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GRASSMANN ANALYTICITY AND EXTENDED SUPERSYMMETRIES

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1. In the superspace approach the role of a real Grassmann variable is played naturally by Majorana spinors

$$\Theta_\alpha = \begin{pmatrix} \theta_\alpha \\ \bar\theta^{\dot\alpha} \end{pmatrix}, \quad \Theta = C\bar\Theta^T, \tag{1}$$

where θ_α, $\bar\theta^{\dot\alpha}$ are conjugated two-component Weyl spinors and C is the charge conjugation matrix. Straightforwardly, in extended N-supersymmetries there appear N Grassman-Majorana variables simultaneously. Even the simplest superfield $\Phi(x, \theta^1, \theta^2, \dots, \theta^N)$ contains a huge number, 2^{4N}, of field components.

The aim of the present note is to introduce a notion of Grassmann analyticity and to show with a simple example, how it helps to reduce the number of Majorana spinor variables. The Cauchy-Riemann analyticity condition

$$\frac{\partial}{\partial \bar z} f(x,y) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(x,y) = 0$$

has the following extension to the Grassmann case

$$\left(\tilde{\mathcal{D}}_\alpha^\theta + i\,\tilde{\mathcal{D}}_\alpha^\eta\right)\Phi(x,\theta,\eta) = 0, \tag{2}$$

where $\tilde{\mathcal{D}}_\alpha^\theta$ ($\tilde{\mathcal{D}}_\alpha^\eta$) is the covariant spinor derivative with respect to the Grassmann variable θ (η). This condition says that superfield Φ is "independent" of the variable $\theta - i\eta$ and it permits us to pass to an N=1 superfield $V(x,\theta)$ depending only on one Majorana variable. Note that in the two-dimensional case a condition of the form (2) was used already in the supersymmetric string theory [1]. Further, already in the N=1 supersymmetry the chirality can be also interpreted as the analyticity. Indeed, the chirality condition

$$\bar{\mathcal{D}}_{\dot\alpha}\,\Psi(x,\theta) = 0 \tag{3}$$

means that $\Psi(x,\theta)$ is dependent only on the left Weyl spinor θ^α being independent of the conjugated right spinor $\bar\theta^{\dot\alpha}$. The solution of the condition (3) is known to have the form (see, e.g., ref.[2])

$$\Psi(x,\theta) = a(x_L) + \theta^\alpha \psi_\alpha(x_L) + \theta^\alpha \theta_\alpha b(x_L)\ , \quad x_L^a = x^a + \tfrac{1}{2}\bar\theta\gamma^a\gamma_5\theta. \tag{4}$$

The solution of the analyticity condition (2) will be given below and will be used for reducing a superfield with two Majorana variables to the one depending only on one Majorana variable.

2. Now we proceed to our aim. The Grassmann analyticity was prompted by the known fact that the complexified N=1 on-shell supermultiplets are also representation of the N=2 supersymmetry with central charge (see /3/ and references therein). This statement turned out to be correct also for off-shell superfields. Indeed, consider an appropriate decomposition of the complex superfield

$$V(x,\theta) = \frac{1}{m}M(x) - i\bar\theta\frac{\Psi^2(x)}{m} + \frac{1}{2}\bar\theta\theta\frac{P^{12}(x)}{m} + \frac{1}{2}\bar\theta\gamma_5\theta S(x) +$$
$$+ \frac{1}{2}\bar\theta i\gamma^a\gamma_5\theta V_a(x) + \bar\theta\theta\bar\theta\left[\psi^1(x) - \frac{1}{2m}\partial\!\!\!/\psi^2(x)\right] + \frac{1}{4}(\bar\theta\theta)^2\left[iP^{11}(x) - \left(\frac{\Box}{2m}+m\right)M(x)\right]. \tag{5}$$

This decomposition contains the complex vector and scalar fields $V_a(x)$, $M(x)$, $S(x)$ forming O(2) singlet, the traceless complex symmetric O(2) tensor $P^{ij}(x)$ and the O(2) doublet of Dirac spinors $\psi^i(x)$. The action for $V(x,\theta)$

$$S = \int d^4x\, d^4\theta\ \frac{1}{2}V^*(x,\theta)\left[\Box + \frac{(\bar{\mathcal{D}}\mathcal{D})^2}{16} + m^2\right]V(x,\theta) = \tag{6a}$$
$$= \int d^4x\left\{-\frac{1}{2}F^*_{ab}(x)F^{ab}(x) + m^2V^*_a(x)V^a(x) + \partial_a M^*(x)\partial^a M(x) - m^2M^*(x)M(x) + \right.$$
$$\left. + m^2S^*(x)S(x) + \frac{1}{2}P^{ij*}(x)P^{ij}(x) + i\bar\psi^k(x)\partial\!\!\!/\psi^k(x) + im\varepsilon^{k\ell}\bar\psi^k(x)\psi^\ell(x)\right\} \tag{6b}$$

is invariant under O(2) supersymmetry transformations

$$\delta V_a = \bar\varepsilon^k i\gamma_a\gamma_5\psi^k + \frac{i}{m}\varepsilon^{k\ell}\bar\varepsilon^k\gamma_5\partial_a\psi^\ell$$

$$\delta\psi^i = i P^{ij}\varepsilon^j + \left(-mM + \tfrac{i}{2}\sigma_{ab}\gamma_5 F^{ab}\right)\varepsilon^i +$$
$$+ \varepsilon^{ik}\left(-im\gamma_5 S + m\gamma^a\gamma_5 V_a - \partial\!\!\!/ M\right)\varepsilon^k$$

$$\delta M = -i\varepsilon^{ij}\bar{\varepsilon}^i\psi^j\ , \quad \delta S = \bar{\varepsilon}^i\gamma_5\psi^i - \varepsilon^{kl}\bar{\varepsilon}^k\gamma_5\frac{\partial\!\!\!/}{m}\psi^l$$

$$\delta P^{ij} = -\bar{\varepsilon}^i(\partial\!\!\!/\psi^j + m\psi^k\varepsilon^{jk}) + \tfrac{1}{2}\delta^{ij}\bar{\varepsilon}^l(\partial\!\!\!/\psi^l + m\psi^k\varepsilon^{lk}) + \qquad (7)$$
$$+ \ (i\leftrightarrow j)\,.$$

In these formulae $F_{ab} = \partial_a V_b - \partial_b V_a$, $\varepsilon^{12} = -\varepsilon^{21} = 1$, $\partial\!\!\!/ = \partial^a\gamma_a$ and the internal symmetry indices are deliberately written down on the same level to stress that we deal here just with O(2) (leaving aside SU(2) at a moment). The parameters of the standard and second supersymmetries are ε^1 and ε^2, respectively. In the superfield form these transformations are written down as

$$\delta V = -i\,\bar{\varepsilon}^i Q^i V$$

with generators
$$Q^1_\beta = i\frac{\partial}{\partial\bar{\theta}^\beta} - (\partial\!\!\!/\theta)_\beta \qquad (8a)$$

$$Q^2_\beta = -2m\,\theta_\beta + \frac{1}{4m}\bar{\mathcal{D}}\mathcal{D}\cdot\mathcal{D}_\beta\ ; \quad \left(\mathcal{D}_\beta = \frac{\partial}{\partial\bar{\theta}^\beta} - i(\partial\!\!\!/\theta)_\beta\right) \qquad (8b)$$

having commutation relations

$$\left\{\bar{Q}^i, Q^k\right\} = 2\delta^{ik}\gamma^a P_a + 2i\,\varepsilon^{ik} Z\,. \qquad (9)$$

It is important that in our O(2) superalgebra there is the central charge Z proportional to the mass parameter

$$Z V = mV\ , \quad Z V^* = -m V^*. \qquad (10)$$

Like the electric charge, operator Z takes opposite values for particles and antiparticles. Presence of the term with three spinor derivatives in Eq. (8b) is somewhat surprising. Now we shall show that these transformations arise quite naturally.

3. To this end, we consider a complex superfield $\Phi(x,\theta,\eta)$ that satisfies the Cauchy-Riemann condition (2), i.e., Φ is analytic. Its supersymmetry generators are naturally defined by

$$Q^1_\alpha = i\frac{\partial}{\partial\bar\theta^\alpha} - (\partial\!\!\!/\theta)_\alpha \ , \quad Q^2_\alpha = i\frac{\partial}{\partial\bar\eta^\alpha} - (\partial\!\!\!/\eta)_\alpha - 2\theta_\alpha Z \tag{11}$$

and obey the commutation relations (9). The central-charge generator Z acts on Φ, Φ^* as in Eq-ns (10) with the change of V by Φ. Then spinor derivatives in the condition (2) are:

$$\tilde{\mathcal{D}}^\theta_\alpha = \frac{\partial}{\partial\bar\theta^\alpha} - i(\partial\!\!\!/\theta)_\alpha + 2i\eta_\alpha Z \ , \quad \tilde{\mathcal{D}}^\eta_\alpha = \frac{\partial}{\partial\bar\eta^\alpha} - i(\partial\!\!\!/\eta)_\alpha \ . \tag{12}$$

The Cauchy-Riemann condition can be solved (analogously to (4)) and

$$\Phi(x,\theta,\eta) = e^{-m\bar\eta\eta}\,\varphi(x^a - \bar\theta\gamma^a\eta\,,\ \theta + i\eta) = \tag{13a}$$

$$= e^{-m\bar\eta\eta}\, e^{-\bar\theta\partial\!\!\!/\eta}\, e^{i\bar\eta\frac{\partial}{\partial\bar\theta}}\ \varphi(x,\theta). \tag{13b}$$

It appears that the superfield $V(x,\theta)$ considered above is expressed in terms of $\varphi(x,\theta)$ as follows:

$$V(x,\theta) = \varphi(x,\theta) + \frac{1}{4m}\,\bar{\mathcal{D}}\mathcal{D}\,\varphi(x,\theta)\,. \tag{14}$$

Then transformations

$$\delta\,\Phi(x,\theta,\eta) = -i\,\bar\varepsilon^i Q^i\,\Phi(x,\theta,\eta) \tag{15}$$

correspond just to the transformations (8) for $V(x,\theta)$. The action (8) can be rewritten as

$$S = \frac{1}{32}\int d^4x\, d^4\theta\, d^4\eta\; \Phi^*(x,\theta,\eta)\,\Phi(x,\theta,\eta) = \tag{16a}$$

$$= \frac{1}{2}\int d^4x\, d^4\theta\; \varphi^*(x,\theta)\left[\Box + \frac{(\bar{\mathcal{D}}\mathcal{D})^2}{8} + \frac{m}{2}\,\bar{\mathcal{D}}\mathcal{D} + m^2\right]\varphi(x,\theta) = \tag{16b}$$

$$= \frac{1}{2}\int d^4x\, d^4\theta\; V^*(x,\theta)\left[\Box + \frac{(\bar{\mathcal{D}}\mathcal{D})^2}{16} + m^2\right]V(x,\theta)\;.$$

Equation (16a) for the Lagrangian density demonstrates once more the analogy with the usual N=1 chiral case /2/ .

4. The representation (13a) suggests the presence of "analytical basis" and "antianalytical basis"

$$x^m - \bar\theta\gamma^m\eta \quad , \quad \theta + i\eta \tag{17a}$$

$$x^m + \bar\theta\gamma^m\eta \quad , \quad \theta - i\eta \tag{17b}$$

in form similar to the left and right bases in N=1 case /4/ and turning into the latter for $\eta \to \gamma_5\theta$ (with an appropriate change of the scale of X due to differentiation in (2)). Really, (17a) and (17b) form invariant superspaces of the N=2 supersymmetry. A hope arises to obtain a purely geometrical formulation of the N=2 supergravity without constraints analogous to that proposed in /4/ for the N=1 case. To this end, one has, in the first place, to extend O(2) to SU(2) and to study the analytical properties of some supermultiplets considered in /5/ .In the case of higher supersymmetries one can think of a deep internal relation with hypercomplex numbers /6/ . We have checked that the hypercomplex coordinates

$$\tilde{x}^m = x^m + i\,\bar\theta\gamma^m\eta_k e_k - \frac{i}{2}\,\bar\eta_k\gamma^m\eta_\ell f_{k\ell p}e_p\,, \quad \tilde\theta = \theta + e_k\eta_k$$
$$(e_k e_\ell = -\delta_{k\ell} + f_{k\ell p}\, e_p) \tag{18}$$

form invariant spaces of the N=4 supersymmetry (e_k are quaternion imaginary units) and of N=8 supersymmetry (e_k are octonion imaginary units). These problems will be discussed in our further publications.

References

1. Ademollo M., Brink L. et al. Nucl.Phys., 1976, B111, p.77.
2. Salam A., Strathdee J. Phys.Rev., 1975, 11D, p.1521.
   Ogievetsky V., Mezincescu L. Usp.Fiz.Nauk, 1975, 117, p.637; transl: Sov.Phys.Usp. 1976, 18, p.960.
3. Fayet P. Prepr.LPTENS 80/7, Paris, 1980.
4. Ogievetsky V., Sokatchev E. Phys.Letters., 1978, 79B, p.222.
5. Breitenlohner P., Sohnius M. Nucl.Phys., 1980, B165, p.483.
   Sohnius M.F., Stelle K.S., West P.C. Prepr.ICTP/79-80/51, London, 1980.
6. Lukierski J. Prepr. UGVA DPT 1979/09 - 216, Geneva, 1979.

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