} . \tag{14}$$

In this way we have connected all components in right- and left-hand side of eq. (14) and so vector superfield V acts as a confining force between the scalar superfields.

The action (13) is equivalent to the action of the super-CP^1 model, which is obtained by direct supersymmetrization ($\partial_\mu \to D_\alpha$, $\phi_i(x) \to \Phi_i(x,\theta)$) of the Bose action (4) and the constraint $\bar{\phi}_i \phi^i = 1$ /6/. This can be generalized also for non-Abelian super-CP^{n-1} models /14/.

So we demonstrated two things:

i) We showed that in the supermodels with topological excitation, which are equivalent to the super-CP^1 model, the central charges appear due to the O(2) extended supersymmetry and the relation:

$$\{D^1_\alpha, D^2_\beta\} = 2iT(\gamma_5\gamma_0)_{\alpha\beta} \tag{15}$$

must be valid. Actually in O(2) extended supersymmetry the central charge T is proportional to the mass parameter /12/. The Lagrangian from /1/ in O(2) real extended supersymmetry has the form:

$$L = \frac{1}{2}\partial_\mu \phi_i \partial_\mu \phi^i + \frac{i}{2}\psi_i \not\partial \psi^i - \frac{1}{2} F_i F^i - F_i V'(\phi) + \frac{i}{2}\psi_i \psi^i V'', \tag{16}$$

where the dummy field $F_i = -V'(\phi_i)$ and we shall assume $V'(\phi_i) = -\frac{i}{2}\phi_i \psi_j \psi^j, V'' = \frac{dV'(\phi_i)}{d\phi_i} = -\frac{i}{2}\psi_j\psi^j$, the O(2) labels $i,j=1,2$. The Lagrangian density (16) is equivalent to the

$$L = \frac{1}{2}\partial_\mu \phi_i \partial_\mu \phi^i + \frac{i}{2}\psi_i \not\partial \psi^i + \frac{1}{8}(\psi_i \psi^i)^2, \tag{17}$$

when we used $\phi_i\phi^i=1$. In the case of equivalence the central charge has the form:

$$T = -\frac{i}{2}\int dx \frac{\partial}{\partial x}(\psi_j \psi^j), \tag{18}$$

which is in agreement with /15/.

ii) Based on the deep mathematical relation between complex numbers and complex supersymmetry we have shown how we could obtain the constraints in extended supersymmetry for N=2 automatically. We get two types of constraints:

1) The first type of constraints is given in complex supersymmetry by the analyticity. Actually for the supersymmetric constraint from /16/ follows:

$$(D^1D^1 - D^2D^2 + 2iD^1D^2)A = 0, \qquad (19)$$

which can be seen from the(1.6a,b) and the anticommutativity of the supercovariant derivatives. It means that the constraint (19) for $A=\frac{1}{2}(C+\bar{C})$ singles out the purely analytical and antianalytical part;

2) The second type of constraints is dictated by the invariance of the superaction (13) under the group U(1) (it has no relation to automorphism group of supersymmetry), where the U(1) vector superfield acts as a binding force.

Let us forget for a moment that we have built the super-CP^1 model. Then we can see that the construction of the complex supersymmetry is quite general and has been extended to four-dimensional space-time in /12/.

We shall generalize this successful coincidence between complex numbers and complex supersymmetry (N=2 extended supersymmetry) for other systems of numbers (quaternions and octonions). The idea to use hypercomplex numbers in supersymmetry is not new /17,12/ but here is presented a possibility for obtaining constraints in extended supersymmetry via this generalization.

As the extension of the real number is the complex number, the extension of complex number is quaternion and the last extension gives octonion, because the Hurwitz theorem is valid /18/. This is also the last step in the extended supergravity models, if we want to have spin 2 for graviton /19/.

The most interesting case is N=8, because in an extended supergravity model one has unified theories of fields incorporating all spins. From a physical point of view in this case there are two most interesting groups: G_2 as the automorphism group of the octonionic algebra and SO(8), which is the group of invariance of a real bilinear form of octonion and also of the norm. But SO(8) actually plays an important role in the extended supergravity with N=8.

Here is the new possibility to construct superaction with extended supersymmetries (N=4,8) using quaternionic or octonionic superfields respectively. The problem is to give constraints of the first type for hypercomplex superfields in such a way to be invariant under the automorphism group SU(2), SO(8). This problem will be discussed in further publication on the basis of Fueter's hypercomplex analysis in extended superspace.

Another possibility for obtaining constraints in extended supersymmetry is to use the constraint of the second type. It gives a possibility to construct supersymmetric SU(5)/SU(4)x xU(1) model, where the global group SU(5) is minimal group for grand unification theory[20].

It also exists the connection between the local gauge groups of CP^{n-1}, HP^{n-1}, CaP(2) models[21] and automorphism groups of extended supersymmetries with $N=2,4,8$ respectively.

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