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Č.Burdík,* P.Exner, M.Havlíček *

HIGHEST-WEIGHT REPRESENTATIONS OF sl(2,C) AND sl(3,C) VIA CANONICAL REALIZATIONS

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* Nuclear Centre, Faculty of Mathematics and Physics of the Charles University, Areál Troja, Povltavská ul., 180 00 Prague 8, Czechoslovakia.

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1. Introduction

In our preceding paper/1/ a set of infinite-dimensional representations of the $sl(n+1,C)$ Lie algebras was presented.*) The representations $\rho_\Lambda^{(n+1)}$ from this set, called maximal representations, are given by means of $\frac{1}{2}n(n+1)$ boson creation-annihilation pairs $\bar{a}_i^{k+1}$, a_j^{k+1}, $k=1,2,\dots,n$; $i,j=1,2,\dots,k$, which obey the standard CCR and n complex parameters $\Lambda = (\Lambda_1,\Lambda_2,\dots,\Lambda_n)$. Since the generators of the maximal representations are expressed as polynomials in the canonical pairs, matrix elements of them may be easily calculated. Sufficient conditions for irreducibility of $\rho_\Lambda^{(n+1)}$ were derived in Ref.1 . If they are satisfied, then $\rho_\Lambda^{(n+1)}$ is representation with the highest weight Λ .

In the present paper these results are illustrated and extended particularly for the simplest cases $sl(2,C) \sim A_1$ and $sl(3,C) \sim A_2$. We shall show that the mentioned irreducibility conditions are in these cases necessary as well. Moreover, our method allows us to construct other set of infinite-dimensional representations of $sl(3,C)$, the so-called mixed representations (Sec.4). They are irreducible highest-weight representations in some of the cases when the maximal as well as the standard highest-weight representations (the so-called elementary representations/2/ or Verma modules/3/, cf. Sec.I.2.6) are reducible.

Let us recall briefly that we use the following Cartan-Weyl basis in $sl(n+1,C)$: $h_i = e_{i+1,i+1} - e_{ii}$, $e_i = e_{i+1,i}$, $f_i = e_{i,i+1}$, together with e_{ij} , $|i-j| > 1$, $i,j = 1,2,\dots,n$, where e_{ij} fulfil the standard commutation relations

$$[e_{ij},e_{kl}] = \delta_{kj}e_{il} - \delta_{il}e_{kj}$$

*) References to assertions and formulae of this paper will be hereafter marked by I .

(cf. (I.1),(I.2)). The operators $\bar{a}_i^{k+1}$, a_j^{k+1} act on the vector space V_{n+1} which is complex envelope of the set of basis vectors denoted by the following symbols

$$\begin{vmatrix} m_{n1} \cdots\cdots\cdots\cdots\cdots\cdots m_{nn} \\ m_{n-1,1} \cdots\cdots m_{n-1,n-1} \\ \cdots\cdots\cdots\cdots \\ m_{11} \end{vmatrix}$$

in the standard way, i.e., $\bar{a}_i^{k+1}$ raises the component m_{ki} by one and multiplies by $\sqrt{m_{ki}+1}$, etc. (cf.(I.7)).

2. Maximal representations of sl(2,C)

2.1 The representation space V_2 is spanned by the vectors $|m_{11}|$; it is more convenient to denote them, e.g., as x_m, $m \in N_0 \equiv \{0,1,2,\ldots\}$. The maximal representation $\rho_\Lambda^{(2)} = \rho_\Lambda$ corresponding to a weight $\Lambda : \Lambda(h_1) = \Lambda_1 \in \mathbf{C}$ is expressed through one pair of creation-annihilation operators $\bar{a}_1^2, a_1^2$ (cf. formulae (I.7),(I.11)) :

$$\begin{aligned} H_1^2 &= -2\bar{a}_1^2 a_1^2 + \Lambda_1 I \quad , \\ E_{21}^2 &= -a_1^2 \quad , \\ E_{12}^2 &= \bar{a}_1^2(\bar{a}_1^2 a_1^2 - \Lambda_1) \quad , \end{aligned} \tag{1}$$

where $E_{ij}^2 \equiv \rho_\Lambda(e_{ij})$, $H_1^2 \equiv \rho_\Lambda(h_1)$, or more explicitly

$$\begin{aligned} &H_1^2 x_m = (\Lambda_1 - 2m)\, x_m \quad , \quad m = 0,1,2,\ldots \quad , \\ &E_{21}^2 x_0 = 0 \quad , \quad E_{21}^2 x_m = -\sqrt{m}\; x_{m-1} \quad , \quad m = 1,2,\ldots \quad , \\ &E_{12}^2 x_m = \sqrt{m+1}\,(m - \Lambda_1)\, x_{m+1} \quad , \quad m = 0,1,2,\ldots \end{aligned} \tag{1a}$$

If Λ_1 is not a non-negative integer, then this representation is irreducible with the highest weight Λ and the highest-weight vector x_0. The irreducible highest-weight representations corresponding to $\Lambda_1 \in N_0$ are finite-dimensional ; according to Proposition I.3.2 they can be obtained by restriction of ρ_Λ to the subspace $V_2^\Lambda \equiv \rho_\Lambda(UL)x_0$ of V_2, UL being the universal enveloping algebra of $L = sl(2,C)$. One finds easily

$$V_2^\Lambda = \mathbb{C}\{ x_m : m \leq \Lambda_1 \} \quad , \quad \Lambda_1 \in N_0 \quad ; \tag{2}$$

thus the irreducible representation $\tilde{\rho}_\Lambda \equiv \rho_\Lambda \restriction V_2^\Lambda$ is $(\Lambda_1 + 1)$-dimensional.

2.2 According to Theorem I.2.4 all irreducible representations of $sl(2,\mathbb{C})$ with the same highest weight Λ are mutually equivalent. Especially, if $\Lambda_1 \notin N_0$, then ρ_Λ has to be equivalent to the elementary representation d_Λ given by the formulae (I.4) :

$$\begin{aligned} d_\Lambda(h)y_m &= (\Lambda_1 - 2m)y_m \quad , \\ d_\Lambda(e)y_m &= -m(m-1-\Lambda_1)y_{m-1} \quad , \\ d_\Lambda(f)y_m &= y_{m+1} \quad , \end{aligned} \tag{3}$$

where $y_0 \equiv 1$, $y_m \equiv f^m$, $m = 1,2,\dots$. Since $h_1 = h$, $e_{21} = e$ and $e_{12} = f$ form the appropriate Cartan-Weyl basis in $sl(2,\mathbb{C})$,both the representations are indeed equivalent :

$$\rho_\Lambda(a) = S_\Lambda d_\Lambda(a) S_\Lambda^{-1} \quad , \quad a \in sl(2,\mathbb{C}) \quad , \tag{4a}$$

where

$$S_\Lambda x_m = (-1)^m (m!)^{3/2} \binom{\Lambda_1}{m} y_m \quad , \quad m = 0,1,2,\dots \tag{4b}$$

Analogously, for $\Lambda_1 \in N_0$ the representation $\tilde{\rho}_\Lambda$ is equivalent to D_Λ which is the irreducible $(\Lambda_1 + 1)$-dimensional component of d_Λ defined by the relations (3) together with $D_\Lambda(f)x_{\Lambda_1} = 0$ (cf. Sec.III of Ref.2). On the other hand, ρ_Λ and d_Λ are inequivalent in this case, because d_Λ is a highest-weight representation while ρ_Λ is not : x_0 is the only eigenvector of H_1^2 annihilated by E_{21}^2 , but $\rho_\Lambda(UL)x_0 = V_2^\Lambda \neq V_2$.

2.3 __Remarks__ : (a) The lowest-weight representations are obtained easily with the help of the automorphism τ of $sl(2,\mathbb{C})$: $\tau(h) = -h$, $\tau(e) = f$, $\tau(f) = e$. Clearly, if $\Lambda_1 \notin N_0$ $(\Lambda_1 \in N_0)$, then the representation $\rho_\Lambda \tau$ $(\tilde{\rho}_\Lambda \tau)$ is irreducible with $-\Lambda$ and x_0 as the lowest weight and the lowest-weight vector, respectively.

(b) One can obtain some other representations starting with the same canonical realizations (I.5) but using a different representation of the canonical pair. For example, representing

q_1^2, p_1^2 by a_1^2, $-\bar{a}_1^2$ instead of $\bar{a}_1^2$, a_1^2 (this choice corresponds to $\beta = \pi/2$ in (I.9)) we obtain

$$\rho'_{\Lambda'}(h) = 2\bar{a}_1^2 a_1^2 + \Lambda'_1 I \quad ,$$

$$\rho'_{\Lambda'}(e) = \bar{a}_1^2 \quad ,$$

$$\rho'_{\Lambda'}(f) = -(\bar{a}_1^2 a_1^2 + \Lambda'_1)\, a_1^2 \quad ,$$

where $\Lambda' : \Lambda'(h) = \Lambda'_1 \equiv \Lambda_1 + 2$. If Λ'_1 is not a non-positive integer, then this representation is irreducible with Λ' as the lowest weight. Furthermore, the representation $\rho'_{\Lambda'}\cdot\tau$ has the highest weight $-\Lambda'$; it is easy to check that it is equivalent to $d_{-\Lambda'}$ for any complex $-\Lambda'_1$ (including non-negative integers).

3. Maximal representations of $sl(3,C)$

3.1 The representation space V_3 is spanned by the vectors $\begin{vmatrix} m_{21} & m_{22} \\ m_{11} & \end{vmatrix}$, $m_i \in N_0$. The maximal representations $\rho_\Lambda^{(3)}$ of $sl(3,C)$ are expressed in terms of three pairs of creation-annihilation operators $\bar{a}_1^3, a_1^3$, $\bar{a}_2^3, a_2^3$ and $\bar{a}_1^2, a_1^2$. We rewrite the formulae (I.11) for $n=2$ so that the generators of the subalgebra $sl(2,C)$ contained in them are exhibited explicitly. The operators $E_{ij}^3 \equiv \rho_\Lambda^{(3)}(e_{ij})$, $H_i^3 \equiv \rho_\Lambda^{(3)}(h_i)$ are then of the form

$$\begin{aligned}
H_1^3 &= -\bar{a}_1^3 a_1^3 + \bar{a}_2^3 a_2^3 + H_1^2 \quad , \\
H_2^3 &= -\bar{a}_1^3 a_1^3 - 2\bar{a}_2^3 a_2^3 - \tfrac{1}{2} H_1^2 + \Lambda_2 + \tfrac{1}{2}\Lambda_1 \quad , \\
E_{21}^3 &= \bar{a}_2^3 a_1^3 + E_{21}^2 \quad , \\
E_{31}^3 &= -a_1^3 \quad , \\
E_{32}^3 &= -a_2^3 \quad , \\
E_{12}^3 &= \bar{a}_1^3 a_2^3 + E_{12}^2 \quad , \\
E_{13}^3 &= \bar{a}_1^3(\bar{a}_1^3 a_1^3 + \bar{a}_2^3 a_2^3 - \tfrac{1}{2} H_1^2 - \Lambda_2 - \tfrac{1}{2}\Lambda_1) + \bar{a}_2^3 E_{12}^2 \quad , \\
E_{23}^3 &= \bar{a}_2^3(\bar{a}_1^3 a_1^3 + \bar{a}_2^3 a_2^3 + \tfrac{1}{2} H_1^2 - \Lambda_2 - \tfrac{1}{2}\Lambda_1) + \bar{a}_1^3 E_{21}^2 \quad ,
\end{aligned} \tag{5}$$

where H_1^2 , E_{21}^2 and E_{12}^2 are given by the relations (1).

According to Theorem I.4.3 this representation is irreducible if the conditions

$$\Lambda_1 \notin N_0 \;,\quad \Lambda_2 \notin N_0 \;,\quad 1 + \Lambda_1 + \Lambda_2 \notin N_0 \tag{6}$$

are satisfied ; it has the highest weight $\Lambda = (\Lambda_1, \Lambda_2)$ and the highest-weight vector $x_0^3 \equiv \begin{vmatrix} 0 & 0 \\ 0 & \end{vmatrix}$.

3.2 __Theorem__ : The conditions (6) are necessary and sufficient for $\rho_\Lambda^{(3)}$ to be irreducible.

__Proof__ : In view of Theorem I.4.3 we have to check the necessary condition only. Let us denote again $V_3^\Lambda \equiv \rho_\Lambda^{(3)}(UL)x_0^3$.

(a) We shall prove first that

$$V_3^\Lambda = V(\Lambda_1) \equiv \mathbf{C}\left\{ \begin{vmatrix} m_{21} & m_{22} \\ m_{11} & \end{vmatrix} \in V_3 : m_{11} \leq \Lambda_1 \right\} \tag{7}$$

$$\text{if} \quad \Lambda_1 \in N_0 \;,\; \Lambda_2 \notin N_0$$

(in fact we need to show only that $V(\Lambda_1)$ is invariant and differs from V_3 if $\Lambda_1 \in N_0$, however, the implication (7) will be useful in the following). Clearly $V_3^\Lambda \subset V(\Lambda_1)$, further $\begin{vmatrix} m & n \\ 0 & \end{vmatrix} \in V_3^\Lambda$ for any $m,n \in N_0$ because

$$y = (E_{21}^3)^n \prod_{j=1}^{m+n} ((j - \Lambda_2)E_{13}^3 - E_{12}^3 E_{23}^3)x_0^3$$

belongs to V_3^Λ according to the definition, and in the same time y is a non-zero multiple of $\begin{vmatrix} m & n \\ 0 & \end{vmatrix}$ as can be verified directly from the relations (5) by the same arguments as those used in the proof of Theorem I.4.3 . Thus the vectors $\begin{vmatrix} m & n \\ s & \end{vmatrix}$ belong to V_3^Λ for all $m,n \in N_0$ and $s = 0$. Assume that this is true also for $s = 1,2,\ldots,r$, $r < \Lambda_1$. It holds

$$E_{12}^3 \begin{vmatrix} m & n \\ r & \end{vmatrix} = \sqrt{m+1}\,\sqrt{n}\,\begin{vmatrix} m+1 & n-1 \\ r & \end{vmatrix} + (r - \Lambda_1)\sqrt{r+1}\,\begin{vmatrix} m & n \\ r+1 & \end{vmatrix}$$

so the assertion is valid for $s = r+1$ as well ; the induction argument clearly breaks at $s = \Lambda_1$, hence (7) is proved. Combining it with Theorem I.2.5 we see that $\rho_\Lambda^{(3)}$ is reducible if $\Lambda_1 \in N_0$.

(b) Let further $\Lambda_2 = r \in N_0$, Λ_1 arbitrary. We decompose the space V_3 into the direct sum $V_3 = \mathbf{C}\{v_r\} \oplus W_r$, where $v_r \equiv \begin{vmatrix} 0 & r+1 \\ 0 & \end{vmatrix}$ and W_r is spanned by the remaining basis vectors. Consequently, each vector $x \in V_3^\Lambda \subset V_3$ can be decomposed uniquely in the form

$$x = \alpha v_r + x' \quad , \quad \alpha \in \mathbb{C} \quad , \quad x' \in W_r \quad .$$

On the other hand, one obtains V_3^Λ acting on the vacuum vector x_0^3 by all polynomials in the operators (5). Each of these polynomials may be arranged using the commutation relations so that in all its monomials the operators E_{ij}^3 , $i < j$, stand in the left of H_i^3 and E_{ij}^3 , $i > j$. Since the latter reproduce or annihilate the vacuum, we may, e.g., write

$$V_3^\Lambda = \mathbb{C}\{ y(m,n,s) \equiv (E_{13}^3)^m (E_{12}^3)^s (E_{23}^3)^n x_0^3 : m,n,s \in N_0 \} . \tag{8}$$

The relations (5) and (1) give

$$y(m,n,s) = \sum_{k=0}^{m} \sum_{j=J_k}^{s} \alpha_{kj} \begin{vmatrix} m+s-k-j & n-s+k+j \\ k+j & \end{vmatrix} , \tag{9}$$

where $J_k = \max(0, s-n-k)$. For the present moment we need not know the coefficients α_{kj} explicitly except for

$$y(0,n,0) = \sqrt{n!} \prod_{i=0}^{n-1} (i-r) \begin{vmatrix} 0 & n \\ 0 & \end{vmatrix} \quad . \tag{10a}$$

The relation (9) implies that only the vector $y(0,r+1,0)$ may have a non-zero component in $\mathbb{C}\{v_r\}$, but $y(0,r+1,0) = 0$ due to (10a) , thus $V_3^\Lambda \subset W_r \neq V_3$.

(c) The remaining case $1 + \Lambda_1 + \Lambda_2 = 1 + r \in N_0$ is more complicated. The above used decomposition of V_3 will be replaced now by $V_3 = U_r \oplus U_r'$ where U_r is the following $(r+3)$-dimensional subspace

$$U_r \equiv \mathbb{C}\left\{ \begin{vmatrix} r+2-j & j \\ j & \end{vmatrix} : j = 0,1,\ldots,r+2 \right\}$$

while U_r' is spanned by the remaining basis vectors. The relation (9) implies easily that a vector $y(m,n,s)$ belongs to U_r iff $n = s$ and $m + n = r+2$ and vice versa, those $y(m,n,s)$ which do not fulfil these conditions belong wholy to U_r' . Consequently, each vector $x \in V_3^\Lambda$ can be decomposed uniquely as follows

$$x = \sum_{n=0}^{r+2} \beta_n y(r+2-n,n,n) + x' \quad , \quad x' \in U_r' \quad .$$

Our aim is to verify that the vectors $y(r+2-n,n,n)$ are linearly dependent and cannot span therefore the $(r+3)$-dimensional subspace U_r ; then $V_3^\Lambda \neq V_3$ immediately follows.

To this purpose we have to express the vectors $y(r+2-n,n,n)$ explicitly. It is a tedious but relatively straightforward procedure so we exhibit here results of its main steps only. Acting on (10a) by E^3_{12} s-times we obtain an expression for $y(0,n,s)$, especially for $n=s$ we get

$$y(0,n,n) = \sqrt{n!}\prod_{i=0}^{n-1}(i-\Lambda_2)\sum_{j=0}^{n}\binom{n}{j}\sqrt{n!(n-j)!}\prod_{k=0}^{j-1}\sqrt{n-k}\ \times \tag{10b}$$
$$\times\prod_{l=0}^{j-1}(l-\Lambda_1)\begin{vmatrix} n-j & j \\ j & \end{vmatrix} .$$

Further we have to apply the operator E^3_{13} $(r+2-n)$-times to (10b). Using the induction argument the relation

$$y(r+2-n,n,n) = n!(r+1-n)!\prod_{i=0}^{n-1}(i-\Lambda_2)\sum_{j=r+1-n}^{r+2}c_{nj}\sqrt{(r+2-j)!}\ \times \tag{10c}$$
$$\times\prod_{l=0}^{j-1}(l-\Lambda_1)\binom{j}{r+1-j}\begin{vmatrix} r+2-j & j \\ j & \end{vmatrix}$$

can be proved, where

$$c_{nj} = \sum_{k=\max(0,j+n-r-2)}^{\min(j,n)}(k+1)\binom{n}{k}\binom{r+2-n}{j-k} \quad . \tag{10d}$$

The last expression can be simplified with the help of the known summing formulae for combination numbers (cf.Ref.4, 0.156) to the form

$$c_{nj} = \frac{1}{j}(r+2+jn)\binom{r+1}{j-1} \quad .$$

Thus we obtain finally

$$y(r+2-n,n,n) = a_n\sum_{j=r+1-n}^{r+2}b_jd_{nj}\begin{vmatrix} r+2-j & j \\ j & \end{vmatrix} , \tag{11a}$$

where

$$a_n = n!(r+1)!\prod_{i=0}^{n-1}(i-\Lambda_2) \quad , \tag{11b}$$

$$b_j = ((r+2-j)!)^{-1/2}\prod_{l=0}^{j-1}(l-\Lambda_1) \tag{11c}$$

and

$$d_{nj} = \frac{r+2+jn}{(j+n-r-1)!} \tag{11d}$$

for $j = r+1-n, r+2-n, \ldots, r+2$, otherwise $d_{nj} = 0$.

In order to show that $y(r+2-n,n,n)$, $r = 0,1,\ldots,r+2$, are linearly dependent it is clearly sufficient to show

$$\det(d_{nj}) = 0 \quad . \tag{12}$$

The determinant certainly does not change when we subtract a suitable linear combination of the last r rows from the second row, thus

$$\det(d_{nj}) = \det(d'_{nj})$$

where $d'_{nj} = d_{nj}$ for $n = 0,2,3,\dots,r+2$ and

$$d'_{1j} = d_{1j} - \sum_{n=2}^{r+2} (-1)^n \frac{(r+1)!}{(r+2-n)!} d_{nj} \quad .$$

Substituting from (11d) into the last expression we obtain

$$d'_{1j} = \sum_{k=\max(0,r-j)}^{r+1} (-1)^k \binom{r+1}{k} \frac{(r+2+j(k+1))\ k!}{(j+k-r)!} \quad .$$

These sums can be easily evaluated (cf.Ref.4, 0.155) as follows

$$d'_{1,r+2} = d'_{1,r+1} = 1 \quad ,$$

$$d'_{1j} = 0 \quad , \quad j = 0,1,\dots,r$$

and since the first row equals $(0,\dots,0,r+2,r+2)$ we see that (12) holds and the proof is therefore completed. ■

3.3 The irreducible highest-weight representations referring to the case when some of the conditions (6) is not fulfilled are obtained by restriction of $\rho_\Lambda^{(3)}$ to the corresponding subspace V_3^Λ (cf.Proposition I.3.2). However, one has to know V_3^Λ explicitly in order to describe the resulting representations $\tilde{\rho}_\Lambda^{(3)} \equiv$ $= \rho_\Lambda^{(3)} \restriction V_3^\Lambda$ fully. There are four subcases here :

(a) $\Lambda_1, \Lambda_2 \in N_0$: we do not discuss this case because the irreducible highest-weight representations here are finite-dimensional and well-known (cf., e.g., Ref.5, Chap.10 or Ref.6, Sec.10.1),

(b) $\Lambda_1 \in N_0$, $\Lambda_2 \notin N_0$: the subspace V_3^Λ is given now by (7) ; this relation together with (1),(5) gives the explicit form of $\tilde{\rho}_\Lambda^{(3)}$,

(c) $\Lambda_1 \notin N_0$, $\Lambda_2 \in N_0$: we do not know V_3^Λ explicitly here but the irreducible representation with the highest-weight $\Lambda = (\Lambda_1,\Lambda_2)$ can be nevertheless constructed using $\tilde{\rho}_{\Lambda'}^{(3)}$, $\Lambda' \equiv (\Lambda_2,\Lambda_1)$ which is known from the case (b) , as we shall see a little later,

(d) $\Lambda_1 \notin N_0$, $\Lambda_2 \notin N_0$, $1 + \Lambda_1 + \Lambda_2 \in N_0$.

In the next section we shall give an alternative method for constructing irreducible highest-weight representations of $sl(3,\mathbb{C})$ in the cases (b) and (c) . It will give also a hint for construction of irreducible highest-weight representations of $sl(n+1,\mathbb{C})$, $n \geq 3$ in some of the cases which are not covered by Theorem I.4.3.

4. Mixed representations of $sl(3,C)$

4.1 As we mentioned in Sec.I.2.7, the canonical realizations[7,8] of $gl(n+1,\mathbb{C})$ which are the essence of our construction are obtained recursively with the help of n canonical pairs, one complex parameter and a realization of $gl(n,\mathbb{C})$. The latter is not necessarily a canonical realization of the same type, one may use instead of it, e.g., a matrix representation. Here we shall employ this possibility for constructing representations of $sl(3,\mathbb{C})$.

We start from the formulae (5), however, instead of the operators (1) we shall use matrices which generate the (2k+1)-dimensional irreducible representation of $sl(2,\mathbb{C})$, $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ The appropriate representation space U_3^{2k} will be of the form $L_3 \otimes \mathbb{C}^{2k+1}$ (cf.Sec.I.4.5), in other words, it will be spanned by the vectors

$$\left\| \begin{matrix} m & n \\ s & \end{matrix} \right\| \quad , \quad m,n \in N_0 \quad , \quad s = -k,-k+1,\dots,k \quad . \tag{13}$$

The mentioned (2k+1)-dimensional matrix representation generates a representation of $sl(2,\mathbb{C})$ on U_3^{2k} in the following way

$$\begin{aligned} E_{12} \left\| \begin{matrix} m & n \\ s & \end{matrix} \right\| &= (k+s) \left\| \begin{matrix} m & n \\ s-1 & \end{matrix} \right\| \quad , \\ E_{21} \left\| \begin{matrix} m & n \\ s & \end{matrix} \right\| &= (k-s) \left\| \begin{matrix} m & n \\ s+1 & \end{matrix} \right\| \quad , \\ H_1 \left\| \begin{matrix} m & n \\ s & \end{matrix} \right\| &= 2s \left\| \begin{matrix} m & n \\ s & \end{matrix} \right\| \quad . \end{aligned} \tag{14}$$

To any $\Lambda = (2k, \Lambda_2)$ we get the representation μ_Λ of $sl(3,\mathbb{C})$ whose generators are obtained by substituting (14) into (5) ; we shall call it <u>mixed representation</u> of $sl(3,\mathbb{C})$. We shall write $\mu_\Lambda(e_{ij}) \equiv E^m_{ij}$, $\mu_\Lambda(h_i) = H^m_i$, whenever it would be necessary to distinguish the generators of μ_Λ from those of the maximal representation ρ_Λ .

4.2 <u>Proposition</u> : Let $2k \in N_0$, $\Lambda_2 \notin N_0$, then the representation $\mu_\Lambda : sl(3,\mathbb{C}) \to \mathcal{L}(U_3^{2k})$ given by (5) and (14) is irreducible ;

it has the highest weight $\Lambda = (2k, \Lambda_2)$ and the highest-weight vector $y_0 \equiv \begin{Vmatrix} 0 & 0 \\ k & \end{Vmatrix}$.

__Proof__ : Clearly $H_1^m y_0 = 2ky_0$, $H_2^m y_0 = \Lambda_2 y_0$ and $E_{ij}^m y_0 = 0$ for $i > j$. The representation μ_Λ has to be equivalent to $\tilde{\rho}_\Lambda = \rho_\Lambda \upharpoonright V_3^\Lambda$ (see (7)) in view of Theorem I.2.4 if the present assertion holds. On the other hand, it is sufficient to verify this equivalence, because $\tilde{\rho}_\Lambda$ is irreducible due to Corollary I.4.2 . Let us define the following isomorphism $S_\Lambda : V_3^\Lambda \to U_3^{2k}$:

$$S_\Lambda \begin{vmatrix} m & n \\ s & \end{vmatrix} = (-1)^{2k-s} \sqrt{2k(2k-1)\ldots(s+1)} \begin{Vmatrix} m & n \\ k-s & \end{Vmatrix} \tag{15a}$$

for $s = 0,1,\ldots,2k$, then the relations

$$\mu_\Lambda(h_i) = S_\Lambda \tilde{\rho}_\Lambda(h_i) S_\Lambda^{-1} ,$$
$$\mu_\Lambda(e_{ij}) = S_\Lambda \tilde{\rho}_\Lambda(e_{ij}) S_\Lambda^{-1} \tag{15b}$$

can be verified directly. ∎

4.3 We have mentioned that irreducible highest-weight representations of $sl(3,\mathbb{C})$ with $\Lambda_1 \notin N_0$, $\Lambda_2 \in N_0$ can be constructed with the help of those previously obtained. To this purpose we define a linear mapping τ of $sl(3,\mathbb{C})$ into itself by

$$\begin{aligned} &\tau(h_1) = h_2 , \quad \tau(h_2) = h_1 , \\ &\tau(e_{21}) = e_{32} , \quad \tau(e_{31}) = -e_{31} , \quad \tau(e_{32}) = e_{21} , \\ &\tau(e_{12}) = e_{23} , \quad \tau(e_{13}) = -e_{13} , \quad \tau(e_{23}) = e_{12} . \end{aligned} \tag{16}$$

__Proposition__ : Let $\Lambda = (\Lambda_1, \Lambda_2)$ where $\Lambda_1 \notin N_0$, $\Lambda_2 = 2k \in N_0$ and denote $\Lambda' = (\Lambda_2, \Lambda_1)$. Then the mappings $\tilde{\rho}_{\Lambda'}^{(3)}\tau$ and $\mu_{\Lambda'}\tau$ are (mutually equivalent) irreducible representations of $sl(3,\mathbb{C})$ with the highest weight Λ and the highest-weight vectors $x_0^3 \in V_3^\Lambda$ and $y_0 \in U_3^{2k}$, respectively.

__Proof__ : One can check easily that τ is an automorphism of $sl(3,\mathbb{C})$. Since the representations $\tilde{\rho}_{\Lambda'}$ and $\mu_{\Lambda'}$ are equivalent due to (15b) the same is true for $\tilde{\rho}_{\Lambda'}\tau$ and $\mu_{\Lambda'}\tau$. The automorphism τ conserves the subalgebra L_+ of $L = sl(3,\mathbb{C})$ so the condition I.2.3(ii) is fulfilled. Finally

$$\tilde{\rho}_{\Lambda'}\tau(h_1)x_0^3 = \tilde{\rho}_{\Lambda'}(h_2)x_0^3 = \Lambda'(h_2)x_0^3 = \Lambda_1 x_0^3 ,$$

$$\tilde{\rho}_{\Lambda}\cdot\tau(h_2)x_0^3 = \Lambda_2 x_0^3 \quad ;$$

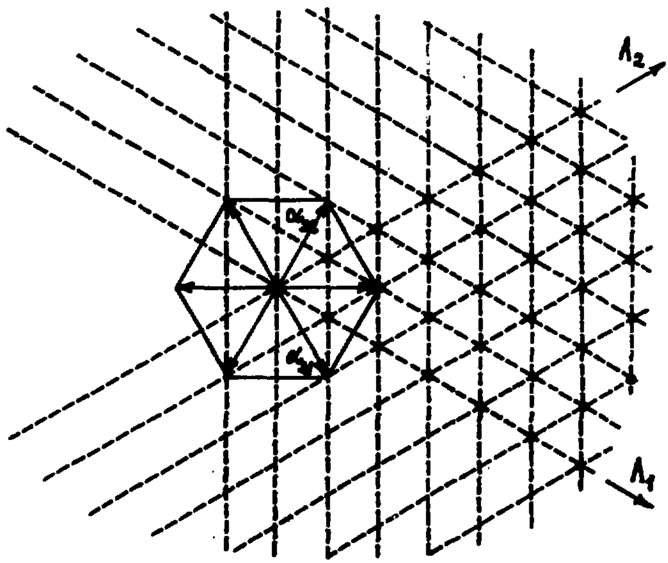
and the analogous relations are valid for $\mu_{\Lambda}\cdot\tau$ too. ■

Remark : *The formulae determining these representations explicitly are obtained immediately from (1),(5),(16) and (5),(14),(16), respectively.*

5. Discussion

In the last section of Ref.1 we compared our results to other known highest-weight representations of $sl(n+1,\mathbb{C})$, especially to the elementary representations (Verma modules). The considerations of the present paper allow us to add some items to this comparison.

The maximal representation $\rho_{\Lambda}^{(3)}$ of $sl(3,\mathbb{C})$ is irreducible under the conditions (6) ; one can check that the same is true for the elementary representation d_{Λ} using Theorem 6 of Ref.2 . This is sketched on Fig.1 : for the weights corresponding to the points



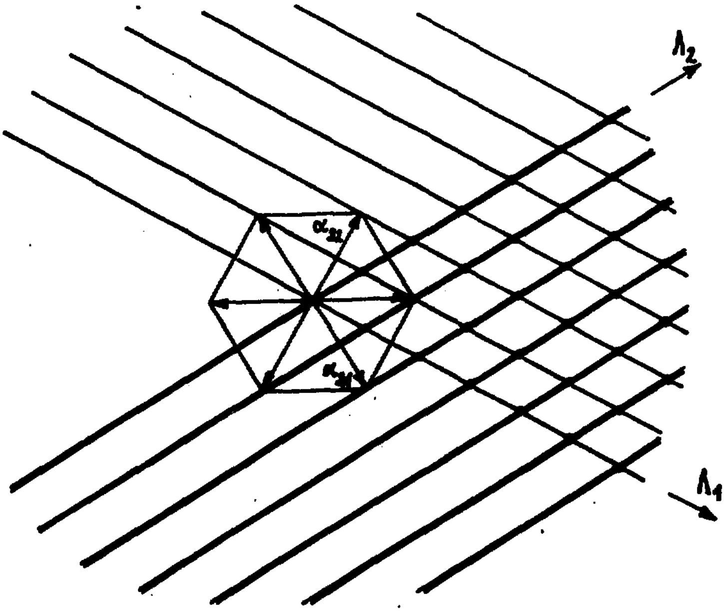
• irreducible highest-weight representations are finite-dimensional

—•— d_{Λ} and ρ_{Λ} reducible

Fig.1

which do not belong to the dashed lines both the types of representations are irreducible. As for the general $sl(n+1,\mathbb{C})$ algebras, our irreducibility condition (I.13) is up to now proved only to be sufficient for $n \geq 3$. On the other hand, our condition is stated directly in terms of the weight *components*, while for the use of the mentioned irreducibility criterion (Theorem 6 of Ref.2) for elementary representations action of the Weyl group elements on a given weight must be analyzed.

We have obtained also other irreducible highest-weight representations of $sl(3,\mathbb{C})$ using procedures described in Secs.3.3(b) and 4.1-4.3 . They are again of the form which makes calculation of matrix elements of the generators quite simple. They cover a part of the cases when the maximal as well as the elementary representations are reducible as shown on Fig.2 . This construction



the representations $\tilde{\rho}_\Lambda \sim \mu_\Lambda$

the representations $\tilde{\rho}_\Lambda^\tau \sim \mu_\Lambda^\tau$

Fig.2

generalizes easily for the highest-weight representations of $sl(n+1,\mathbb{C})$ when some of the Λ_i's are non-negative integers and the rest of the conditions (I.13) is fulfilled. The remaining cases need a different approach.

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