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THE SCHRÖDINGER EQUATION IN THE QUANTUM FIELD THEORY WITH NONLOCAL INTERACTIONS

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1. Introduction

Construction of a self-consistent theory of nonlocal interactions of quantized fields /1/ became possible owing to the following two ideas.

First, the form factors must be entire analytical functions in the momentum space and must decrease rapidly enough in the Euclidean space. Second, the form factor must be quantized, i.e., it is necessary to introduce supplementary degrees of freedom, which determine the regularization in order to enable the transition to the Euclidean metric and re-establishment of the form factor in the limit of cancelled regularization. Development of these ideas allowed the construction of the finite and unitary S-matrix for arbitrary enough interaction Lagrangians in each perturbation order /1/.

What is incomplete in this construction? The following problem arose from investigations of the causality conditions. The coefficient functions of the S-matrix in the configuration space turn out to be analytical functions. Analytic methods used in the investigation of local properties of analytical functionals make it impossible, in principle, to determine a space localization of studied functionals within an accuracy of a certain distance given by the nonlocality /2/. Obviously, the use of nonanalytical methods is needed. However, these methods are not developed. The results obtained by the projecting sequences of functions /1/ rouse doubts because, as is shown in ref. /3/, there are examples of explicitly nonlocal functionals which are, as local ones, characterized by the projecting sequences of functions. Therefore, the existence of the microcausality condition, understood as a strict equality of the corresponding functional outside of the light cone, remains an open problem in the theory with nonlocal interactions.

On the other hand, causality is nothing else than correctness of the Cauchy problem of the quantum-field Schrödinger equation (or equation of Tomanaga and Schwinger). However, the utilization of the regularization procedure in the construction of S-matrix both in the local and nonlocal theories reduces to that the S-matrix is not a solution of the corresponding equation and is determined by a series of limits. It seems therefore that the natural properties of the Shcrödinger equation solutions as unitarity, and causality are to be proved separately.

Difficulties in nonlocal theory arise usually when a nonlocal form factor is introduced into the interaction Lagrangian, but the Schrödinger field equation (or Tomonaga-Schwinger equation) remains local

$$i \frac{\delta}{\delta\sigma(x)} \Psi\left[\sigma(x)\right] = \mathcal{H}_I(x)\,\Psi(\sigma(x)), \tag{1.1}$$

where, for example,

$$\mathcal{H}_I(x) = g\left[\int dx'\, V(x-x')\,\varphi(x')\right]^4.$$

The integrability conditions are violated within this approach and numerous other difficulties arise (see ref. /4/, for example).

We assume that while introducing nonlocality in the interaction Hamiltonian the equation of Tomonaga and Schwinger must be treated also nonlocally but with retardation. In this way only, in our opinion, one can get rid of difficulties caused by the integrability conditions in the nonlocal theory.

In this paper we shall show that the S-matrix describing nonlocal interactions of quantized fields /1/ solves the Cauchy problem of the evolution equation (or Schrödinger equation in the interaction picture at imaginary time, i.e., in the Euclidean metric) with retardation. In this way supplementary degrees of freedom with respect to the Fock space of physical particles need not be introduced.

Strictly speaking, formulation of such an equation with the correctly stated Cauchy problem at imaginary time does not answer directly the question about the causality condition of the S-matrix in Minkowski space. However, a simple analytical connection between S-matrices both in Euclidean and Minkowski spaces without any doubts means that causality of the evolution equation must ensure absence of any physically observable noncausal phenomena.

2. Field Operator at Imaginary Time

We shall consider the theory of a one component scalar field $\varphi(x)$ describing particles with mass m. The field operator $\varphi(x)$ may be written in a standard way (see [5], for example)

$$\varphi(x)=\varphi(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int\frac{d\vec{k}}{\sqrt{2\omega}}\left(a_{\vec{k}}e^{-i\omega t+i\vec{k}\vec{x}}+a^{+}_{\vec{k}}e^{i\omega t-i\vec{k}\vec{x}}\right), \quad (2.1)$$

where $$\omega=(m^2+\vec{k}^2)^{1/2}.$$

Creation $a^{+}_{\vec{k}}$ and annihilation $a_{\vec{k}}$ Boson operators satisfy ordinary commutation rules:

$$[a_{\vec{k}},a_{\vec{k}'}]=[a^{+}_{\vec{k}},a^{+}_{\vec{k}'}]=0\,, \qquad [a_{\vec{k}},a^{+}_{\vec{k}'}]=\delta^{(3)}(\vec{k}-\vec{k}'). \quad (2.2)$$

We assume that there exists a single vacuum state $\Psi_0=|0\rangle$ which obeys the conditions:

$$\langle 0|0\rangle=1\,,\qquad a_{\vec{k}}|0\rangle=0 \qquad \forall\,\vec{k}. \quad (2.3)$$

State vectors of scalar particles are represented by rays in the Fock space which is,as usually,constructed over the basis

$$\Psi^{(n)}_{\vec{k}_1,\dots\vec{k}_n}=\frac{1}{\sqrt{n!}}a^{+}_{\vec{k}_1}\dots a^{+}_{\vec{k}_n}|0\rangle \quad (2.4)$$

where n=0,1,2,... .

Now we pass to imaginary time $t\to -i\tau$, or to the Euclidean metric. In the constructive quantum field theory the physical Hilbert space F of a free field in Minkowski space is considered as a subspace of the Hilbert space $\mathcal{N}$ of a free Euclidean field (see ref.[6]).Especially,in order to obtain

Euclidean Green functions and to construct the scattering matrix /7/ in creation and annihilation operators $a^+_{\vec{k}}$ and $a_{\vec{k}}$ the supplementary degree of freedom $a^+_{\vec{k}} \to a^+_{\vec{k},\varepsilon}$ ($a_{\vec{k}} \to a_{\vec{k},\varepsilon}$) connected with imaginary time is introduced so that commutation rules are of the form

$$\left[a_{\vec{k},\varepsilon}\, ,\, a^+_{\vec{k}',\varepsilon'}\right] = \delta^{(3)}(\vec{k}-\vec{k}')\,\delta(\varepsilon-\varepsilon') = \delta^{(4)}(k_E - k_E'), \qquad (2.5)$$

where $\quad k_E = (\varepsilon,\, \vec{k})$.

Here we shall not enlarge the number of degrees of freedom of a scalar field and we shall construct a space of our Euclidean states over the same basis (2.4).

So, we introduce the free field $\phi(x_E)$, where $x_E = (\tau, \vec{x})$ in the Euclidean space, i.e., at imaginary time by the replacement $t \to -i\tau$ in the expression (2.1) for the field operator $\varphi(t,\vec{x})$:

$$\phi(x_E) = \phi(\tau,\vec{x}) = \varphi(-i\tau,\vec{x}) =$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{\sqrt{2\omega}} \left(a_{\vec{k}}\, e^{-\omega\tau - i\vec{k}\vec{x}} + a^+_{\vec{k}}\, e^{\omega\tau + i\vec{k}\vec{x}} \right). \qquad (2.6)$$

Let us introduce the T-product operation, i.e., the imaginary-time τ ordering operation. In representation (2.6) the parameter τ is introduced explicitly, so the T product operation is defined straightforwardly:

$$T\Big(\phi(x_{1E})\cdots\phi(x_{nE})\Big) = \phi(x_{nE})\cdots\phi(x_{1E}) \qquad (2.7)$$

$$(\tau_1 \le \tau_2 \le \tau_3 \le \ldots \le \tau_n).$$

Further, we shall define the two-point Euclidean Green function which will be called causal

$$\Delta_c(x_{1E} - x_{2E}) = \langle 0| T\Big(\phi(x_{1E})\,\phi(x_{2E})\Big)|0\rangle =$$

$$= \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2\omega}\, e^{-\omega|\tau_1-\tau_2| + i\vec{k}(\vec{x}_1-\vec{x}_2)} =$$

$$= \frac{1}{(2\pi)^4} \int \frac{d^4k_E \; e^{ik_E(x_{1E}-x_{2E})}}{m^2 + k_E^2} =$$

$$= \frac{m}{(2\pi)^2} \; \frac{\mathcal{K}_1\left(m\sqrt{(x_{1E}-x_{2E})^2}\right)}{\sqrt{(x_{1E}-x_{2E})^2}} \, , \qquad (2.8)$$

where $k_E = (\varepsilon, \vec{k})$, $k_E^2 = \varepsilon^2 + \vec{k}^2$ and $\mathcal{K}_1(z)$ is the Macdonald function. The obtained function $\Delta_c(x_E)$ represents an analytical continuation to $t \to -i\tau$ of the causal Green function in Minkowski space:

$$\Delta_c(x_1-x_2) = \frac{1}{(2\pi)^4 i} \int \frac{d^4k \; e^{-ik(x_1-x_2)}}{m^2 - k^2 - i\varepsilon} \, .$$

Now let us consider the commutator of the fields $\phi(x_E)$ and the function $\Delta_{(-)}(\cdots)$:

$$[\phi(x_{1E})\phi(x_{2E})] = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2\omega}\, e^{i\vec{k}(\vec{x}_1-\vec{x}_2)} \cdot 2\,\mathrm{Sh}\,\omega(\tau_1-\tau_2) \qquad (2.9)$$

$$\Delta_{(-)}(x_{1E}-x_{2E}) = \langle 0|\phi(x_{1E})\phi(x_{2E})|0\rangle = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2\omega}\, e^{i\vec{k}(\vec{x}_1-\vec{x}_2) - \omega(\tau_1-\tau_2)} \, .$$

We see easily that integrals in the right-hand side of the formula (2.9) do not exist under an arbitrary choice of the difference $(\tau_1-\tau_2)$. Therefore, only the T-product of operators $\phi(x_E)$ has a reasonable mathematical meaning.

Let us introduce operators $R[\phi]$ of the following type:

$$R[\Phi]=\sum_n \frac{1}{n!}\int dx_{1E}\ldots\int dx_{nE}\, R_n(x_{1E},\ldots,x_{nE})\cdot$$
$$\cdot\, T\big(\Phi(x_{1E})\ldots\phi(x_{nE})\big). \qquad (2.10)$$

Operators $R[\Phi]$ **are defined by a set of functions** $\{R_n(x_{1E},\ldots x_{nE})\}$ **, the properties of which will be elaborated below.**

Let us define the operation of conjugation

$$\big(\Phi(x_E)\big)^{*}=\big(\varphi^{+}(i\tau,\vec{x})\big)^{+}=\varphi(-i\tau,\vec{x})=\Phi(x_E). \qquad (2.11)$$

Then for the operator R **in (2.10) we obtain the following expression**

$$R^{*}[\Phi]=\sum_n \frac{1}{n!}\int dx_{1E}\ldots\int dx_{nE}\, R^{*}(x_{1E},\ldots,x_{nE})\cdot \qquad (2.12)$$
$$\cdot\, T\big(\Phi(x_{1E})\ldots\phi(x_{nE})\big).$$

Now we determine the operation of multiplication of two operators $R_1[\Phi]$ **and** $R_2[\Phi]$ **of the type (2.10) by definition:**

$$R_1[\Phi]R_2[\Phi]\underset{\mathrm{df}}{=}T\big(R_1[\Phi]R_2[\Phi]\big)= \qquad (2.13)$$
$$=\sum_{n_1,n_2}\frac{1}{n_1!\,n_2!}\int dx_{1E}\ldots\int dx_{n_1E}\int dy_{1E}\ldots\int dy_{n_2E}\, R_{n_1}(x_{1E},\ldots,x_{n_1E})\cdot$$
$$\cdot R_{n_2}(y_{1E},\ldots y_{n_2E})\,T\big(\Phi(x_{1E})\ldots\phi(x_{n_1E})\phi(y_{1E})\ldots\phi(y_{n_2E})\big)$$

3. The State Space at Imaginary Time

Let us define the state space $\mathcal{N}$ of the system as a set of vectors of the type

$$\Psi = R[\Phi]\,|0\rangle\,, \tag{3.1}$$

where R has the form (2.10). It will be assumed that the state (3.1) is given if all the functions $R_n(x_{1E},\dots,x_{nE})$ determining the operator $R[\Phi]$ in (2.10) are known.

Notice that any state in the Fock space F must be represented in the way (3.1) in the case of the corresponding choice of the set of functions $\{R_n(x_{1E},\dots,x_{nE})\}$ in the operator $R[\Phi]$ since there exists a simple linear connection between sets of basic vectors:

$$\left\{\begin{array}{r} |0\rangle \\ a^+_{\vec{k}_1}|0\rangle \\ a^+_{\vec{k}_1}a^+_{\vec{k}_2}|0\rangle \\ a^+_{\vec{k}_1}\dots a^+_{\vec{k}_3}|0\rangle \\ \dots\dots\dots \\ \dots\dots\dots \\ a^+_{\vec{k}_1}\dots\dots a^+_{\vec{k}_n}|0\rangle \end{array}\right\} \rightleftarrows \left\{\begin{array}{r} |0\rangle \\ \phi(x_{1E})|0\rangle \\ T(\phi(x_{1E})\,\phi(x_{2E}))|0\rangle \\ T(\phi(x_{1E})\dots\phi(x_{3E}))|0\rangle \\ \dots\dots\dots \\ \dots\dots\dots \\ T(\phi(x_{1E})\dots\phi(x_{nE}))|0\rangle \end{array}\right\} \tag{3.2}$$

The appropriate formula of this correspondence can be easily found, if necessary.

The most important here is the following: the Fock space F consists of vectors of the type

$$\Psi_F = \sum_n \frac{1}{n!}\int d\vec{k}_1\dots\int d\vec{k}_n\, f_n(\vec{k}_1,\dots\vec{k}_n)\frac{1}{\sqrt{n!}}\,a^+_{\vec{k}_1}\dots a^+_{\vec{k}_n}\,|0\rangle \tag{3.3}$$

but the space $\mathcal{N}$ of vectors

$$\Psi=\sum_n \frac{1}{n!}\int dx_{1E}\cdots\int dx_{nE}\, R_n(x_{1E},\ldots,x_{nE})\cdot$$

$$\cdot\, T\big(\phi(x_{1E})\ldots\phi(x_{nE})\big)|0\rangle \tag{3.4}$$

Obviously, from representations (3.3) and (3.4) it follows that the space F is a subspace of $\mathcal{N}$, since some set of mutually different vectors (different sets of functions $R_n(\cdots)$) from $\mathcal{N}$ corresponds to each vector from F.

In the scattering problem the initial state of the type (3.3) is given at some $\tau_0=-T$, where $T\to\infty$. In this case such a state can be written in the form

$$\Psi(\tau_0)=\sum_n \frac{1}{n!}\int d\vec{x}_1\ldots\int d\vec{x}_n\, \mathcal{T}_n(\vec{x}_1,\ldots,\vec{x}_n)\cdot$$

$$\cdot\, T\big(\phi(\tau_0,\vec{x}_1)\ldots\phi(\tau_0,\vec{x}_n)\big)|0\rangle, \tag{3.5}$$

where $\mathcal{T}_n(\vec{x}_1,\ldots,\vec{x}_n)$ is connected simply with $f_n(\vec{k}_1,\ldots,\vec{k}_n)$ in (3.3). The state (3.5) can be also rewritten in the form (3.4), where

$$R_n(x_{1E},\ldots,x_{nE})=\mathcal{T}_n(\vec{x}_1,\ldots,\vec{x}_n)\prod_{j=1}^{n}\delta(\tau_j-\tau_0).$$

The norm of state vectors from $\mathcal{N}$ is given by

$$\|\Psi\|^2=(\Psi,\Psi)=\langle 0|R^*[\phi]R[\phi]|0\rangle=$$

$$=\sum_{n_1,n_2}\frac{1}{n_1!\,n_2!}\int dx_{1E}\ldots\int dx_{n_1E}\int dy_{1E}\ldots\int dy_{n_2E}\, R^*_{n_1}(x_{1E},\ldots,x_{n_1E})\cdot \tag{3.6}$$

$$\cdot\, R_{n_2}(y_{1E},\ldots y_{n_2E})\langle 0|T\big(\phi(x_{1E})\ldots\phi(x_{n_1E})\phi(y_{1E})\ldots\phi(y_{n_2E})\big)|0\rangle.$$

It is known (see /6,8/, for example) that for a free Euclidean field there exists such a Gaussian positive measure $d\mu_f$ that

$$\int d\mu_f\, f(x_{1E})\cdots f(x_{nE}) = \langle 0|T(\phi(x_{1E})\cdots\phi(x_{nE}))|0\rangle . \qquad (3.7)$$

Then the Euclidean norm (3.6) is

$$\|\Psi\|^2 = \int d\mu_f\, |R[f]|^2 < \infty , \qquad (3.8)$$

where the functional $R[f]$ is given by

$$R[f] = \sum_n \frac{1}{n!}\int dx_{1E}\cdots\int dx_{nE}\, R_n(x_{1E},\dots,x_{nE})\, f(x_{1E})\cdots f(x_{nE}) .$$

Therefore, functions $R_n(x_{1E},\cdots,x_{nE})$ must be such that the norm (3.8) is finite.

So, we have constructed the state space $\mathcal{N}$, which includes all vectors of the type (3.1) for which the norm (3.6) (or (3.8)) is finite. Furthermore, the operators $R[\phi]$ (2.10) have been introduced for which the operations of conjugation (2.12) and multiplication (2.13) are defined.

4. The Interaction Hamiltonian and The Evolution Equation

The dynamics of a quantum field system is described by the Schrödinger equation. In the interaction picture it has the form

$$i\frac{\partial}{\partial t}\Psi(t) = H_I(t)\,\Psi(t) \qquad (4.1)$$

with some initial condition:

$$\Psi(t_0) = \Psi_0 . \qquad (4.2)$$

Passing to imaginary time $t \to -i\tau$ we get

$$\frac{\partial}{\partial\tau}\Psi(\tau) = -H_I(\tau)\Psi(\tau)$$

$$\Psi(\tau_0) = \Psi_0 . \qquad (4.3)$$

It is customary to call the equations (4.3) as evolution equations.

In the local quantum field theory interaction Hamiltonians are usually of the form of polynomials in the field operators. For example, for the self-interacting scalar field we have

$$H_I(t) = g\int d\vec{x}\, \varphi^4(t,\vec{x}) \tag{4.4}$$

or in the imaginary-time formulation

$$H_I(\tau) = g\int d\vec{x}\, \phi^4(\tau,\vec{x}). \tag{4.5}$$

If we look for a solution of the equation (4.3) with the Hamiltonian (4.5) in the form (3.1), then we have standard problem of the local quantum field theory with all its difficulties.

Consider now the quantum field theory with the nonlocal interaction. Introduce the spreaded field:

$$\phi_a(x) = \phi_a(\tau,\vec{x}) = \int dy_E\, a(y_E^2)\,\phi(\tau+\xi, \vec{x}+\vec{y}), \tag{4.6}$$

where $y_E = (\xi, \vec{y})$, $y_E^2 = \xi^2 + \vec{y}^{\,2}$ and $a(y_E^2)$ is a real function, the properties of which will be discussed later.

Let us calculate the causal Green function of the spreaded field (having in mind that the T-ordering symbol concerns the field $\phi(x)$):

$$\mathcal{D}(x_{1E}-x_{2E}) = \langle 0|T\left(\phi_a(x_{1E})\,\phi_a(x_{2E})\right)|0\rangle = \int dy_{1E}\int dy_{2E}\, a(y_{1E}^2)\,a(y_{2E}^2)\cdot$$

$$\cdot\langle 0|T\left(\phi(x_{1E}+y_{1E})\,\phi(x_{2E}+y_{2E})\right)|0\rangle = \int dy_{1E}\int dy_{2E}\, a(y_{1E}^2)\,a(y_{2E}^2)\cdot \tag{4.7}$$

$$\cdot\int \frac{d^4k_E}{(2\pi)^4}\,\frac{e^{-ik_E(x_{1E}+y_{1E}-x_{2E}-y_{2E})}}{m^2+k_E^2} = \int \frac{d^4k_E}{(2\pi)^4}\,\frac{\left[K(k_E^2)\right]^2}{m^2+k_E^2}\,e^{-ik_E(x_{1E}-x_{2E})},$$

where
$$\mathcal{K}(\kappa_E^2)=\int dy_E\, a(y_E^2)\, e^{-i\kappa_E y_E}. \tag{4.8}$$

We shall assume that the function $a(y_E^2)$ is chosen so that the $\mathcal{K}(\kappa^2)$ is an entire analytic function in the complex κ^2 - plane and that it increases so rapidly as $\kappa_E^2\to\infty$ that

$$\mathcal{D}(0)=\int\frac{d^4\kappa_E}{(2\pi)^4}\,\frac{[\mathcal{K}(\kappa_E^2)]^2}{m^2+\kappa_E^2}<\infty\,. \tag{4.9}$$

Instead of the interaction (4.5) we write

$$H_I[\tau,\varphi]=g\int d\vec{x}\; T\{\varphi_a^4(\tau,\vec{x})\}= \tag{4.10}$$
$$=g\int d\vec{x}\prod_{j=1}^{4}\int dy_{jE}\, a(y_{jE}^2)\,T\Big\{\prod_{k=1}^{4}\varphi(\tau+\tau_k,\vec{x}+\vec{y}_k)\Big\}.$$

Therefore, the interaction Hamiltonian (4.10) belongs to the class of operators $R[\varphi]$ (2.10). Notice that the interaction Hamiltonian may be chosen in the normal form:

$$H_I[\tau,\varphi]=g\int d\vec{x}:T\{\varphi_a^4(\tau,\vec{x})\}:= \tag{4.11}$$
$$=g\int d\vec{x}\;T\{\varphi_a^4(\tau,\vec{x})-6\varphi_a^2(\tau,\vec{x})\mathcal{D}(0)+3\mathcal{D}^2(0)\},$$

where $\mathcal{D}(0)$ is given by (4.9). In this case
$$\langle 0|:H_I[\tau,\varphi]:|0\rangle=0.$$

Using the representation of the state vector $\Psi(\tau)$ (3.1) we shall write the evolution equation (4.3) for the nonlocal interaction (4.10) or (4.11) in the following form:

$$\frac{\partial}{\partial\tau}R[\tau,\varphi]|0\rangle=-T\{H_I[\tau,\varphi]R[\tau,\varphi]\}|0\rangle. \tag{4.12}$$

The operation of multiplication in the right-hand side of (4.12) is defined by (2.13). The initial condition for equation (4.12) is

$$\Psi(\tau_0)=R[\tau_0,\phi]|0\rangle = R_0[\phi]|0\rangle . \tag{4.13}$$

So, the obtained evolution equation is retarded, since the "time" τ in the nonlocal Hamiltonian $H_I(\tau)$ (4.10,4.11) may precede the times in field operators $R[\tau,\phi]$. However, the T-ordering operation in equation (4.12) arranges the times appropriately.

Equation (4.12) with the initial condition (4.13) may be rewritten as

$$\frac{\partial}{\partial\tau} R[\tau,\phi]=-T\left\{H_I[\tau,\phi]R[\tau,\phi]\right\}, \tag{4.14}$$

$$R[\tau_0,\phi]= R_0[\phi]. \tag{4.15}$$

Notice that if the solution (4.15) is written in the form $R(\tau,\tau_0)$ then it does not satisfy the condition

$$R(\tau,\tau_0)=R(\tau,\tau_1)R(\tau_1,\tau_0) \qquad (\tau_0<\tau_1<\tau),$$

where multiplication is understood in the ordinary sense. This is an immediate consequence of nonlocality of the theory.

Nevertheless, the Cauchy problem for equation (4.14) can be formulated.

If we introduce the space cut-offs $g\to g(\vec{x})$ in order to get rid of the difficulties caused by the Haag theorem, the Hamiltonian $H_I[\tau,\phi]$ represents an operator in the considered space $\mathcal{N}$. We have

$$H_I[\tau,\phi]=\int d\vec{x}\,g(\vec{x}):T\left\{\phi_u^4(\tau,\vec{x})\right\}: \tag{4.16}$$

Considering the norm of the state

$$\Psi=H_I[\tau,\phi]|0\rangle$$

we get

$$\|\Psi\|^2 = \langle 0 | T\{ H_I[\tau,\phi] H_I[\tau,\phi]\} | 0\rangle =$$

$$= \int d\vec{x}_1 \int d\vec{x}_2\, g(\vec{x}_1) g(\vec{x}_2) \cdot 6\, \mathcal{D}^2(\vec{x}_1 - \vec{x}_2) \leq 6 \cdot \mathcal{D}^2(0) \left[\int d\vec{x}\, g(\vec{x}) \right]^2 < \infty .$$

Therefore, the ultraviolet catastrophe is absent in the considered nonlocal theory.

Let us turn now to the solution of equation (4.14) with the initial condition (4.15). The solution is given by

$$R[\tau,\phi] = T\left\{ exp\left[-\int_{\tau_0}^{\tau} d\tau' H_I[\tau',\phi] \right] R_0[\phi] \right\}. \tag{4.17}$$

For the state vector we get

$$\Psi(\tau) = R[\tau,\phi] | 0\rangle . \tag{4.18}$$

The norm of state $\Psi(\tau)$ (4.18) equals

$$\|\Psi(\tau)\|^2 = \int d\mu_f\, exp\left\{ -2\int_{\tau_0}^{\tau} d\tau' H_I[\tau',f] \right\} \cdot |R_0[f]|^2 \tag{4.19}$$

If the Hamiltonian $H_I[\tau,f]$ in (4.19) is chosen in the form (4.16), then owing to (4.11) we have

$$H_I[\tau,f] \geq -6\,\mathcal{D}^2(0) \int d\vec{x}\, g(\vec{x}).$$

For the norm (4.19) we get

$$\|\Psi(\tau)\| \leq \|\Psi_0\|\, exp\left\{ 6(\tau-\tau_0)\mathcal{D}^2(0) \int d\vec{x}\, g(\vec{x}) \right\} . \tag{4.20}$$

So, we see that the solution exists in the state space $\mathcal{N}$. Uniqueness of the obtained solution follows straightforwardly from (4.17).

The obtained solution (4.17) produces the S-matrix that coincides completely with the S-matrix constructed and investigated in ref./1/.

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