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INTERACTING INSTANTONS, 1/N EXPANSION AND THE GLUON CONDENSATE

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Instanton calculations in the dilute gas approximation (DGA) /1/ require usually an infrared cutoff ρ_c , to be fixed by some diluteness criterion. One feels uncomfortable with this, not only from the principal point of view, but also from a practical one.

On the one hand, the relation between the naive expressions for the would-be vacuum energy $\mathcal{E}$ (or pressure P) and for the square field strength, respectively,

$$P = -\mathcal{E} = 2 \int_0^{\rho_c} \frac{d\rho}{\rho} d_0(\rho) \tag{1}$$

and

$$\langle \frac{\alpha_s}{\pi} F^a_{\mu\nu} F^a_{\mu\nu} \rangle = 16 \int_0^{\rho_c} \frac{d\rho}{\rho} d_0(\rho) \tag{2}$$

violates what is required by the trace anomaly /2/

$$\mathcal{E} = \frac{\pi^2}{2} \frac{\beta(g)}{g} \langle \frac{\alpha_s}{\pi} F^a_{\mu\nu} F^a_{\mu\nu} \rangle \simeq -\frac{b}{32} \langle \frac{\alpha_s}{\pi} F^a_{\mu\nu} F^a_{\mu\nu} \rangle , \quad b = \frac{11N}{3} \tag{3}$$

(in one-loop approximation). In any case, a factor b/4 $\sim$ O(N) seems to be missing in this relation /3/.

On the other hand, nothing prevents in the usual DGA instantons (and antiinstantons) from approaching each other too much. So there is no control guaranteeing that interactions are really small, and that the semiclassical approximation is working for g $\to$ 0. From different considerations, that suggest some kind of repulsive interaction at small distances (cf. Ref. /4/), we decide to abstract a hard core potential

$$\mathcal{U} = \begin{cases} 0 & |z_1 - z_2|^4 > a' \rho_1^2 \rho_2^2 \\ \infty & |z_1 - z_2|^4 < a' \rho_1^2 \rho_2^2 \end{cases} , \tag{4}$$

where a' in advance can be estimated roughly from the desired smallness of the known dipole interaction as $a' \gtrsim 100$ to be sharpened later on. Taking this hard core seriously, at least the before mentioned discrepancy is removed, giving a satisfactory behaviour for $g \longrightarrow 0$.

Moreover, the instanton calculation scheme, based on the hard core interaction, allows one to discuss unambiguously the N dependence of different instanton effects (or contributions) and to decide what instantons are able to explain and what not. In general, neither they are suppressed uniformly as e^{-N} nor they can account for what is known about the physical vacuum; rather they contribute next-to-leading order terms in 1/N in a sense to be explained later. However, they seem to play a definite role /5/ in the transition from weak to strong coupling behaviour /6/, at least they serve to fix the intermediate coupling range.

For the moment let us disregard dipole interactions. Then under the hard core interaction the instanton density /7/ (R denotes the regularization scheme, P.V. means Pauli-Villars)

$$d_0(\rho) = C_N^R \, \varkappa_0^{2N} \, \rho^{-4} \, e^{-x(\rho)} \tag{5}$$

with $$C_N^R = \frac{4.6}{\pi^2} \frac{e^{-1.68N}}{(N-1)!(N-2)!} \left(\frac{\Lambda^{P.V.}}{\Lambda^R}\right)^b , \qquad x(\rho) = b \ln \frac{1}{\rho \Lambda^R}$$

becomes changed /8/ into a damped one,

$$d(\rho) = d_0(\rho)\, e^{-P\bar{v}} = d_0(\rho)\, e^{-A(P)\rho^2} \tag{6}$$

with $\bar{v} = \frac{\pi^2}{2} a' \bar{\rho}^2 \rho^2$. This can be seen from the identity

$$d(\rho,z) = \frac{1}{Z_{hc}(\phi,V)} \frac{\delta}{\delta\phi(z,\rho)} Z_{hc}(\phi,V)\Big|_{\phi=1} = d_0(\rho) \frac{Z_{hc}(\hat{\phi}(\cdot|z,\rho),V)}{Z_{hc}(\phi,V)}\Big|_{\phi=1} = d_0(\rho) \frac{e^{P(V-v)}}{e^{PV}} \tag{7}$$

involving the hard core gas generating functional $Z_{hc}(\phi,V)$ and

$$\hat{\phi}(z',\rho'|z,\rho) = \phi(z',\rho')\, e^{-\mathcal{U}(z',\rho',z,\rho)}$$

P is the hard core pressure

$$P = \frac{1}{V} \ln Z_{hc}(1, V) \quad . \tag{8}$$

Introduction of a fugacity $\lambda = e^{\zeta}$ changes density and pressure, giving rise to a differential equation /8/

$$\frac{dP(\lambda)}{d\lambda} = \sum_{\varepsilon} \int_0^{\infty} \frac{d\rho}{\rho} \, d_0(\rho) \, e^{-R(P(\lambda))\rho^2} , \qquad \varepsilon = \pm \tag{9}$$

with the condition $P(\lambda = 0) = 0$, to be solved for $P = P(\lambda = 1)$; it has the solution (h denotes a known function)

$$\frac{P(e^{\zeta})}{\Lambda^4} = \left[\frac{b}{4} \, 2 \, e^{\zeta} \, h\left(C_N x_0^{2N}, a'\right) \right]^{\frac{4}{b}} \quad . \tag{10}$$

On the other hand, P is related to the r.m.s. instanton radius $\bar{\rho}$ via

$$P = (b-4)\left(\pi^2 a' \bar{\rho}^4\right)^{-1} \quad . \tag{11}$$

The packing fraction, usually required to be $f_0 \leq 1$, is found

$$f_0 = \left(1 - \frac{2}{b}\right) \frac{2}{a'} \quad . \tag{12}$$

In fact it should be bounded by some number $O(N^0)$; with the rough bound given before, we have $f_0 \lesssim 0.02$. We considered the zero mode factor x_0^{2N} as constant under the ρ integrals, eventually to be determined later by demanding $x_0 \simeq x(\bar{\rho})$. Then a' becomes (from (10) and (11)) dependent on x_0

$$\frac{1}{a'} = \Gamma\left(\frac{b}{2} - 1\right) \frac{b}{2} \left(\frac{b}{2} - 2\right)^{-\frac{b}{2}} \frac{\pi^2}{2} C_N \, e^{-x_0} \, x_0^{2N} \tag{13}$$

showing, how $a' \longrightarrow \infty$ for $x_0 \longrightarrow \infty$ ($g \longrightarrow 0$, $\bar{\rho} \longrightarrow 0$).
Finally we find the susceptibility

$$\pi^2 \chi_0 = f_0 \, \frac{x_0}{N^2 - 1} \quad . \tag{14}$$

Later on the condition will become clear under which a' can be fixed as an N independent quantity. If this is the case then $\pi^2 \chi_0$ turns out of order $O(1/N)$, which tells, that the corrections due to the dipole-interaction can be viewed as an $1/N$ expansion /8,9/ and that a strongly paramagnetic instanton gas (necessary for the Princeton bag /10/ by instanton suppression) is incompatible with large N. In an earlier paper /11/ we have critically examined the wanted first order phase transition and found as a necessary condition for its appearance $f_0 > 0.1$. From our present knowledge this is far beyond the validity of the DGA.

How dipole-type interactions between instantons and anti-instantons can be dealt with by means of a functional averaging procedure in its Gaussian approximation is explained in detail in Refs. /8,9/ . In accordance with this approach the vacuum pressure can be represented by

$$P = P_{\ell c} - \frac{3}{16\pi^2}(N^2-1)\frac{1}{\bar{\rho}^4}\int_0^\infty dx\, x^3 \ln\left[1 - F^4(x)/(\pi^2\chi_0)^2\right], \quad (15)$$

where the function $F(x) = \frac{4}{x^2}\left(1 - \frac{x^2}{2}K_2(x)\right)$ has been pulled out of the integrand of the susceptibility χ_0 using the mean value theorem. From eq. (15) we deduce the correction of the instanton density due to dipole interactions

$$d(\rho) = d_0(\rho)e^{-P\bar{v}}\left\{1 + \frac{3}{16}\frac{\rho^4}{\bar{\rho}^4}x_0\pi^2\chi_0\int_0^\infty \frac{F^4(x)\,x^3dx}{1 - F^4(x)/(\pi^2\chi_0)^2}\right\}. \quad (16)$$

Having the smallness of $\pi^2\chi_0$ in mind we can restrict ourselves to the leading orders (in 1/N if a'=const) and get

$$d(\rho) = d_0(\rho)e^{-P\bar{v}}\left\{1 + \frac{3K}{16}\frac{\rho^4}{\bar{\rho}^4}x_0\pi^2\chi_0 + \mathcal{O}\left(1/N^2\right)\right\}, \quad (17)$$

where $K \approx 1.33$.

We may use the above formulae to investigate vacuum fluctuations of varying size $\bar{\rho}$ in isolation, letting x_0 (or a') run down from infinity, e.g.,in order to study the instanton related modification of the β-function /5/ in the intermediate coupling region, and to decide whether a reasonable dilute gas succeeds to interpolate between the weak and strong coupling behaviour. The coupling constant is renormalized nonperturbatively through the vacuum permeability $\mu = 1+2\pi^2\chi_0$ here in the form

$$g^2(\bar{\rho}) = \mu(\bar{\rho})\, g^2_{HF}(\bar{\rho}) \quad . \quad (18)$$

So we obtain (writing $x_0 = N\hat{x}_0$)

$$
\begin{aligned}
-\frac{\beta(g)}{g} &= \frac{g^2}{8\pi^2}\,\frac{b}{2}\,\frac{1}{\mu}\left(1-\frac{\partial\mu}{\partial x_0}\,\frac{8\pi^2}{g^2}\right) \\
&= \frac{g^2}{8\pi^2}\,\frac{b}{2} + \frac{11}{3} f_0(\hat{x}_0-2) + \frac{11K}{8} f_0^2\,\hat{x}_0^2(\hat{x}_0-2).
\end{aligned}
\tag{19}
$$

If we fix N very large and let vary $x_0 = x(\bar{\rho})$ then equs. (12,13) show the β - function exhibiting a step-function like jump to the strong coupling curve, idealized as

$$
-\frac{\beta(g)}{g} = \begin{cases} \frac{11N}{6}\,\frac{g^2}{8\pi^2} \\ \ln N g^2 \end{cases} \qquad \hat{x}_0 = \frac{8\pi^2}{Ng^2} \gtrless \hat{\xi}\ , \tag{20}
$$

where $\hat{\xi}$ is found from

$$
e^{-\hat{\xi}}\,\hat{\xi}^2\left(\frac{\Lambda^{P.V.}}{\Lambda^{\bar{R}}}\right)^{\frac{11}{3}} e^{-1.51} = 1 \quad . \tag{21}
$$

In the case of lattice regularization, $\Lambda^{P.V.}/\Lambda^{L} = 39$ /12/, we find $\hat{\xi} = 17.67$ corresponding to $\bar{\rho}\,\Lambda^{P.V.} = .315$; for N=3 this means g=1.22. This behaviour is expected from the large N limit of some soluble model gauge systems /13/, and can be conjectured from the Hamiltonian treatment of SU(N) lattice theory, too /14/ . In the instanton language this kind of behaviour was noticed in Ref. /15/ . From the jump

$$
\begin{aligned}
\Delta\left(\frac{\beta}{g}\right)\Big|_{\hat{\xi}} &= \frac{11}{3} f_0(\hat{\xi}-2) + \frac{11K}{8} f_0^2(\hat{\xi}-2)\,\hat{\xi}^2 \\
&= \ln 8\pi^2\,\hat{\xi}^{-1} - \frac{11}{6}\,\hat{\xi}^{-1}
\end{aligned}
\tag{22}
$$

we get the before mentioned N independent estimate of $f_0 = 9.7\cdot 10^{-3}$, i.e., a'=205. In the Fig. the curves of the limiting case $N \to \infty$ as well as for N=3 are displayed for comparison. The crossover, marking the point beyond of which the DGA obviously loses sense, corresponds for N=3 to $x_0 = 62.2$ and $\mu = 1.25$ (only!).

But has the instanton gas given at this extreme (a' $\simeq$ 200 for <u>all</u> N), where instanton effects are not suppressed as e^{-N}, any chance to describe the vacuum, characterized as gluon condensate /3/ ? To decide this question we consider the N dependence of some relevant quantities. As far as their large

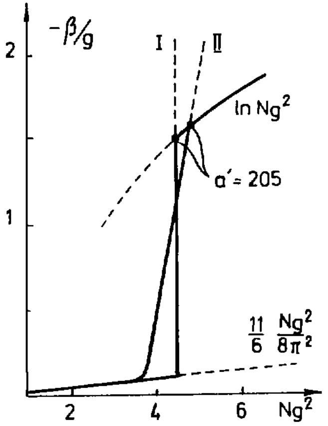

Fig.: Instanton driven β-function.
Curve I : $N \to \infty$;
Curve II: $N=3$ /8/ .

N behaviour is concerned, dipole interactions do not change the results. Therefore, they can be omitted in the following. Essentially thanks to the equs. (9,10), replacing the naive "definition" (1) of P, the relation (3) is established between the instanton contributions to $\mathcal{E}$ and $\langle \frac{\alpha_s}{\pi} F^a_{\mu\nu} F^a_{\mu\nu} \rangle$, since the chemical potential ζ in equ. (10) acts as a constant source for the instanton number or $\alpha_s F^2$. The result is in accordance with what is expected from renormalization group arguments /16/ . However, it is disturbing that the N dependence is down by 1/N compared with what should be expected /17/:

$$\mathcal{E}_{vac} \sim N^2, \qquad \langle \frac{\alpha_s}{\pi} F^a_{\mu\nu} F^a_{\mu\nu} \rangle \sim N. \qquad (23)$$

Attempts to enforce this by assuming $a' \bar{\varrho}^4 \sim N^{-1}$ in (11) run into troubles with the lower bound for a' and/or the limit $g^2 N \longrightarrow$ const., combined with $\Lambda \longrightarrow$ const.

Therefore we conclude, that dilute instanton configurations give only nonleading (in 1/N) contributions to both quantities (23).

It is instructive to consider also the dependence on the Θ angle. Apart from the overall N dependence Witten /18/ argues for $\mathcal{E}_{vac} \sim f(\theta/N)$. If we consider the instanton-antiinstanton gas with uniform hard core pressure at angle θ, equs. (9, 10) become modified to

$$\frac{dP/\Lambda^4}{d\lambda} = 2 \cos\theta \int \frac{d\varrho}{\varrho}\, d_o(\varrho)\, e^{-H(P(\lambda))\varrho^2} \qquad (24)$$

giving

$$P(\theta) = (\cos\theta)^{\frac{4}{b}}\, P(\theta=0) \sim \left(1 - \frac{2}{b}\theta^2\right) P(\theta=0) \,, \qquad (25)$$

whereas the assumption of hard core interactions only among instantons and antiinstantons, respectively, leads to

$$\frac{dP_\pm/\Lambda^4}{d\lambda} = e^{\pm i\vartheta} \int \frac{d\rho}{\rho} \, d_0(\rho) \, e^{-H(P_\pm(\lambda))\rho^2} \tag{26}$$

for the partial pressures, such that

$$P(\theta) \simeq P_+ + P_- \simeq P(\theta=0) \cos\frac{4}{b}\theta \, . \tag{27}$$

Both options give for the topological correlation function

$$\int d^4x \langle Q(x)\, Q(0)\rangle = -\frac{d^2}{d\theta^2} P(\theta)\Big|_{\theta=0} \sim \begin{cases} P/N & \text{acc.to equ.(25)} \\ P/N^2 & \text{acc.to equ.(27),} \end{cases} \tag{28}$$

respectively. In view of Witten's arguments /18/, the last option seems to be more credible. Then instanton contributions to all quantities characterizing the gluon condensate would be down by $1/N$ compared with what is expected for general reasons /17,19/ .

Let us close with some speculations. The last conclusion, that hard core interactions should be considered only between instantons and antiinstantons, respectively, resembles to what has been learned recently about combination of multiinstanton and multiantiinstanton configurations in the CP^{N-1} model /20/. The partition function was found to be $Z=Z_+Z_-$, which means the partition function of selfdual and antiselfdual configurations without mutual interactions. Assuming that something like this proves true in the Yang-Mills theory, too, we may think to isolate, as demonstrated above, fluctuations of a given size $\bar\rho$ corresponding to some x_0 both in Z_+ and Z_-. To a certain extent they will be well separated (anti)instantons. Then the mechanism for the rise in the β-function works well yet, but both the (maximally dense) dilute gases contribute to ε, $\langle\alpha_s F^2\rangle$ and $\int dx \langle Q(x)\, Q(0)\rangle$ with the correct θ dependence, but only $O(1/N)$ compared to some kind of leading contribution. It is tempting to conjecture, that the dissociation of (anti)instantons, each into N (anti)instanton constituents /21/, would provide the necessary proliferation of degrees of freedom possibly happening at x_0, marked by the above estimated jump in the β-function.

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