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ON ORIGIN OF NONLOCAL CONSERVED CURRENTS FOR GENERALIZED NONLINEAR SIGMA MODELS

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1. INTRODUCTION

In the last years completely integrable field theoretical models, on a classical and on a quantum level, are under intensive investigation. The existence of infinite set of conserved quantities is one of characteristics of such models. In the quantum two-dimensional case availability of such quantities makes it possible to derive the explicit form of S-matrix [1,2]. Simultaneously with the local conserved charges, for the O(N) nonlinear sigma model there exists and infinite set of nonlocal conserved charges on a classical [3] and quantum [4] level. These nonlocal conserved quantities can be also used for construction of S-matrix [4]. A simple constructive proof of existence of one infinite series of classical conserved currents for the generalized nonlinear sigma models was given in ref. [5]. The nonlocal currents in the four-dimensional case were found also for the self-dual sector of the Yang-Mills field [6], for arbitrary Yang-Mills field [7], and the Yang-Mills field defined on contours [8]. Such currents exist also for the two-dimensional supersymmetric O(N) [9] and supersymmetric generalized [10] nonlinear sigma models.

For certain models it was established that higher conserved currents are a consequence of symmetry of the equation of motion with respect to some infinite-parameter Abelian group [11]. In the case of two-dimensional generalized non-linear sigma models it is shown in ref. [12] that the nonlocal conserved currents follow from the dual symmetry of the equation of motion.

In the present paper the origin of nonlocal currents in the case of generalized nonlinear sigma models is investigated. Such field transformations, for which the action is not invariant but is changed with an integral of full divergence of some function are considered. In the last case, according to the generalized Noether theorem [13], to any one-parameter transformation of such a kind there corresponds one conserved current. For the generator functions of transformations under consideration we have a system of nonhomogeneous first-order linear partial differential equations. These equations are a consequence of the dual symmetry discussed in ref. [12]. The solutions of these equations give the representation of the generator function of the transformations which give the nonlocal currents. The explicit form of these generators is found only in the two-dimensional case, but they can be found also in space with more than two dimensions. These representations in general are nonlinear and nonlocal. Consequently, the nonlocal currents are generated from nonlinear and nonlocal [14] transformations, however these nonlocal transformations are not symmetry of the equation of motion.

The results of this paper are extended for supersymmetric case in ref [15].

2. EQUATIONS FOR FUNCTIONS GENERATING NONLOCAL CURRENTS

Consider the action for the generalized nonlinear sigma models in h-dimensional space

$$S = \frac{\lambda^{2-h}}{2} \int d^h x \, \mathrm{tr}\{\partial^\mu g^{-1}(x) \partial_\mu g(x)\} = -\frac{\lambda^{2-h}}{2} \int d^h x \, \mathrm{tr}\{g^{-1}\partial^\mu g g^{-1}\partial_\mu g\}, \tag{2.1}$$

where $g(x) \in O(N)$, $U(N)$ or $GL(N)$, i.e., the general linear group in N dimensional isotopic space, and λ is a parameter with dimensionality of length. In the case when $g^{-1}(x) = g(x)$, i.e., $g_{jk}(x) = \delta_{jk} - 2P_{jk}$, where $P^2 = P$ we deal with $O(N)$, CP^N models or the field defined on Grassmann manifolds [16]. From (2.1) we have the Euler-Lagrange equations of motion, which can be written in the form

$$\partial^\mu A_\mu(x) = 0, \tag{2.2}$$

where the following notation is used

$$A_\mu(x) = g^{-1}\partial_\mu g(x) = -\partial_\mu g^{-1}(x) g(x). \tag{2.3}$$

From (2.3) it follows that

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0, \tag{2.4}$$

i.e., the curvature tensor vanishes identically. In (2.4)

$$D_\mu(x) = \partial_\mu + A_\mu(x) \tag{2.5}$$

means the matrix "covariant" derivative.

Consider the following infinitesimal "global" gauge transformations

$$g'(x) = U(x) g(x) U^{-1}(x) = g(x) + [\zeta^{(k)}(x), g(x)]\omega_k + O(\omega^2), \tag{2.6}$$

where ω_k are independent of x infinitesimal parameters, $\zeta^{(k)}(x)$ are generators of transformations $U(x)$, i.e.,

$$\zeta^{(k)}(x) = \frac{\partial U(x)}{\partial \omega^k}\Big|_{\omega=0}, \qquad (k = 1, 2, \dots). \tag{2.7}$$

The variation of action (2.1) with respect to the transformations (2.6) is

$$\begin{aligned} \delta S &= \lambda^{2-h} \int d^h x \, \mathrm{tr}\{A^\mu(x) \delta A_\mu(x)\} = \\ &= \lambda^{2-h} \int d^h x \, \mathrm{tr}\{A^\mu(x) (\partial_\mu \zeta^j(x) + [A_\mu(x), \zeta^j(x)])\}\omega_j = \\ &= \lambda^{2-h} \int d^h x \, \mathrm{tr}\{A^\mu(x) \partial_\mu \zeta^j(x)\}\omega_j, \end{aligned} \tag{2.8}$$

where

$$\delta A_\mu(x) = \delta g^{-1}\partial_\mu g + g^{-1}\partial_\mu \delta g = \{\partial_\mu \zeta^j + [A_\mu, \zeta^j]\}\omega_j \tag{2.9}$$

is substituted and $[B, C]$ denotes the matrix commutator.

Consequently, only such transformations for which $\partial_\mu \zeta^j = 0$ are invariance of action (2.1), i.e., $\delta S = 0$. In our paper a more general class transformations, for which $\delta S \neq 0$ is considered. The explicit form of these transformations can be determined from the following propositions:

<u>Proposition I</u>: The change of action (2.1) under transformations (2.6) must be represented as an integral of full divergence of some arbitrary function, i.e.,

$$\delta S = \lambda^{8-h} \int d^h x \; tr\{\partial^\mu K^j_\mu (x, g, \partial_\nu g)\} \omega_j , \qquad (2.10)$$

where $K^j_\mu (x, g, \partial_\nu g)$ is one h-vector function whose form is given by the second proposition.

Proposition II: We restrict ourselves only to such h-vector functions K^j_μ which can be represented in the following form

$$K^j_\mu = A^\nu (x) \chi^j_{\mu\nu}(x) , \qquad (j = 1, 2, \dots) , \qquad (2.11)$$

where $\chi^j_{\mu\nu} (x) = -\chi^j_{\nu\mu}(x)$, i.e., is an antisymmetric second rank tensor with $N\times N$ matrix components. The functions $\chi^j_{\mu\nu}(x)$ are connected with $\zeta^j(x)$ by the following theorem:

Theorem I: The necessary and sufficient condition for equivalence of (2.8) and (2.10), where $K^j_\mu(x)$ is given by (2.11), is that the functions $\zeta^j(x)$ and $\chi^j_{\mu\nu}(x)$ are coupled by the following system of differential equation

$$\partial_\mu \zeta^j(x) = D^\nu \chi^j_{\mu\nu}(x) . \qquad (2.12)$$

Indeed, substituting (2.12) into (2.8) and taking into account zero curvature, i.e., $F_{\mu\nu}=0$ we have (2.10) where K_μ is given by (2.11). And vice versa: substituting (2.11) into (2.10) taking into account (2.4) and comparing with (2.8) we have (2.12).

For the transformations (2.6) whose generators (2.7) satisfy eqs. (2.12), i.e., which change the action with integral of full divergence, there takes place the generalized Noether theorem [13].

Noether theorem: To any one-parameter transformation whose generators satisfy eqs. (2.12), i.e., which give the variation of action in the form of integral of full divergence, there corresponds one quantity

$$J^j_\mu (x) = tr\{A_\mu(x) \zeta^j(x) + A^\nu(x) \chi^j_{\mu\nu}(x)\}, \quad (j = 1,2\dots) \qquad (2.13)$$

which is conserved in a weak sense, i.e., $\partial^\mu J^j_\mu(x) = 0$, if the equations of motion (2.2) are satisfied.

Consequently, for any two functions ζ^j and $\chi^j_{\mu\nu}$ satisfying equation (2.12) we have one conserved current. Equation (2.12) is a first-order partial differential equation and consequently it has infinitely many solutions. As we will see, these solutions, in general, are nonlinear and nonlocal functionals of the field $g(x)$. From the latter it follows that we are dealing with nonlinear and nonlocal [14] transformation (2.6) whose generator functions are given by (2.7), i.e., they are solutions of (2.12). Conseqiently, the problem of finding of the nonlocal currents (2.13) and generators of the transformations giving these currents reduces to the problem of solving of eq. (2.12).

3. SOLUTION OF EQUATIONS FOR GENERATOR FUNCTIONS

Equations (2.12) have the following trivial solution

$$\zeta^j(x) = \text{const}, \quad \chi^j_{\mu\nu}(x) = 0. \qquad (3.1)$$

For this solution the current (2.13) coincides, up to a multiplictive constant, with (2.3). According to (3.1) transformations (2.6) are linear and local.

To find other solutions of (2.12) the following constraints

$$D^\mu \partial_\mu \zeta^j(x) = 0 , \qquad (3.2)$$

and

$$\partial_\mu D^\lambda \chi^j_{\nu\lambda} - \partial_\nu D^\lambda \chi^j_{\mu\lambda} = 0 , \qquad (3.3)$$

on the functions ζ^j and $\chi^j_{\mu\nu}$ must be taken into account. Equation (3.2) is a consequence of zero curvature (2.4) and (3.3) is the integrability condition for the system (2.12).

In a subsequent consideration we restrict ourselves only to the two-dimensional case, for which

$$\chi^j_{\mu\nu}(x) = \epsilon_{\mu\nu} \chi^j(x) , \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \qquad (3.4)$$

and, consequently, (2.12) and (3.3) take the forms

$$\partial_\mu \zeta(x) = \epsilon_{\mu\nu} D^\nu \chi(x) , \qquad (3.5)$$

and

$$D^{\mu}\partial_{\mu}\chi(x) = 0. \quad (3.6)$$

Hence for the two-dimensional case constraints (3.2) and (3.3) coincide. Moreover, it is convenient to change the parameters ω_i to matrix parameters ω. Then the conserved current (2.13) takes the form

$$J_{\mu}(x) = [A_{\mu}(x), \zeta(x)] + \epsilon_{\mu\nu}[A^{\nu}(x), \chi(x)]. \quad (3.7)$$

The fact that in the two-dimensional space the functions $\zeta(x)$ and $\chi(x)$ are Lorentz scalar matrices and satisfy the same second order equations (3.2) and (3.6) is useful for finding solutions of eqs. (3.5). Indeed, if we know solution $\chi^{(0)}(x)$ of (3.2) then by substitution of this function into the r.h.s. of (3.5) we are able to determine the function $\zeta = \chi^{(1)}$ from the equation

$$\partial_{\mu}\chi^{(1)}(x) = \epsilon_{\mu\nu}[D^{\nu}, \chi^{(0)}(x)]. \quad (3.8)$$

Because of (3.5) it follows that $\chi^{(1)}$ also satisfies equation (3.2) and consequently can also be inserted into the r.h.s. of (3.5). In such a way we are able to construct one infinite sequence of functions $\chi^{(k)}$ $(k = 0, 1, \dots)$ satisfying eq. (3.2) and functions $\chi^{(k)}$ and $\chi^{(k-1)}$ are coupled with equation (3.5) i.e.,

$$\partial_{\mu}\chi^{(k)} = \epsilon_{\mu\nu}[D^{\nu}, \chi^{(k-1)}], \quad (k = 1, 2, \dots). \quad (3.9)$$

If we have found M linear independent * solutions $\chi^{(0)}_{m}(x)$ $(m = 1, \dots, M)$ which are not coupled with multiple action of (3.5), we are able to construct by the given method M linear independent infinite sequences $\{\chi^{(k)}_{m}(x)\}$ $(k = 0, 1, \dots; m = 1, \dots, M)$. It can be pointed out that when $\zeta(\chi^{(k)})$ is given from eq.

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* We say that two sequences $\{\chi^{(k)}_{m}\}$ and $\{\chi^{(k)}_{\ell}\}$ are linear independent if any of them contains not less than one element $\chi^{(p)}_{\ell(m)}$ linear independent of all elements of other series $\{\chi^{(k)}_{m(\ell)}\}$.

eq. (3.5) or (3.9) we find $\chi(\chi^{k-1})$. In such a way sequences $\{\chi^{(k)}\}$ can be extended to negative k, i.e., we have

$$\dots, \chi^{(-k)}_{m}, \dots, \chi^{(-1)}_{m}, \chi^{(0)}_{m}, \chi^{(1)}_{m}, \dots, \chi^{(k)}_{m}. \quad (m = 1, \dots, M). \quad (3.10)$$

Because of (3.9) we have M infinite series of conserved currents

$$J^{(m,k)}_{\mu}(x) = [A_{\mu}(x), \chi^{(k)}_{m}(x)] + \epsilon_{\mu\nu}[A^{\nu}(x), \chi^{(k-1)}_{m}(x)], \quad (k = 0, \pm 1, \dots) \quad (m = 1, 2, \dots, M). \quad (3.11)$$

To find the solutions $\chi^{(0)}_{m}$ of eq. (3.2) from which we start the construction of sequency (3.10), like in the super-symmetric case /15/, let us consider a more strong condition

$$[D_{\mu}, \chi^{(0)}_{m}(x)] = \partial_{\mu}\chi^{(0)}_{m} + [A_{\mu}(x), \chi^{(0)}_{m}] = 0, \quad (m = 1, \dots, M). \quad (3.12)$$

It can be pointed our that eq. (3.2) is equivalent to the following first-order equations

$$D_{\mu}\chi^{(0)}_{m}(x) = C\, j^{(m)}_{\mu}(x), \quad (3.13)$$

where C is a constant N×N matrix and $j^{(m)}_{\mu}(x)$ is some conserved current. If $j^{(m)}_{\mu}(x)$ is one of currents (3.11) then from the solution of eq. (3.13) we have the same sequence from which $j^{(m)}_{\mu}$ is constructed. However, if $j^{(m)}_{\mu}(x)$ has another nature, then from the solution of eq. (3.13) new linear independent sequences can be constructed and consequently new linear independent infinite series of conserved currents. We do not consider here the last possibilities, and restrict ourselves only to the case C = 0, i.e., with eq. (3.12).

For the system of eqs. (3.12) we have three linear independent solutions which are not coupled with the multiple action of (3.5):

a) The trivial solution

$$\chi_1^{(0)} = 0. \tag{3.14}$$

b) Nontrivial solutions which can be written in the form

$$\chi_a^{(0)}(x) = U_a C, \qquad (a = 2,3), \tag{3.15}$$

where U_a $(a = 2,3)$ satisfies eq. (3.12) and C_a are $N \times N$ constant matrix. Equations (3.12) have the following two solutions

$$U_2(x) = g^{-1}(x), \tag{3.16}$$

and

$$U_3(x) = V\,W, \tag{3.17}$$

where

$$V = P \exp \int_{-\infty}^{x_0} dy_0 g^{-1} \partial_0 g(y_0, x_1),$$

$$W = P \exp \int_{-\infty}^{x_1} dy_1 (V g^{-1} \partial_1 g\, V^{-1})(x_0, y_1). \tag{3.18}$$

Here P is the Wilson ordering operator.

From (3114) and (3.15)-(3.18) by the above given method we construct three linear-independent sequences (3.10). From the fact that $\chi_m^{(0)}$ $(m = 1,2,3)$ are solutions of (3.12) it follows that for $k \geq 1$ all elements of thus found sequences coincide. But these sequences are distinguished by $k \leq 0$ and consequently they are linear independent.

The explicit form of $\{\chi_2^{(k)}\}$ and $\{\chi_3^{(k)}\}$ is found with the use of symmetry of eq. (3.5) or (3.9) which is discussed in Appendix A. This symmetry allows us to write $\{\chi_2^{(k)}\}$ and $\{\chi_3^{(k)}\}$ if we find $\{\chi_1^{(k)}\}$. It can be checked that the general form of $\{\chi_1^{(k)}\}$ is given by

$$\begin{aligned}
\chi_1^{(0)} &= 0, \\
\chi_1^{(1)} &= C_1, \\
\chi_1^{(2)} &= \int_{-\infty}^{x_1} dy_1 \, [g^{-1}\partial_0 g(x_0, y_1), C_1] + C_2, \\
&\ \vdots \\
\chi_1^{(k)} &= -\int_{-\infty}^{x_1} dy_1 \{\partial_0 \chi^{(k-1)} + [g^{-1}\partial_0 g, \chi^{(k-1)}]\}(x_0, y_1) + C_k,
\end{aligned} \tag{3.19}$$

where $C_1, C_2, \ldots$ are $N \times N$ constant matrices. The constants C_k in (3.19) in the current (3.11) give the terms $A_\mu C_k$ which are conserved separately because of the equation of motion (2.2). Consequently, without loss of generality, constants C_k $k \geq 2$ can be omitted.

The functions $\chi^{(k)}(x_0, \infty)$ coinside with nonlocal conserved charges found in papers [3,5].

From (A.3) it follows that

$$\{\chi_2^{(-k)} = U_2 \tilde{\chi}_1^{(k)}\}, \quad \{\chi_3^{(-k)} = U_3 \tilde{\chi}_1^{(k)}\}, \tag{3.20}$$

where $\tilde{\chi}_1^{(k)}$ can be obtained from (3.12) with substitution

$$A_\mu \to -\tilde{A}_\mu^{(a)} = -U_a A_\mu U_a^{-1}, \qquad (a = 2,3)$$

where U_a are given by (3.15-18).

Substitute (3.19) and (3.20) in (3.11) we have three (N^2-1) parameter linear independent infinite series of conserved currents.

In the case of spaces with more than two dimensions eqs. (2.12), (3.2) and (3.3) must be solved. In these cases we are not able to construct numerable sequences of solutions of these equations.

4. CONCLUSION

In this way we are able to find infinite set of conserved nonlocal currents and the "symmetry" transformations which generate these currents. This "symmetry" is of a generalized type in the sense that the variation of action is full divergence of the functions (2.11). Corresponding transformations (2,6) are, in general, nonlinear and nonlocal ones (3.19). These transformations form an infinite-parameter group. The algebra of generators of this group can be constructed from (3.19) and (3.20) using the Poisson brackets. The structure of this algebra depends on the choice of constant parameters **C** in (3.19) and (3.20).

Because of the fact that the function K_μ (2.11) depends on the derivatives of the field $g(x)$, the transformations under consideration are not symmetry of equations of motion, i.e.,

$$\partial^\mu \delta A_\mu(x) \neq 0$$

except of the case of variations given by linear and local transformations with generatory $\zeta = \text{const}$. The latter can be checked directly by substituting (2.9) into the above equation and using (2.12).

The nonlocal conserved charges corresponding to the nonlocal currents (3.11) can be found from generators $\chi^{(k)}(x)$ (3.19) and (3.20) with substitution $x_1 = \infty$

APPENDIX

Equation (2.12) or (3.5) possesses the following symmetry. Suppose that there exists a nonsingular matrix satisfying the equation

$$D_\mu U(x) = (\partial_\mu + A_\mu(x))\, U(x) = 0. \qquad (A.1)$$

Then eq. (3.5) with the substitution

$$\zeta = U\tilde{\zeta} \qquad \text{and} \qquad \chi = U\tilde{\zeta}, \qquad (A.2)$$

is written in the following form

$$\partial_\mu \tilde{\chi} = \epsilon_{\mu\nu}(\partial^\nu + \tilde{A}^\nu(x))\, \tilde{\zeta}(x) , \qquad (A.3)$$

where

$$\tilde{A}_\mu(x) = -U^{-1} A_\mu(x)\, U(x) . \qquad (A.4)$$

From (A.1) and (A.4) it follows that

$$\partial^\mu \tilde{A}_\mu(x) = -U^{-1}\partial^\mu A_\mu U = 0, \qquad (A.5)$$

if eq. (2.2) is satisfied and

$$\tilde{F}_{\mu\nu} = [\tilde{D}_\mu , \tilde{D}_\nu] = U^{-1}[D_\mu , D_\nu]\, U . \qquad (A.6)$$

Consequently, $\tilde{\zeta}$ and $\tilde{\chi}$ must also satisfy the second order eqs. (3.2) and (3.6). It can be pointed out that (A.2) are not similarity transformations.

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