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ON THE SUPER-SIGMA MODELS AND SUPER-SINE-GORDON MODEL

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1. INTRODUCTION

The study of models in which fields appear in a nonlinear representation played an important role in several areas in particle physics. Great progress has been achieved in two-dimensional field theories with the bosonic sector only. However, models with classical fermion fields are of great interest as well.

As is well known, it is most natural to consider classical fermion fields as anticommuting variables and via a supersymmetric extension to obtain both the bosonic and fermionic sector. Some models containing anticommuting fields in an extension of this type have been considered: the supersymmetric sine-Gordon model /1/, the supersymmetric σ model /1,2/ and CP^{n-1} model /3/ and also other nonlinear supermodels.

Recently a general method has been proposed to build large classes of bidimensional supersymmetry models, whose equation of motion can be interpreted as the integrability condition of a set of linear equations /4/.

There is shown a close relationship between the group-theoretical structure and geometric interpretation in the supersymmetric case using the modern mathematical language of the differential forms.

In this language the equations of motion in the Bose bidimensional nonlinear models appear as the integrability condition of the following linear set /5/:

$$dv = \omega v, \tag{1.1}$$

where v is a column of 0-forms, $\omega = \omega_\mu dx^\mu, \mu = 0,1$ is a matrix of 1-forms (in the two-dimensional space-time (x_0, x_1)) and d is the exterior derivative.

The integrability condition of the set (1.1) has the form:

$$ddv = (d\omega - \omega \wedge \omega)v = 0, \tag{1.2}$$

that is, $d\omega - \omega \wedge \omega = 0$, where the symbol $\wedge$ means the exterior product of differential forms /6/.

The aim is to find the expression for ω, as a function of certain fields such that eq. (1.2) gives second-order differential equations in these fields.

For this purpose the 1-form matrix ω is assumed with values in the algebra **G** of the Lie group G:

$$\omega = \omega_a I^a , \qquad (1.3)$$

where ω^a are 1-forms and I^a are the generators of the group G, in some representation. Then eq. (1.2) can be written in the form:

$$d\omega^c = \frac{1}{2} c^c_{ab} \omega^a \wedge \omega^b , \qquad (1.4)$$

(Maurer-Cartan eq.), where the structure constants of G are defined by the commutation relations:

$$[I_a , I_b] = c^c_{ab} I_c . \qquad (1.5)$$

An important property of eq. (1.2) is gauge invariance; it means eq. (1.2) is invariant under the local gauge transformation

$$\omega \to \omega' = g^{-1} \omega g - g^{-1} dg \qquad (1.6)$$

for any $g(x) \in G$.

There can be considered a subgroup H of the group G such, that $G = H + F$ and $F = G/H$ is the symmetric coset space. The corresponding symmetric decomposition of the Lie algebra **G** is $\boldsymbol{G} = \boldsymbol{H} + \boldsymbol{F}$, **H** is the subalgebra, and the following commutation relations hold:

$$[\boldsymbol{H}, \boldsymbol{H}] \subset \boldsymbol{H} ,$$

$$[\boldsymbol{H}, \boldsymbol{F}] \subset \boldsymbol{F}, \qquad (1.7)$$

$$[\boldsymbol{F}, \boldsymbol{F}] \subset \boldsymbol{H}.$$

In terms of the matrices ω the decomposition can be written:

$$\omega = \omega^i H_i + \omega^\alpha F_\alpha , \qquad (1.8)$$

where H_i , F_α are generators of **H**, **F** respectively. By substituting the decomposition (1.8) in eq. (1.2) we obtain the relations:

$$d\omega^{a} = c^{a}_{i\beta}\,\omega^{i} \wedge \omega^{\beta}, \tag{1.9a}$$

$$d\omega^{i} = \frac{1}{2} c^{i}_{jk}\,\omega^{j} \wedge \omega^{k} + \frac{1}{2} c^{i}_{a\beta}\omega^{a} \wedge \omega^{\beta}, \tag{1.9b}$$

where $c^{a}_{i\beta}, c^{i}_{jk}, c^{i}_{a\beta}$ are the structure constants, defined by the commutation relations:

$$[H_i, H_j] = c^{k}_{ij} H_k, \quad [H_i, F_a] = c^{\beta}_{ia} F_\beta, \quad [F_a, F_\beta] = c^{i}_{a\beta} H_i .$$

S.Sciuto proposed a general method to build a supermodel in this formalism /7/. This method is given by a direct supersymmetric extension of the calculus of differential forms to superspace. The extension is based on the definition of the "truncated" 1-forms (t1-forms) /7/:

$$\hat{E} = d\theta^{a} E_{a}, \tag{1.10}$$

where $d\theta^{a}$ are the differentials of the Grassmann variables, $\theta^{a}(a=1,2)$ and $E_a(x,\theta)$ are the superfields (bosonic or fermionic), defined on the four-dimensional superspace.

Using the language of truncated forms, the linear set associated with a super-bidimensional model can be written as

$$\hat{D} V = \hat{\Omega} V; \tag{1.11}$$

V is a column of 0 forms, $\hat{\Omega}$ is a matrix of t1-forms and $\hat{D}$ is the truncated exterior derivative:

$$\hat{D} = d\theta^{a} D_{a} = d\theta^{a}\left(\frac{\partial}{\partial \bar{\theta}} + i\partial\!\!\!/\theta\right)_{a}, \quad \partial\!\!\!/ = \gamma.\partial, \tag{1.12}$$

The integrability condition of the set (1.11) is

$$\hat{D}\hat{\Omega} - \hat{\Omega} \wedge \hat{\Omega} \doteq 0, \tag{1.13}$$

where the symbol $\doteq$ means that the coefficient of $d\theta_1 \wedge d\theta_2$ on the left-hand side must vanish.

R.D.`Auria and S.Sciuto have shown /4/ that in the supersymmetric models the t1-form matrix $\hat{\Omega}$ takes values in a graded Lie algebra $\hat{G}$ and eq. (1.13) is invariant under the local supergauge transformation. The important thing is that the decomposition $\hat{G} = G + Q$ into the even and odd part is automatically symmetric in the sense of the (anti-) commutation relations:

$$[\mathbf{G},\mathbf{G}]\subset\mathbf{G},$$
$$[\mathbf{G},\mathbf{Q}]\subset\mathbf{Q},$$
$$\{\mathbf{Q},\mathbf{Q}\}\subset\mathbf{G}. \qquad (1.14)$$

The actual construction of a supermodel with associated linear set is only a matter of fixing a graded group $\hat{G}$ and of choosing a suitable matrix $\hat{\Omega}$, which takes values in $\mathbf{G}$, such that $\hat{\Omega}$ contains a submatrix ω corresponding to the Bose model. For this aim we have to consider the following decomposition of $\hat{\mathbf{G}}$ [4]:

$$\hat{\mathbf{G}}=\mathbf{G}+\mathbf{Q},$$
$$\mathbf{G}=\mathbf{H}+\mathbf{F}, \qquad (1.14')$$
$$\mathbf{Q}=\mathbf{Q}'+\mathbf{Q}'',$$

with the (anti-) commutation relations:

$$[\mathbf{G},\mathbf{Q}]\subset\mathbf{Q},\ [\mathbf{H},\mathbf{Q}']\subset\mathbf{Q}',\ [\mathbf{H},\mathbf{Q}'']\subset\mathbf{Q}'',$$
$$\{\mathbf{Q}',\mathbf{Q}'\}\subset\mathbf{F},\ \{\mathbf{Q}'',\mathbf{Q}''\}\subset\mathbf{F}, \qquad (1.15)$$
$$\{\mathbf{Q}',\mathbf{Q}''\}\subset\mathbf{H},$$

plus the commutation relation (1.7).

After this short review in using the differential-form formalism for bidimensional nonlinear models and supermodels, we concentrate our attention on the two of them: the classical O(3) invariant σ model (equivalent to the $\mathbf{CP}^1$ model) and the sine-Gordon model and their supersymmetric extension.

The reason for studying these two models together are as follows:

i) both these models have geometric nature,

ii) the equivalence between the O(3) symmetric nonlinear σ model and the sine-Gordon model exists [8] in the Bose case and must exist in a supersymmetric extension,

iii) the O(3) symmetric nonlinear model has the most striking similarities with four-dimensional gauge theories [9],

iv) in the superalgebra of the super-sine-Gordon model a "central charge" appears [10] and, analogously, it must be in the super-O(3) σ model.

In this paper we shall proceed as follows: we repeat the basic facts about the O(3) invariant σ model in the Bose sector in Sec. 2. Using the language of differential forms, whose short review is given in the introduction, we show here the connection with the sine-Gordon model.

The super-O(3) sigma model will be described in a self-consistent way in Sec. 3. There the invariance of this model under complex supersymmetry will be shown and constraints will be obtained straight-forwardly from the gauge invariant action.

In Sec. 4 the correspondence between the super-O(3) σ model and the super-sine-Gordon model will be shown, using the equivalence between the super-O(3) σ model and super- $\mathbf{CP}^1$ model. Working with the Lagrangian formalism and Noether's supercurrents the presence of the central charges will also be demonstrated.

Some of our results will partly overlap with works of the other authors, however, we think that in this work we obtain new relations, which could be usefull.

2. THE O(3) INVARIANT NONLINEAR σ MODEL AND SINE-GORDON MODEL IN TWO DIMENSIONS

The action of the O(3) σ model has the form:

$$S = \frac{1}{2} \int d^2x \, (\partial_\mu \phi^a \cdot \partial_\mu \phi^a) = \frac{1}{2} \int d^2x \sum_{a=1}^{n=3} (\partial_\mu \phi^a)^2 , \tag{2.1}$$

where the classical fields $\phi^a(x_0, x_1)$, a =1, 2, 3, fullfil the constraint:

$$\sum_{a=1}^{n=3} (\phi^a)^2 = \phi^a \cdot \phi^a = 1. \tag{2.2}$$

The corresponding Lagrangian density takes the simple form:

$$L = \frac{1}{2} \partial_\mu \phi^a \cdot \partial_\mu \phi^a + \frac{\Lambda}{2} (\phi^a \cdot \phi^a - 1), \tag{2.3}$$

Λ is a Lagrange multiplier. The corresponding Euler-Lagrange equation of motion has the form:

$$\Box \phi^b + (\partial_\mu \phi^a \cdot \partial_\mu \phi^a) \phi^b = 0, \quad \phi^a \cdot \phi^a = 1. \tag{2.4}$$

Using light-cone coordinates $x_\pm = \frac{1}{2}(x_0 \pm x_1)$,as is usually done in two-dimensional space-time, eq. (2.4) can be written as (see Appendix)

$$\phi^{b}_{+-} + (\phi^{a}_{+} \cdot \phi^{a}_{-})\phi^{b} = 0,$$

with the notation $\partial_{\pm}\phi^{a} = \phi^{a}_{\pm}$.

Eq. (2.4) is invariant under the local scale transformation:

$$x'_{\pm} = |F_{\pm}(x_{\pm})|\, x_{\pm} \tag{2.5}$$

and F_{+} (F_{-}) is a function of $x_{+}(x_{-})$ respectively, $F_{\pm}(x_{\pm}) \neq 0$. This invariance implies that the energy-momentum density

$$\theta_{\mu\nu} = \partial_{\mu}\phi^{a} \cdot \partial_{\nu}\phi^{a} - \frac{1}{2} g_{\mu\nu} \partial_{\Lambda}\phi^{a} \cdot \partial^{\Lambda}\phi_{a}$$

is traceless and has only two independent components. The energy momentum conservation says that

$$\{\frac{1}{2}\phi^{a}_{-} \cdot \phi^{a}_{-}\}_{+} = 0 = \{\frac{1}{2}\phi^{a}_{+} \cdot \phi^{a}_{+}\}_{-},$$

where symbols $\pm$ mean again the derivation of the expression in the brackets.

Hence $\phi^{a}_{\pm} \cdot \phi^{a}_{\pm} = \sum_{a=1}^{3} (\partial_{\pm}\phi^{a})^{2} = f^{2}_{\pm}(x_{\pm})$ where $f_{+}(f_{-})$ is a new function of $(x_{+}(x_{-})$ respectively, but we may choose:

$$|F_{\pm}(x_{\pm})| = |f_{\pm}(x_{\pm})|.$$

This ammounts to an identically vanishing momentum density and a constant energy density ($= \frac{1}{2}$) in the new coordinates $x'_{\pm}$. In this new system only one of the O(3) invariants formed from ϕ^{a} and $\phi^{a}_{\pm}$ is undetermined, namely $(\phi^{a}_{+} \cdot \phi^{a}_{-})$, and the following relations hold:

$$\phi^{a} \cdot \phi^{a} = 1,\ \ \phi^{a}_{+} \cdot \phi^{a}_{+} = 1,\ \ \phi^{a}_{-} \cdot \phi^{a}_{-} = 1,\ \ \phi^{a}_{\pm} \cdot \phi^{a} = 0,\ \ -1 \leq (\phi^{a}_{+} \cdot \phi^{a}_{-}) \leq 1, \tag{2.6}$$

We denote $\phi^{a}_{+} \cdot \phi^{a}_{-} = \cos\phi$, where $\phi \neq \phi^{a}$ is a scalar field.

From the geometrical point of view the action (2.1) can be interpreted (for arbitrary $n \geq 1$) in the following way/11/:

Let G be a compact Lie group and H some closed subgroup of G. Then the action (2.1) describes the model of fields $\phi^{a}(x_{0}, x_{1})$, $a = 1, \ldots, n$, which takes values in the coset space G/H. The group G acts on such fields according to the transformation law:

$$\phi'^{a}(x) = g\phi^{a}(x), \quad g \in G, \tag{2.7}$$

and the action (2.1) is invariant under the transformation law (2.7). The expression $(\partial_\mu\phi^a\cdot\partial_\mu\phi^a)$ then denotes the length squared of ϕ^a with respect to the chosen G-invariant metric in G/H.

In our case for the O(3) invariant σ model $G=SO(3)$ and $H=SO(2)$; the action has the form (2.1) and $\phi^a\in SO(3)/SO(2)$ that is the symmetric coset space.

If we take $G=SU(2)$ and $H=U(1)$, then $SU(2)/U(1)$ can be identified with the one-dimensional complex projective space CP^1. So we obtain the CP^1 model, which involves the complex fields $\phi_i(x)$, $i=1,2$ satisfying a constraint $\phi_i^*\cdot\phi_i=1$. In addition a local U(1) gauge invariance

$$\phi_i(x)\to e^{i\Lambda(x)}\phi_i(x) \tag{2.8}$$

is imposed, for arbitrary space-time dependent $\Lambda(x)$. The Lagrangian of the CP^1 model with this invariance has the form:

$$L=\frac{1}{2}[\partial_\mu\phi_i^*\cdot\partial_\mu\phi_i+(\phi_i^*\cdot\partial_\mu\phi_i)(\phi_j^*\cdot\partial_\mu\phi_j)]. \tag{2.9}$$

The local U(1) gauge invariance can be made more obvious by introducing an Abelian gauge field A_μ and writing

$$A_\mu=\frac{i}{2}(\phi_i^*\partial_\mu\phi_i-\partial_\mu\phi_i^*\cdot\phi_i), \tag{2.10}$$

which transforms under (2.8) like

$$A'_\mu=A_\mu-\partial_\mu\Lambda. \tag{2.11}$$

Then the action can be written in gauge invariant form:

$$S=\frac{1}{2}\int d^2x(D_\mu\phi_i)^*\cdot(D_\mu\phi_i);\quad D_\mu=\partial_\mu+iA_\mu . \tag{2.12}$$

This model has the obvious symmetry group SU(2), corresponding to rotations of the ϕ_i (some authors have termed it the SU(2) σ model). The equivalence with O(3) σ model is given by introduction of the vector $\vec{\phi}=\phi_i^*\vec{\sigma}_{ik}\phi_k$ and constraints $\phi_i^*\cdot\phi_i=1$ become $\phi_a\cdot\phi_a=1$. It means that ϕ_i is the spinor representation of ϕ_a ($\vec{\sigma}$ -Pauli matrices).

Now we express the O(3) σ model in terms of the differential forms. Eq. (1.4) for the group SO(3) is:

$$d\omega^c = \frac{1}{2}\epsilon^{abc}\,\omega^a \wedge \omega^b\ , \qquad (2.13)$$

where indices a, b, c equal 1, 2, 3.

By selecting ω^3 as the connection of the subgroup $H = SO(2)$, we obtain from eqs. (1.9 a,b):

$$d\omega^\alpha = -\epsilon^{\alpha\beta}\,\omega^3 \wedge \omega^\beta, \qquad (2.14a)$$

$$d\omega^3 = \frac{1}{2}\epsilon^{\alpha\beta}\omega^\alpha \wedge \omega^\beta = \omega^1 \wedge \omega^2\ , \qquad (2.14b)$$

where $\alpha, \beta = 1, 2$ and $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha},\ \epsilon^{12} = 1.$

Using light-cone coordinates and performing a Lorentz boost $(x^+ \to \Lambda x^+,\ x^- \to \Lambda x^-)$ on the 1-forms ω^α, ω^3 we get /5/.

$$\omega^\alpha(\Lambda) = \Lambda^{-1} f^\alpha dx^- + \Lambda g^\alpha dx^+\ , \qquad (2.15a)$$

$$\omega^3(\Lambda) = \omega^3_- dx^- + \omega^3_+ dx^+ . \qquad (2.15b)$$

By substituting eqs. (2.15a, b) into eqs. (2.14a, b) we obtain two sets of decoupled constraint equations:

$$\partial_- g^\alpha = -\epsilon^{\alpha\beta}\omega^3_- g^\beta, \qquad (2.16a)$$

$$\partial_+ f^\alpha = -\epsilon^{\alpha\beta}\omega^3_+ f^\beta, \qquad (2.16b)$$

which result in $f^\alpha f^\alpha = $ const. and $g^\alpha g^\alpha = $ const. Using scale invariance and the fact that only one of the remaining invariants is undetermined, namely, the scalar product $f^\alpha g^\alpha$, we can write:

$$f^\alpha f^\alpha = g^\alpha g^\alpha = 1, \quad f^\alpha f^\alpha_\pm = 0 = g^\alpha g^\alpha_\pm\ , \quad f^\alpha g^\alpha = \cos\phi. \qquad (2.17)$$

We can choose

$$f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad g = \left(\frac{\phi^a_+ \cdot \phi^a_-}{\sqrt{1 - (\phi^a_+ \cdot \phi^a_-)^2}} \right), \qquad (2.18)$$

and the relations (2.17) are fullfilled. From eqs. (2.16a,b) the determination of ω^3 follows:

$$\omega_+^3 = 0, \qquad \omega_-^3 = -\frac{(\phi_+^a \cdot \phi_-^a)_-}{\sqrt{1-(\phi_+^a \cdot \phi_-^a)^2}}. \tag{2.19}$$

The 1-forms ω^1, ω^2 are given by the relation (2.15a):

$$\left.\begin{aligned} \omega^1 &= \Lambda^{-1} dx^- + \Lambda(\phi_+^a \cdot \phi_-^a) dx^+ \\ \omega^2 &= \Lambda\sqrt{1-(\phi_+^a \cdot \phi_-^a)^2}\, dx^+ \end{aligned}\right\} \tag{2.20}$$

and the 1-form ω^3 follows from (2.15b) and (2.19):

$$\omega^3 = -\frac{(\phi_+^a \cdot \phi_-^a)_-}{\sqrt{1-(\phi_+^a \cdot \phi_-^a)^2}} dx^-. \tag{2.21}$$

The equations of motion (2.14a, b) become:

$$\partial_- \omega_+^3 - \partial_+ \omega_-^3 = \omega_-^1 \omega_+^2 - \omega_+^1 \omega_-^2 , \tag{2.22}$$

what gives

$$\left\{ \frac{(\phi_+^a \cdot \phi_-^a)_-}{\sqrt{1-(\phi_+^a \cdot \phi_-^a)^2}} \right\}_+ = \sqrt{1-(\phi_+^a \cdot \phi_-^a)^2} \quad . \tag{2.23}$$

Using the relations in (2.6) it can be shown that eq. (2.22) is equivalent to the equation

$$\phi_{+-}^b + (\phi_+^a \cdot \phi_-^a)\phi^b = 0.$$

For this purpose the calculation from ref.[8] must be used.
Moreover from eq. (2.22) we obtain the following:

$$\left\{ \frac{(\phi_+^a \cdot \phi_-^a)_-}{\sqrt{1-(\phi_+^a \cdot \phi_-^a)^2}} \right\}_+ = -\{\partial_- \arccos(\phi_+^a \cdot \phi_-^a)\}_+ = -\partial_+ \partial_- \phi = \sqrt{1-\cos^2\phi},$$

$$\partial_+ \partial_- \phi = -\sin\phi , \tag{2.24}$$

but it is the known sine-Gordon equation in terms of light-cone variables.

Thus we have shown the known result [8] that the O(3) σ model is equivalent to the sine-Gordon model, which is given by the action:

$$S = \int d^2x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{\beta} (\cos \beta\phi - 1) \right], \qquad (2.25)$$

where we have put $m = \beta = 1$.

This result means that the sine-Gordon model in this realization can be interpreted as a $SO(3)/SO(2)$ model. Using the homomorphism $SO(3) \sim SU(2)$ the sine-Gordon model can be also interpreted as the $SU(2)/SO(2)$ model, if we choose ω in eq. (1.2) in the following way [4]:

$$\omega = \begin{pmatrix} \frac{i}{2}\Lambda & -\phi_- \\ \phi_+ & -\frac{i}{2}\Lambda \end{pmatrix} dx^- + \frac{i}{2\Lambda} \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} dx^+. \qquad (2.26)$$

Then from eq. (1.2) the sine-Gordon equation follows in the form:

$$\partial_+ \partial_- \phi = -\frac{1}{2} \sin 2\phi .$$

This realization of the sine-Gordon model is equivalent to its interpretation as an $SU(2)/U(1)$ model, which is the CP^1 model.

3. THE SUPERSYMMETRIC O(3) INVARIANT NONLINEAR σ MODEL AND THE ROLE OF THE COMPLEX SUPERSYMMETRY

At first we shall describe the construction of the supersymmetric extension of this model in a self-contained way.

We assume the four-dimensional superspace (x_μ, θ_a), where the Bose coordinates are $x_\mu, \mu = 0, 1$, and the Fermi coordinates θ_a, $a = 1, 2$. The scalar superfield

$$\Phi(x, \theta) = \phi(x) + i\theta^a \psi_a(x) + \frac{i}{2} \theta^a \theta_a F(x)$$

defined on the superspace (x, θ) is "equivalent" to the supermultiplet of the ordinary Bose fields $\phi(x)$, $F(x)$ and the Fermi field $\psi(x)$.

The supersymmetric and Lorentz invariant action for the O(3) σ model has the form [1,2]:

$$\frac{1}{2} \int d^2x \, d^2\theta \, \epsilon_{\alpha\beta} (D_\alpha \Phi^a \cdot D_\beta \Phi^a) \qquad (3.1)$$

and the constraint is given by

$$\Phi^a \cdot \Phi^a = 1, \quad a = 1, 2, 3. \tag{3.2}$$

From the action (3.1) the Lagrangian density for the ordinary fields is obtained as

$$L = \frac{1}{2} \partial_\mu \phi^a \cdot \partial_\mu \phi^a + \frac{i}{2} \psi^a \cdot \not\partial \psi^a - \frac{1}{2} F^a \cdot F^a \tag{3.3}$$

and the constraint (3.2) gives:

$$\begin{aligned} &\phi^a \cdot \phi^a = 1, \\ &\phi^a \cdot \psi^a = 0, \\ &\phi^a \cdot F^a = \frac{i}{2} \psi^a \cdot \psi^a . \end{aligned} \tag{3.4}$$

Using the constraints (3.4) in the Lagrangian density (3.3) we can write the Lagrangian density for the super-O(3) σ model in the form:

$$L = \frac{1}{2} \partial_\mu \phi^a \cdot \partial_\mu \phi^a + \frac{\Lambda}{2} (\phi^a \cdot \phi^a - 1) + \frac{i}{2} \psi^a \cdot \not\partial \psi^a + \frac{1}{8} (\psi^a \cdot \psi^a)^2 . \tag{3.5}$$

The corresponding equations of motion are

$$\Box \phi^b + (\partial_\mu \phi^a \cdot \partial_\mu \phi^a) \phi^b = 0, \tag{3.6a}$$

$$i \not\partial \psi^b + \frac{1}{2} (\psi^a \cdot \psi^a) \psi^b = 0, \tag{3.6b}$$

where the form of these equations in light-cone variables with exact derivation is given in Appendix.

Therefore the presence of fermions in the action (3.1) does not modify the first-order equations for the bosons.

One can show that there exists a conserved supercurrent:

$$J_\mu = \not\partial \phi^a \cdot \gamma^\mu \psi^a \tag{3.7}$$

and the energy momentum tensor has the form:

$$\theta_{\mu\nu} = \partial_\mu \phi^a \cdot \partial_\nu \phi^a - \frac{1}{2} g_{\mu\nu} \partial_\Lambda \phi^a \cdot \partial^\Lambda \phi^a + \frac{i}{4} [\psi^a \cdot (\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu) \psi^a - - g_{\mu\nu} \psi^a \cdot \not\partial \psi^a].$$

It is natural to associate a conserved-current invariance with a continuous set of transformations. The action defined by (3.1) and the constraint (3.2) are invariant under the supersymmetry transformation:

$$\delta\phi^a = i\epsilon\psi^a,$$
$$\delta\psi^a = (\not\partial\phi^a + F)\epsilon, \qquad (3.8)$$
$$\delta F^a = i\epsilon\not\partial\psi^a .$$

It can be observed that this model has an enlarged supersymmetry algebra [2]. In fact, the action (1) is invariant under supersymmetry transformation with internal symmetry O(2), whose Noether's supercurrents are given by

$$\tilde{J}_\mu = \epsilon^{abc}\phi^a\not\partial\phi^b\gamma_\mu\psi^c . \qquad (3.9)$$

The structure of an O(2) extended supersymmetry (or complex supersymmetry) implies furthermore the existence of a conserved vector current

$$V_\mu = \epsilon^{abc}\phi^a\psi^b\gamma_\mu\psi^c \qquad (3.10)$$

and axial-vector current $A_\nu = \epsilon_{\mu\nu}V^\nu$.

The manifestly O(2) invariant super-σ model can be built if two Majorana spinors θ^i_α, i =1, 2 are considered as Fermi coordinates. In this enlarged superspace the constraints

$$\Phi^a(x,\theta^i)\cdot\Phi^a(x,\theta^i) = 1 \qquad (3.11)$$

contain many degrees of freedom and it is difficult to construct a physically sensible theory involving this field. Therefore E.Witten [2] had to assume a supersymmetric constraints, that would remore the extra degrees of freedom.

Now we shall demonstrate the way in which one must not impose any necessary supersymmetric constraints.

At first we shall show that the action (3.1), invariant under the real extended supersymmetry N=2, is also invariant under the U(1) complex supersymmetry. Here real means that the Fermi variables are real anticommuting spinors and complex means that Fermi variables are complex combination of two real anticommuting spinors. For this purpose we shall rewrite the action (3.1) for the super-CP^1 model.

In an analogy with the Bose case we shall define the complex superfields $\Phi_i(x,\theta)$, i =1, 2, whose components ϕ_i, ψ_i, F_i are complex fields which transform according to the fundamental representation of SU(2), while θ is a real two-component spinor . Then transformation $\Phi^a = \Phi_i \sigma^a_{ik} \Phi_k$ gives the equivalence between the action (3.1) and supersymmetric CP^1 action which follows:

$$S = \frac{1}{4} \int d^2x d^2\theta \; \overline{\nabla \Phi_i} \cdot \nabla \Phi_i , \tag{3.12}$$

where $\nabla = D - A$ is supersymmetric U(1) gauge invariant covariant derivative. Here A is a fermionic real superfield, which transforms as an Abelian gauge field under U(1) gauge transformation.

The constraints $\Phi^a \cdot \Phi^a = 1$ have the form $\bar{\Phi}_i \cdot \Phi_i = 1$ and because A has no kinetic term in the superaction (3.12) it can be eliminated by using the equation of motion:

$$A = \overline{\Phi}_i \cdot D \Phi_i \tag{3.13}$$

and supersymmetric CP^1 action has the form:

$$S = \frac{1}{4} \int d^2x d^2\theta [\overline{D\Phi}_i \cdot D\Phi_i - (\bar{\Phi}_i \cdot D\Phi_i)(\bar{\Phi}_j \cdot D\Phi_j)] .$$

This superaction with constraints is equivalent to the super-O(3) σ model.

This supermodel can be obtained directly from the complex supersymmetry. For this we shall define the complex scalar superfield $C(x,\theta,\bar{\theta}) = C(x, \theta_1 + i\theta_2, \theta_1 - i\theta_2)$. The supersymmetry transformation on the superspace $(x, \theta, \bar{\theta})$ acts in the following way:

$$\delta x_\mu = -\frac{i}{2}[\epsilon \gamma_\mu \bar{\theta} + \bar{\epsilon} \gamma_\mu \theta], \quad \delta\theta = \epsilon , \quad \delta\bar{\theta} = \bar{\epsilon}$$

and on the superfield $C(x, \theta, \bar{\theta})$ acts as follows:

$$\delta C = [\epsilon Q + \bar{\epsilon} \bar{Q}] C, \tag{3.14}$$

where $Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - \frac{i}{2}(\partial\!\!\!/ \bar{\theta})_\alpha$, $\bar{Q}_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - \frac{i}{2}(\partial\!\!\!/ \theta)_\alpha$.

These supercharges anticommute with the covariant derivatives:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}(\partial\!\!\!/\bar{\theta})_\alpha, \quad \bar{D}_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{i}{2}(\partial\!\!\!/\theta)_\alpha .$$

Actually, the complex covariant derivatives are just complex combinations of real derivatives. This coincidence with complex combinations of real spinors and real derivatives into complex one has deep reason in a self-duality condition /13/:

$$\bar{D}_\alpha C(x,\theta,\bar{\theta}) = 0. \tag{3.15}$$

This condition (3.15) plays a role of the invariant constraint

$$C(x,\theta,\bar{\theta}) = C(x - \frac{i}{2}\bar{\theta}\gamma\theta, \theta). \tag{3.16}$$

It shows that the graded Lie algebra with complex supersymmetry can be realized in a smaller parametric superspace $(x-\frac{i}{2}\bar{\theta}\gamma\theta,\theta)$ with the complex Bose variable, but independent on $\bar{\theta}$ in Fermi variable.

The complex supersymmetry then can be written as

$$\delta C = [\epsilon_\alpha \frac{\partial}{\partial\theta^\alpha} - i\bar{\epsilon}_\alpha (\partial\!\!\!/\theta)_\alpha] C$$

and for complex ordinary fields ϕ_c, ψ_c, F_c we obtain

$$\begin{aligned} \delta\phi_c &= i\epsilon\bar{\psi}_c , \\ \delta\psi_c &= \epsilon F_c + \partial\!\!\!/\phi_c \bar{\epsilon}, \\ \delta F_c &= i\bar{\epsilon}\partial\!\!\!/\psi_c \end{aligned} \tag{3.17}$$

and the action (3.12) is invariant under this transformations.

The identification with the fields ϕ_i, ψ_i, F_i, $i = 1,2$, from the action (3.12) can be made by using the relations /13/:

$$\phi_c = \phi_1 + i\phi_2, \quad \psi_c = \psi_1 + i\psi_2, \quad F_c = F_1 - iF_2 . \tag{3.18}$$

We remark that for the complex conjugate superfield $\bar{C}$ the opposite self-duality condition is valid:

$$D_\alpha \bar{C} = 0. \tag{3.19}$$

We remember that in the Bose version of the CP^1 model a gauge field $A(x)$ appears. In the supersymmetric version of the CP^1 model we have to introduce a vector superfield $V(x,\theta,\bar{\theta})$, which in the Wess-Zumino gauge has the form:/14/:

$$V = \frac{1}{2}\bar{\theta}\gamma_\mu \theta A_\mu + \text{"coefficients with two or more } \theta\text{"}, \tag{3.20}$$

where for simplicity we have not written other components explicitly.

The superfields C and V transform under an infinitesimal U(1) gauge transformation as follows:

$$\delta C = i\Lambda C, \quad \delta\bar{C} = -i\bar{\Lambda}\bar{C}, \quad \delta V = i(\Lambda - \bar{\Lambda}),$$

where $\Lambda(x_\mu - \frac{i}{2}\bar{\theta}\gamma_\mu \theta, \theta)$ is a chiral gauge superfield.

A complex supersymmetric and gauge invariant action is given by /3/:

$$S = \frac{1}{8}\int d^2x\, d^2\theta\, d^2\bar{\theta}(-V + C\bar{C}e^V), \tag{3.21}$$

from which the equation of motion follows:

$$\bar{C}C = e^{-V} = 1 - V, \tag{3.22}$$

that implies for the first terms in θ-expansion the following constraints on the ordinary fields:

$$\bar{\phi}\cdot\phi = 1, \quad \bar{\psi}\cdot\phi = \psi\cdot\bar{\phi} = 0.$$

In this way we have connected all components in right- and left-hand side of eq. (3.22) and so vector superfield V acts as a confining force between the scalar superfields.

So, in a manifestly complex supersymmetry formalism the constraints are obtained straightforwardly from the gauge invariant action.

4. THE CORRESPONDENCE BETWEEN THE SUPER-O(3) σ MODEL AND SUPER-SINE-GORDON MODEL

In the introduction we remarked to give the supersymmetric extension of G/H nonlinear models, we had to look for the graduation that the (anti-) commutation relations (1.15) are valid. In Sec. 2 there was shown that a realization of the sine-Gordon model is equivalent to the $SU(2)/U(1)$ model, which is equivalent to $SO(3)/SO(2)$ σ model.

If we want to show equivalence between the super-O(3) σ model and the super-sine-Gordon model, then we have to choose supersymmetric extension of $SU(2)/U(1)$ model. Then super-O(3) σ model, which is equivalent to the super-CP^1 model (see Sec. 3), will be also equivalent to the super-sine-Gordon model by the following way:

The super-sine-Gordon model can be interpreted as the integrability condition of this linear set [7]:

$$D_\alpha V = \Omega_\alpha V$$

$$V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} 0 & 0 & -\frac{i}{\sqrt{2}}\sin\Phi \\ 0 & 0 & -\frac{i}{\sqrt{2}}\cos\Phi \\ -\frac{i}{\sqrt{2}}\sin\Phi & -\frac{i}{\sqrt{2}}\cos\Phi & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & -D_2\Phi & 0 \\ -D_2\Phi & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{4.1}$$

where Φ, V_1, V_2 are bosonic superfields, V_3 is a fermionic superfield and the truncated 1-form $\hat{\Omega} = d\theta_1 \Omega_1 + d\theta_2 \Omega_2$.

By decomposing the superfields in component fields we get the matrix Ω, corresponding to the supersymmetric extension of the Bose $SU(2)/U(1)$ model, which has the form:

$$\Omega = \begin{pmatrix} \omega & Q \\ -Q^+ & 0 \end{pmatrix}, \tag{4.2}$$

where the submatrix ω has the form (2.26) as in the Bose case and

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = e^{-i(2x^- + \frac{x^+}{2})} \begin{pmatrix} \sqrt{2}\psi_2 \, dx^- + \frac{1}{\sqrt{2}} \psi_1 \cos\phi \, dx^+ \\ \frac{\psi_1}{\sqrt{2}} \sin\phi \, dx^+ \end{pmatrix}. \tag{4.3}$$

Substituting the matrix Ω in the equation

$$d\Omega - \Omega \wedge \dot{\Omega} = 0,$$

we obtain equations of motion of the super-sine-Gordon model:

$$\partial_+ \partial_- \phi = -\frac{1}{2}(\sin 2\phi + i\bar{\psi}\psi \sin\phi),$$

$$\partial_- \psi_1 = -\cos\phi\, \psi_2, \qquad (4.4)$$

$$\partial_+ \psi_2 = -\cos\phi\, \psi_1 .$$

In this way the equivalence between the super- O(3) σ model and the super-sine-Gordon model is shown.

Now we explain why in super-sine-Gordon the central charges appear via this proved eguivalence.

In Sec. 3 it was shown that the super-O(3) σ model is invariant under O(2) extended supersymmetry or complex supersymmetry.

In this case the supersymmetry algebra can be modified to include central charges, as it has been shown mathematically by R.Haag et al. /15/:

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}(\gamma\cdot P\gamma_0)_{\alpha\beta} + 2i\epsilon_{ij}\, T(\gamma_5\gamma_0)_{\alpha\beta}, \qquad (4.5)$$

where the O(2) labels i,j run from 1 to 2.

This is in agreement with the super-sine-Gordon model, where the superalgebra is modified to include central charge because certain surface term is nonvanishing /10/.

The conserved supercurrent in the super-sine-Gordon model has the form /16/:

$$\partial_\mu [(\not\partial\phi - V'(\phi))\gamma^\mu \psi] = 0. \qquad (4.6)$$

From (4.6) it is possible to obtain supercharges and show, that the anticommutator has the form:

$$\{Q_\alpha, Q_\beta\} = 2(\gamma\cdot P\gamma_0)_{\alpha\beta} + 2i\,T(\gamma_5\gamma_0)_{\alpha\beta} . \qquad (4.7)$$

The extra term $T = \int dx V'(\phi)\frac{d\phi}{dx}$ appears in (4.7) due to the nontrivial boundary condition. T, being the surface term, must commute with all other generators of the algebra. In this way appearing of the central charge in the supermodel with topological excitation is connected with complex supersymmetry.

We can demonstrate it on the level of the Lagrangian formalism: in the two-dimensional space time massless free-field supersymmetric Lagrangian density has the following form:

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{i}{2}\psi \partial\!\!\!/ \psi - \frac{1}{2}F^2 . \tag{4.8}$$

In such a theory the interaction can be added
i) by a constraint on the fields,
ii) by a potential.
The sigma models are the typical representants of the first case. The representant of the second case in O(2) extended supersymmetry has the form:

$$L = \frac{1}{2}\partial_\mu \phi_i \cdot \partial_\mu \phi_i + \frac{i}{2}\psi_i \cdot \partial\!\!\!/ \psi_i - \frac{1}{2}F_i \cdot F_i - F_i V'(\phi_i) + \frac{i}{2}\psi_i \cdot \psi_i V'', \tag{4.9}$$

where the dummy fiel $F_i = -V'(\phi_i)$ and we shall assume $V'(\phi_i) = -\frac{i}{2}\phi_i \psi_j \cdot \psi_j$, $V'' = \frac{dV'(\phi_i)}{d\phi_i} = -\frac{i}{2}\psi_j \psi_j$ and the O(2) labels i,j =1, 2. Then the Lagrangian density (4.9) is equivalent to the

$$L = \frac{1}{2}\partial_\mu \phi_i \cdot \partial_\mu \phi_i + \frac{i}{2}\psi_i \cdot \partial\!\!\!/ \psi_i + \frac{1}{8}(\psi_i \cdot \psi_i)^2 . \tag{4.10}$$

In the case of equivalence the central charge has the form

$$T = -\frac{i}{2}\int dx \frac{\partial}{\partial x}(\psi_j \cdot \psi_j), \tag{4.11}$$

which is in agreement with [17].

In this way we have shown that the supersymmetry algebra of the super O(3) σ model given in [3] must be modified to include the central charge T.

5. COMMENTS •

In this work the equivalence between the super-O(3) σ model (it is super-CP^1 model) and the super-sine-Gordon model has been explicitly shown. In fact this equivalence was shown due to the equivalence between super-O(3) invariant σ model and the super-CP^1 model. It was shown in [4] that the direct supersymmetrization of the SO(3)/SO(2) model gives the super-sine-Gordon model. This corresponds to another possibility of supersymmetric extension of the sine-Gordon model, then only via direct supersymmetrization ($\partial_\pm \to D_{\frac{1}{2}}$, $\phi(x) \to \Phi(x,\theta)$) as in [1].

The role of conformal supersymmetry in the super-O(3) σ model is the bosonification /17/. It was shown in the first paper of ref. /1/ that the relation:

$$\psi^a = \partial\!\!\!/ \phi^a \chi \qquad a = 1,2,3 \tag{5.1}$$

is valid and the arbitrary spinor χ must satisfy the conformal supersymmetry condition

$$\gamma^\nu \gamma^\mu \partial_\nu \chi = 0. \tag{5.2}$$

Inserting the relation (5.1) into the Lagrangian density (3.5) it collapsed in the bosonized form. It is in agreement with the first paper of ref. /17/, where it is easy demonstrated in the SU(2) language.

In this paper we also demonstrated the crucial role of the complex supersymmetry in these models. The appearing of the constraints in N=2 extended supersymmetry and the generalisation for other extended supersymmetries will be discussed in further publication.

7. APPENDIX

In two-dimensional space-time (x_0, x_1) it is usual to us light-cone variables:

$$x_\pm = \frac{1}{2}(x_1 \pm x_0), \quad \partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{\partial}{\partial x^1} \pm \frac{\partial}{\partial x^0}.$$

Analogically in the four-dimensional superspace (x_μ, θ_α), $\mu = 0,1$, $\alpha = 1,2$ we can define for Fermi variables

$$\theta_\pm = \frac{1}{2}(1 \pm \gamma_5)\theta.$$

The notation of ref. /1/ has been used; we remember that θ_α is a Majorana spinor of anticommuting variables and $\theta^\alpha = \bar{\theta}_\alpha = \theta_\beta (\gamma^0)^{\beta\alpha}$ with $\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\partial\!\!\!/ = \gamma^0\partial_0 + \gamma^1\partial_1 = \begin{pmatrix} 0 & \partial_+ \\ \partial_- & 0 \end{pmatrix}$.

The analog of light cone derivatives are covariant derivatives D_1, D_2:

$$D_1 = -\frac{\partial}{\partial \theta_2} + i\theta_2 \partial_+ ,$$

$$D_2 = -\frac{\partial}{\partial\theta_1} + i\theta_1\partial_- .$$

We can see: $D_1D_1 = -i\partial_+$, $D_2D_2 = i\partial_-$. A naive supersymmetrization can be done, performed by substituting ∂_+, ∂_- with D_1, D_2 into equations of motion. We obtain for presented models the following relations (a-Bose case, b-Fermi-Bose case):

1a) equation of motion of the O(3) σ model

$$\partial_+\partial_-\phi^a + (\partial_+\phi^b \cdot \partial_-\phi^b)\phi^a = 0,$$

1b) *equation of motion of the* super O(3) σ model

$$D_2D_1\Phi^a + (D_2\Phi^b D_1\Phi^b)\Phi^a = 0,$$

where $\Phi^a(x,\theta) = \phi^a(x) + i(\theta_1\psi_2^a(x) - \theta_2\psi_1^a(x)) + i\theta_1\theta_2F^a(x)$.

The relations are valid:

$$D_2D_1\Phi^a = iF^a + \theta_2\partial_+\psi_2^a - \theta_1\partial_-\psi_1^a - \theta_1\theta_2\partial_-\partial_+\phi \; .$$

$$D_2\Phi^b \cdot D_1\Phi^b = -\psi_2^b\psi_1^b + \psi_2^b\cdot\theta_1F^b - \psi_2^b\cdot\theta_2\partial_+\phi^b - \theta_2\psi_1^b\cdot F^b + \theta_2\theta_1F^b\cdot F^b$$

$$-\theta_1\partial_-\phi^b\cdot\psi_1^b - \theta_1\theta_2\partial_-\phi^b\cdot\partial_-\phi^b.$$

The equation of motion has the form:

$$\partial_-\psi_1^a - i(\psi_1^b\cdot\psi_2^b)\psi_2^a = 0,$$

$$\partial_+\psi_2^a - i(\psi_1^b\cdot\psi_2^b)\psi_1^a = 0,$$

$$\partial_+\partial_-\phi^a + (\partial_+\phi^b\cdot\partial_-\phi^b)\phi^a = 0$$

and the constraints:

$$\phi^a\cdot\phi^a = 1,$$

$$\phi^a\cdot\psi^a = 0,$$

$$\phi^a\cdot F^a = i\psi_1^b\cdot\psi_2^b.$$

2a) equation of motion of the sine-Gordon model

$$\partial_+\partial_-\phi = -\sin\phi,$$

2b) equation of motion of the supersine-Gordon model

$$D_2 D_1 \Phi = -\sin\Phi,$$

$$\partial_-\psi_1 + \cos\phi\,\psi_2 = 0,$$

$$\partial_-\psi_2 + \cos\phi\,\psi_1 = 0,$$

$$\partial_+\partial_-\phi = -\frac{1}{2}\sin 2\phi - i\psi_1\psi_2\sin\phi.$$

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