сообщения
объединенного
института
ядерных
исследований
дубна

E2-80-636

P.Exner*, G.I.Kolerov

FEYNMAN MAPS WITHOUT IMPROPER INTEGRALS

* On leave from the Nuclear Centre, Faculty of Mathematics and Physics of the Charles University, Areál Troja, Povltavská ul., 18000 Prague 8, Czechoslovakia.

1980

I. INTRODUCTION

Mathematical concepts born in physics often prove their fertility in two steps: first on a more or less formal level and after that by finding a suitable mathematical framework and examining the original idea rigorously. So the delta function represents itself an extremely useful computational tool, on the other hand, full power of this concept (which certainly goes beyond Dirac's intention) was not revealed before formulation of the distribution theory. One can therefore understand easily why there is so much temptation in mathematical theory of Feynman integrals, or more exactly, in various attempts to construct such a theory. A substantial progress achieved in this field during recent years is reported, e.g., in the monographs /1,2/ and review papers /3-5/*.

The efforts are mainly concentrated around the problem of expressing dynamics by means of the path integrals. As to the simplest case of a single spinless particle corresponding to the free Hamiltonian $H_0 = -\frac{\hbar^2}{2m}\Delta$, which interacts with an external field described by a potential V, the celebrated Feynman result (cf. /6/; /7/, chap. 3) states that the wave function at a given time t is given by **

$$(\exp(-\frac{i}{\hbar}(H_0 + V)\,t\,)\,\psi)(x) = \int_{\Gamma_x} \exp(\frac{i}{\hbar} S(\gamma))\,\psi(\gamma(0))\,\mathcal{D}\gamma, \qquad (1)$$

where ψ is a wave function at the initial time $t=0$,

---

* We pretend neither to an exhaustive exposition of the path-integration problems nor to completeness of the list of references - at the present time it seems to be the task rather for a monograph writer

** For the sake of simplicity we shall further always set $\hbar = m = 1$.

$$S(\gamma)=\int_0^t[\frac{1}{2}m|\dot{\gamma}(\tau)|^2 - V(\gamma(\tau))]d\tau \tag{2}$$

is the classical action along the path γ and Γ_x is the space of all paths ending in the point x *. The central problem is to give meaning to the formal expression on the rhs of eq. (1), or more generally, to

$$\int_{\Gamma_x} \exp(\frac{i}{2s}\int_0^t |\dot{\gamma}(\tau)|^2 d\tau) f(\gamma)\mathfrak{D}\gamma . \tag{3}$$

where f is a complex-valued function on the path space Γ_x and s is a real parameter. First possible way was proposed by Feynman in his original paper.He replaces the set of all paths by a subset of polygonal paths(velocity of the particle is assumed to be constant in the time intervals $(ti/n, t(i+1)/n)$, $i=0,1,\dots,n-1$), in which case (3) can be defined naturally as an integral over the corresponding finite-dimensional vector space of paths. The construction is completed by taking the limit $n\to\infty$.It is clear, that we need not assume equidistant partitions of $[0,t]$ only; every sequence of partitions such that the subinterval lengths tend to zero would serve as well. The most important property of this definition is the following: for cylindrical functions (which are, roughly speaking, those depending on "finite number of variables" only) the relation

$$\int_{\Gamma_x} \exp(\frac{i}{2s}\int_0^t |\dot{\gamma}(\tau)|^2 d\tau) f(\gamma(\tau_0),\dots,\gamma(\tau_{n-1}))\mathfrak{D}\gamma = \tag{4}$$

---

*This is the standard quantum-mechanical convention. On the other hand, people more inclined to the probability theory often write <u>the same</u> formula using the space of paths with fixed origins - cf. /8/, sec.X.11, /1/, p. 291 and Note I added in proof.

$$= \prod_{i=0}^{n-1} (2\pi \, is(\tau_{i+1} - \tau_i))^{-3/2} \int_{R^{3n}} \exp\left(\frac{i}{2s} \sum_{i=0}^{n-1} |\gamma^{i+1} - \gamma^i|^2 (\tau_{i+1} - \tau_i)^{-1}\right) f(\gamma^0, \dots, \gamma^{n-1}) d\gamma^0 \dots d\gamma^{n-1}$$

is valid, where $0 = \tau_0 < \tau_1 < \dots < \tau_n = t$. This relation interprets in a so natural way that it seems to be reasonable to require validity of the analogous formula for every definition of the functional integral (3).

Let us notice that one of the constructions of the Wiener measure starts just from the formula (4) with $s = -i$ (cf. [9] or [8], sec.X. 11). In this case, however, nonexistence of the Lebesgue-type measure in an infinite-dimensional path space ([1], Appendix A, [10], chap. 1) does not hinder from treating this functional integral in terms of the measure theory: loosely speaking, singularities of the exponential term and of $\mathfrak{D}\gamma$ cancel one another, and the formal expression $\exp\left(-\frac{1}{2}\int_0^t |\dot{\gamma}(\tau)|^2 d\tau\right)\mathfrak{D}\gamma$ can be replaced by $dw(\gamma)$, where w is the Wiener measure. On the other hand, these considerations do not apply to the Feynman integral, where the exponential term behaves in a different way. One might overcome this difficulty by defining the F-integral as a limit (with s arriving to the real axis from below), if appropriate path space measures would exist in the open lower complex half-plane of s (this is essentially the proposal of Gel'fand and Yaglom[11]). Unfortunately, there are no such measures as shown by Cameron[12]: a finite measure $\mu^{(s)}$, such that integrals of all cylindrical functions w.r.t. $\mu^{(s)}$ are expressed by the rhs of eq. (4), exists iff $s = -i\sigma$, $\sigma = 0$.

The Wiener integral itself can be also used for treating the F-integral: either the latter is determined directly by some sort of analytic continuation of the former (an extensive list of references concerning this matter is given in ref.[2]) or the problem under consideration is reformulated (essentially again by analytic continuation) so that the F-integrals are replaced by W-integrals. The last mentioned method represents a backbone of Euclidean approach to constructive quantum field theory which developes so successfully in recent years[13,14].

The second group of definitions follows the original idea of Feynman and determines the integrals (3) "sequentially", i.e., as a limit of some sequence of "finite-dimensional" integrals (see again[2] for further references). In this way Nelson[9] was first able to derive a rigorous version of eq. (1) (the analogous relation for the heat equation, the so-called Feynman-Kac formula, was deduced by Kac in 1951 - cf.[8] sec.X.11). In order to make use from the Lie-Trotter formula Nelson was forced to define the rhs of eq. (1) in a way which differs slightly from the Feynman's heuristic proposal: the function $\exp(-\int_0^t V(\gamma(\tau))\,d\tau)$ was replaced in the n-th approximative integral by the Riemannian sum $\exp(-\sum_{j=0}^{n-1} V(\gamma(\tau_{j+1}))(\tau_{j+1}-\tau_j))$, where $\tau_j = jt/n$. In this way validity of eq. (1) was established for potentials V belonging to $L^2(\mathbf{R}^3)+L^\infty(\mathbf{R}^3)$. This result was further discussed and extended (see, e.g.[15] for the case of harmonic oscillator). Generally speaking, the sequential definitions (handled more or less rigorously) are the most popular in physical literature.

Recently a new group of definitions has appeared which makes use of Fourier transformation. First to be mentioned among them is that of DeWitt-Morette[16,17] which replaces the nonexisting Feynman measure by a "prodistribution" determined by its Fourier transform (equal to $\exp(-\frac{i}{2}W(x',x'))$; here W is a bilinear form on the dual X' of the path space X, which is assumed to be a locally convex Haussdorf space). This method combined with the product operator formalism can be applied to the calculations of various path-integral expressions of physical interest[1]. More important for us is the definition of Albeverio and Hoegh-Krohn[2,3]. It assumes the path space to be a Hilbert space $\mathcal{H}$, in the same time the class of "integrable" functions is restricted to Fourier transforms of finite complex measures on $\mathcal{H}$. The F-integral is at that price expressed by a simple and elegant formula. In this framework a rigorous version of eq. (1) can be derived for potentials which are Fourier transforms of finite measures on the configuration space (cf.[2,18] - the mentioned relation is usually called Feynman-Ito formula, because the F-integral in the sense of[2] coincides with the path integral defined by Ito[19] in probabilistic terms).

The original definition of Albeverio and Hoegh-Krohn (AH) does not make possible to "integrate" some physically important functions, as for example the exponential function corresponding to the harmonic-oscillator potential. It has led these authors to the more general definition of F-integral w.r.t. a (not necessarily positive) quadratic form, in which the exponential term entering in the "measure" can contain a part corresponding to the potential energy as well. An alternative way how to handle some non-bounded potentials was proposed by Truman[18], which extended the original definition of ref.[2] by means of polygonal-path approximations. He showed also that these finite-dimensional approximations are expressed through integrals of exponent of the exact classical action along the polygonal paths; in this sense his approach is more close to the heuristic considerations of Feynman than Nelson-type approximations*.

The same author has formulated also other appealing idea[21] by generalizing the definition of F-integral from ref.[20] to the concept of Feynman maps. By this notion a certain family of maps from a set of functions on the path space $\mathcal{H}$ into $\mathbf{C}$ is understood, which is indexed by numbers s from the lower complex halfplane; the cases $s=1,-i$ refer to the F-integral and W-integral, respectively. This approach makes possible to treat both the important path integrals on the same footing (for a certain class of functions). Except of that it unifies in some sense the sequential methods with those based on analytic continuation.

On the other hand, some objections can be stated. Firstly, the finite-dimensional approximations used in the definition of the F-maps in ref.[21] are not given by means of some AH-type expressions, but via integrals analogous to the rhs of eq. (4). Consequently, if the "integrated" function is such that its cylindrical approximations are not L-integrable (such situation occurs frequently and represents no pathology), then the approximations to the F-map value contain improper integrals. It certainly means no harm as far as we know how to calculate them. However, principal values of multidimensional integrals represent an extremely touchy business (cf. a simple example in sec. 1.2 of ref.[1], which shows that two quite reasonable choices of the limiting procedure in a two-dimensional integral can give completely different results) and we prefer to stay on the solid ground

---

*We need not worry within this framework, whether a change of the Riemannian approximation to the action will not change the value of the resulting path integral.

of the measure theory. Secondly, the definition under consideration uses for approximative purposes only those polygonal paths, which refer to equidistant partitions of the given time interval. This seems to be a discriminative assumption: it may happen that a function "integrable" w.r.t. the given prescription would not occur to be "integrable" in the approximation carried out using arbitrary polygonal paths. This circumstance is stressed by absence of the dominated convergence theorem for F-integrals (see below), which could assure independence on the choice of polygonal-path approximation.

The above considerations determine the main line of this paper. We shall examine here the AH-type definition of the F-maps; its "polygonal" extensions, properties and applications are left to the next paper. First we review for the further use results about the algebra of "Fresnel-integrable" functions [2,3,18]; we hope that some more complete or alternative proofs presented here could excuse extensive character of this part. In particular, the important assertion about injectivity of Fourier transformation on $\mathfrak{M}(\mathcal{H})$ is proved very briefly in ref.[2] especially the implication: if $\{y:(x,y)\leq a\}=0$ for all $x\in\mathcal{H}$, $a\in \boldsymbol{R}$, then $\mu(A)=0$ for all closed convex A, is in no case obvious for non-positive μ. We present below other proof which uses properties of promeasures extracted from ref.[22]. In the third section we define the F-maps and discuss their properties. Some of them are connected closely to those obtained in ref.[21], only presentation (and consequently, some of the assumptions) differs. The other are new, as, e.g., the "Fubini theorem" for F-maps.

We present also a simple example illustrating that the AH-integral does not fulfill the dominated convergence theorem. This invalidates the theorem concerning the classical limit of quantum mechanics deduced in ref.[18], the proof of which is based on this very assumption *. Fortunately, there are other methods how to treat this problem - see refs.[23-25] and references quoted therein.

---

* There is a weak form of the dominated convergence theorem[21], however, its assumptions are such that they hardly could be verified in the cases of physical interest.

2. THE ALGEBRA $\mathcal{F}(\mathcal{H})$

Let $\mathcal{H}$ be a real separable Hilbert space of paths (to be specified later) and $\mathfrak{M}(\mathcal{H})$ the set of all complex Borel measures on $\mathcal{H}$ with $|\mu|(\mathcal{H}) < \infty$. Here $|\mu|$ is the total variation of μ: $|\mu|(A) = \sup\{\sum_k |\mu(A_k)| : \{A_k\}$ finite system of disjoint Borel sets, $\cup_k A_k = A\}$; it is a non-negative measure on $\mathcal{H}$. Any linear combination of $\mu, \nu \in \mathfrak{M}(\mathcal{H})$ belongs again to $\mathfrak{M}(\mathcal{H})$, since the definition of $|\mu|$ implies easily $|\alpha\mu|(A) = |\alpha|\,|\mu|(A)$, $|\mu+\nu|(A) \le |\mu|(A) + |\nu|(A)$ for all $\alpha \in \mathbf{C}$ and any $A \in \mathcal{B}$, the system of Borel sets in $\mathcal{H}$. Let $\{\mu_n\}$ be a sequence $\subset \mathfrak{M}(\mathcal{H})$ such that $|\mu_n - \mu_m|(\mathcal{H}) \to 0$ with $n,m \to \infty$. Using standard arguments one can prove that $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ exists for each Borel A and that μ defined in this way belongs to $\mathfrak{M}(\mathcal{H})$. Thus the space $\mathfrak{M}(\mathcal{H})$ equipped with the norm $|.|(\mathcal{H})$ is Banach.

We shall show further that $\mathfrak{M}(\mathcal{H})$ can be equipped naturally with an algebraic structure. To this purpose assume first $\mu, \nu \in \mathfrak{M}(\mathcal{H})$. The product measure $\mu \otimes \nu$ on $(\mathcal{H}\times\mathcal{H}, \mathcal{B}\otimes\mathcal{B})$ is defined in the standard way; it is finite because $|\mu \otimes \nu|(\mathcal{H}\times\mathcal{H}) = |\mu|(\mathcal{H})\,|\nu|(\mathcal{H})$ (cf. /26/, sec. III.11, lemma 11). Further $f: \mathcal{H}\times\mathcal{H} \to \mathbf{C}$ belongs to $L(\mathcal{H}\times\mathcal{H}, \mu\otimes\nu)$ iff $|f| \in L(\mathcal{H}\times\mathcal{H}, |\mu\otimes\nu|)$ and the "complex Fubini theorem" holds in this case (/26/, sec. III.11, th. 13).

$$\int_{\mathcal{H}\times\mathcal{H}} f(\gamma,\gamma')\, d(\mu\otimes\nu)(\gamma,\gamma') = \int_{\mathcal{H}} d\mu(\gamma) \int_{\mathcal{H}} f(\gamma,\gamma')\, d\nu(\gamma') = \int_{\mathcal{H}} d\nu(\gamma') \int_{\mathcal{H}} f(\gamma,\gamma')\, d\mu(\gamma). \tag{5}$$

Notice that in the mentioned theorem finiteness of μ,ν is substantial in contrast to the usual Fubini theorem where both the measures are non-negative but may be σ-finite (/26/, sec. III.11, /22/, sec. IX.2).

Let us define convolution of $\mu, \nu \in \mathfrak{M}(\mathcal{H})$: we denote $A-\gamma = \{\gamma'-\gamma : \gamma' \in A\}$ and set

$$(\mu * \nu)(A) = \int_{\mathcal{H}} \mu(A-\gamma)\, d\nu(\gamma), \qquad A \in \mathcal{B}. \tag{6}$$

The following properties are easily derived: (i) $\mu * \nu$ is a complex Borel measure: it holds $\mu(A-\gamma) = \int_{\mathcal{H}} \chi_{A-\gamma}(\gamma')\, d\mu(\gamma') =$

$=\int_{\mathcal{H}} \chi_A(\gamma+\gamma')\, d\mu(\gamma')$, χ_A being the characteristic function of A. The Fubini theorem (5) then implies

$$(\mu*\nu)(A)=\int_{\mathcal{H}\times\mathcal{H}} \chi_A(\gamma+\gamma')\, d(\nu\otimes\mu)(\gamma,\gamma'). \tag{7}$$

The mapping ϕ of $\mathcal{H}\times\mathcal{H}$ onto itself, $\phi(\gamma,\gamma')=(\gamma+\gamma',\gamma-\gamma')$, is continuous in the product topology of $\mathcal{H}\times\mathcal{H}$ and $\phi^{-1}(A\times\mathcal{H})=\{(\gamma,\gamma'):\gamma+\gamma'\in A\}$ is Borel in $\mathcal{H}\times\mathcal{H}$ for any $A\in\mathcal{B}$ (cf. Appendix A), thus the last integral makes sense. The set function $\mu*\nu:\mathcal{B}\to\mathbf{C}$ is obviously σ-additive and $(\mu*\nu)(\emptyset)=0$.
(ii) The mapping $(\mu,\nu)\mapsto\mu*\nu$ is clearly bilinear and commutative: $(\mu*\nu)(A)=(\nu*\mu)(A)$ follows from (7). We shall verify its associativity. The relation (7) together with the image measure theorem (or change-of-variable theorem, cf., e.g., ref. [26], sec. III.10, prop. 8) imply

$$(\mu*\nu)(A)=\int_{\mathcal{H}\times\mathcal{H}} \chi_{A\times\mathcal{H}}(\gamma,\gamma')\, d(\mu\otimes\nu)(\phi^{-1}(\gamma,\gamma'))=$$

$$=\int_{A\times\mathcal{H}} d(\mu\otimes\nu)(\phi^{-1}(\gamma,\gamma')),$$

where ϕ is the mapping defined above. Then for any $f\in L(\mathcal{H},\mu*\nu)$ we have

$$\int_{\mathcal{H}} f(\gamma)\, d(\mu*\nu)(\gamma)=\int_{\mathcal{H}\times\mathcal{H}} f(\gamma)\, d(\mu\otimes\nu)(\phi^{-1}(\gamma,\gamma'))$$

and using once more the image measure theorem we get

$$\int_{\mathcal{H}} f(\gamma)\, d(\mu*\nu)(\gamma)=\int_{\mathcal{H}\times\mathcal{H}} f(\gamma+\gamma')\, d(\mu\otimes\nu)(\gamma,\gamma')=$$

$$=\int_{\mathcal{H}} d\mu(\gamma)\int_{\mathcal{H}} d\nu(\gamma')\, f(\gamma+\gamma').$$

In particular, we obtain

$$(\mu*(\nu*\rho))(A)=\int_{\mathcal{H}}\mu(A-\gamma)\, d(\nu*\rho)(\gamma)=\int_{\mathcal{H}} d\nu(\gamma)\int_{\mathcal{H}} d\rho(\gamma')\,\mu(A-$$

$$-\gamma-\gamma')=\int_{\mathcal{H}}(\mu*\nu)(A-\gamma')\, d\rho(\gamma')=((\mu*\nu)*\rho)(A)$$

for any $A\in\mathcal{B}$.

(iii) Finally, the inequality

$$|\mu * \nu|(\mathcal{H}) \leq |\mu|(\mathcal{H})\,|\nu|(\mathcal{H}) \qquad (9)$$

holds, which in particular shows that $*$ maps $\mathfrak{M}(\mathcal{H}) \times \mathfrak{M}(\mathcal{H})$ into $\mathfrak{M}(\mathcal{H})$. It can be obtained with the help of the following expression for total variation: $|\mu|(A) = \sup\{|\int_A g(\gamma)\, d\mu(\gamma)|:$ g Borel, $|g(\gamma)| \leq 1\}$ (ref. [26], sec. III.2; ref. [27], § 29). The relation (8) implies.

$$\Big|\int_{\mathcal{H}} g(\gamma)\, d(\mu * \nu)(\gamma)\Big| \leq \int_{\mathcal{H}} \Big|\int_{\mathcal{H}} g(\gamma+\gamma')\, d\nu(\gamma')\Big|\, d|\mu|(\gamma) \leq$$

$$\leq \int_{\mathcal{H}} d|\mu|(\gamma) \int_{\mathcal{H}} |g(\gamma+\gamma')|\, d|\nu|(\gamma') \leq |\mu|(\mathcal{H})\,|\nu|(\mathcal{H})$$

if $|g(\gamma)| \leq 1$ so (9) is valid. Thus we arrive to the following assertion:

<u>Proposition 1</u>: The space $\mathfrak{M}(\mathcal{H})$ equipped with the norm $|\cdot|(\mathcal{H})$ and the product $*$ is a commutative Banach algebra.

Up to now we have not made use of the Hilbert structure of $\mathcal{H}$. Assume now the set $\mathfrak{F}(\mathcal{H}) = \{f: f(\gamma) = \int_{\mathcal{H}} e^{i(\gamma,\gamma')} d\mu(\gamma'),$ $\mu \in \mathfrak{M}(\mathcal{H})\}$, where $(.,.)$ is the inner product in $\mathcal{H}$. Continuity of $(\gamma,.)$ implies continuity of $e^{i(\gamma,.)}$ so the latter is Borel measurable and f is well-defined for each $\mu \in \mathfrak{M}(\mathcal{H})$. Further $\mathfrak{F}(\mathcal{H})$ is a vector space w.r.t. pointwise addition and scalar multiplication.

We shall show that the B-algebra structure of $\mathfrak{M}(\mathcal{H})$ can be isomorphically transferred to $\mathfrak{F}(\mathcal{H})$. The crucial point here is to prove <u>bijectivity</u> of the mapping $\mu \mapsto f$; in view of linearity it is sufficient to check that $f = 0$ implies $\mu_f = 0$. In order to perform this we use the following assertions addopted from [22], chap. IX:

(a) To every measure μ on $\mathcal{H}$ there exists a promeasure $\tilde{\mu}$ on $\mathcal{H}$ associated with μ. The mapping $\mu \mapsto \tilde{\mu}$ from the set of all (positive) bounded measures is injective (§6.1, prop. 1; it follows from the Prokhorov theorem).

(b) If $\tilde{\mu}$ is the promeasure associated with a (positive) bounded measure μ and $\mathfrak{F}$ denotes the Fourier transformation, then $\mathfrak{F}\mu = \mathfrak{F}\tilde{\mu}$ (§6.3).

(c) The mapping $\tilde{\mu} \mapsto \mathfrak{F}\tilde{\mu}$ from the set of the promeasures on $\mathcal{H}$ into the set of functions on $\mathcal{H}$ is injective (§6.3).

All these assertions are valid generally for measures and promeasures on locally convex topological spaces. On the other hand, the concept of a measure in chap. IX of ref. /22/ differs from that used here (these measures are not set functions but linear functionals on certain function spaces), so one has to check that a bounded measure in the sense of ref. /22/ corresponds injectively to every finite Borel measure on separable $\mathcal{H}$. This assertion is proved in Appendix B; it resembles the uniqueness part of the Riesz-Markov theorem, however, the space $\mathcal{H}$ is not general locally compact and the measures involved are complex.

If $f=0$, then $\frac{1}{2}(f(\gamma)+f(-\gamma)) = \frac{1}{2i}(f(\gamma)-f(-\gamma)) = 0$ for all $\gamma\in\mathcal{H}$ so

$$\int_{\mathcal{H}} \cos(\gamma,\gamma')\, d\mu_f(\gamma') = \int_{\mathcal{H}} \sin(\gamma,\gamma')\, d\mu_f(\gamma') = 0.$$

If g is real-valued and ν complex, then $\int g\, d\nu = 0$ implies $\int g\, d\mathrm{Re}\,\nu = \int g\, d\mathrm{Im}\,\nu = 0$, thus the above equalities give

$$\int_{\mathcal{H}} e^{i(\gamma,\gamma')} d\,\mathrm{Re}\,\mu_f(\gamma') = \int_{\mathcal{H}} e^{i(\gamma,\gamma')} d\,\mathrm{Im}\,\mu_f(\gamma') = 0 \qquad (*)$$

for all $\gamma\in\mathcal{H}$. We shall assume, e.g., the signed measure $\rho=\mathrm{Re}\,\mu_f$; the argument concerning $\mathrm{Im}\,\mu_f$ would be the same. Let $\rho=\rho_1-\rho_2$ be the Jordan decomposition of ρ. The positive measures ρ_1,ρ_2 have disjoint supports so that $\rho_1\neq\rho_2$ unless both these measures are zero. In the first case the promeasures $\tilde{\rho}_i$ associated with ρ_i are different due to (a), further (b) and (c) together with linearity of the Fourier transformation imply $\mathcal{F}\rho=\mathcal{F}\rho_1-\mathcal{F}\rho_2=\mathcal{F}\tilde{\rho}_1-\mathcal{F}\tilde{\rho}_2\neq 0$. This contradicts to (*), thus $\rho=\mathrm{Re}\,\mu_f=0$. Analogously $\mathrm{Im}\,\mu_f=0$ holds so $\mu_f=0$.

The abbreviation μ_f for the measure corresponding to $f\in\mathcal{F}(\mathcal{H})$ makes therefore sense and we shall use it whenever it will prove to be convenient. The above-mentioned statement together with other properties of $\mathcal{F}(\mathcal{H})$ are given by the following

<u>Proposition 2</u>: The space $\mathcal{F}(\mathcal{H})$ is a functional Banach algebra with unity w.r.t. the norm $\|.\|_0$: $\|f\|_0=|\mu_f|(\mathcal{H})$. Each $f\in\mathcal{F}(\mathcal{H})$ is norm continuous and bounded, $\|f\|_\infty\le\|f\|_0$. If $h:\mathbf{C}\to\mathbf{C}$ is an entire function and $f\in\mathcal{F}(\mathcal{H})$, then the composed mapping $h\circ f$ belongs to $\mathcal{F}(\mathcal{H})$ as well.

Proof: The Fourier transformation $\mathcal{F}$ is linear and maps $\mathfrak{M}(\mathcal{H})$ bijectively onto $\mathcal{F}(\mathcal{H})$, further this isomorphism is isometric, $|\mu|(\mathcal{H})=||\mathcal{F}\mu||_0$, so the space $\mathcal{F}(\mathcal{H})$ is Banach. The convolution is transformed by $\mathcal{F}$ into pointwise multiplication: the relations (5), (8) give

$$(\mathcal{F}(\mu*\nu))(\gamma)=\int_{\mathcal{H}} e^{i(\gamma,\gamma')}d(\mu*\nu)(\gamma')=$$

$$=\int_{\mathcal{H}\times\mathcal{H}} e^{i(\gamma,\gamma'+\gamma'')}d(\mu\otimes\nu)(\gamma',\gamma'')=(\mathcal{F}\mu)(\gamma)(\mathcal{F}\nu)(\gamma)$$

for all $\mu,\nu\in\mathfrak{M}(\mathcal{H}),\gamma\in\mathcal{H}$. Further the B-algebra $\mathcal{F}(\mathcal{H})$ contains unity, because the Dirac measure μ_e: $\mu_e(\{0\})=1$, $\mu_e(\mathcal{H}-\{0\})=0$ belongs to $\mathfrak{M}(\mathcal{H})$ and $(\mathcal{F}\mu_e)(\gamma)=1$ for all $\gamma\in\mathcal{H}$. Since $\mathcal{H}$ is first countable w.r.t. the norm topology (even second countable), a function $f:\mathcal{H}\to\mathbf{C}$ is continuous if it is sequentially continuous. Let γ be the norm limit of a sequence $\{\gamma_n\}\subset\mathcal{H}$, then $(\gamma,\gamma')=\lim_{n\to\infty}(\gamma_n,\gamma')$ so that $\exp(i(\gamma,\gamma'))=\lim_{n\to\infty}\exp(i(\gamma_n,\gamma'))$ for all $\gamma'\in\mathcal{H}$. If $f\in\mathcal{F}(\mathcal{H})$ the dominated convergence theorem implies

$$f(\gamma)=\int_{\mathcal{H}}\exp(i(\gamma,\gamma'))\,d\mu_f(\gamma')=\lim_{n\to\infty}\int_{\mathcal{H}}\exp(i(\gamma_n,\gamma'))\,d\mu_f(\gamma')=$$

$$=\lim_{n\to\infty}f(\gamma_n).$$

Consequently, f is norm continuous. Further the inequality $|f(\gamma)|\le\int_{\mathcal{H}}d|\mu_f|(\gamma')$ gives $||f||_\infty=\sup_{\gamma\in\mathcal{H}}|f(\gamma)|\le|\mu_f|(\mathcal{H})=||f||_0$. Finally, let h be expressed by the series $h(z)=\sum_{n=0}^{\infty}a_n z^n$ with the infinite radius of convergence, then $h\circ f=\sum_{n=0}^{\infty}a_n f^n$. The sequence $\{a_n f^n\}_{n=0}$ is absolutely summable, $\sum_{n=0}^{\infty}||a_n f^n||_0\le\sum_{n=0}^{\infty}|a_n|\,||f||_0^n<\infty$, and because $\mathcal{F}(\mathcal{H})$ is Banach, it is also summable, i.e., $\sum_{n=0}^{N}a_n f^n$ converges in $||.||_0$-norm to some element of $\mathcal{F}(\mathcal{H})$ (cf. /8/, th. III.3).

3. THE FEYNMAN MAPS

Now we are in position to formulate the main definition. Let us denote $\mathbf{C}_F=\{z\in\mathbf{C}:\ z\neq 0,\mathrm{Im}\,z\le 0\}$ and $\mathbf{C}_F^o=\{z:\ \mathrm{Im}\,z<0\}$. To any $s\in\mathbf{C}_F$ we define the mapping $I_s:\mathcal{F}(\mathcal{H})\to\mathbf{C}$ by

$$I_s(f)=\int_{\mathcal{H}}\exp\left(-\frac{is}{2}||\gamma||^2\right)d\mu_f(\gamma) \tag{10}$$

and call it F_s-map. In particular, $I_1(.)\equiv I(.)$ is called F-integral; this definition coincides precisely with the first definition of the F-integral in ref.[2]*. Basic properties of the F-maps are the following:

(i) $I_s(f)$ is well-defined: the Hilbert space norm is continuous so $g_s(.)=\exp(-\frac{is}{2}\|.\|^2)$ is continuous too, and therefore Borel measurable; further $|g_s(y)|\le\exp(\frac{1}{2}\|y\|^2\,\mathrm{Im}\,s)\le 1$ implies $g_s\in L(\mathcal{H},\mu_f)$ for each $f\in\mathcal{F}(\mathcal{H})$. Moreover, $I_s(.)$ is a linear functional, which is obviously bounded, $\|I_s\|=1$, and normalized, because to the unit function e the normalized $\{0\}$-supported Dirac measure corresponds so that $I_s(e)=1$. On the other hand, $I_s(.)$ is not positive w.r.t. the natural involution in $\mathcal{F}(\mathcal{H})$ unless s is purely imaginary; it is clear from the relations (11,12) below.

(ii) Let us take a finite-dimensional $\mathcal{H}=\boldsymbol{R}^n$ with the standard norm $|.|$ and express

$$I'_s(f)=(2\pi is)^{-n/2}\int_{\boldsymbol{R}^n}\exp(\frac{i}{2s}\|x\|^2)\,f(x)\,dm(x) \tag{11}$$

for $f: f(x)=\int_{\boldsymbol{R}^n} e^{i(x,y)}\,d\mu_f(y)$, $\mu_f\in\mathfrak{M}(\boldsymbol{R}^n)$, where m is the Lebesgue measure on $\boldsymbol{R}^n$. The integral (11) is assumed to exist for all $s\in\mathbf{C}_F$, i.e., $f\in L(\boldsymbol{R}^n)$. If $s\in\mathbf{C}^o_F$, then by Fubini theorem one obtains

$$I'_s(f)=(2\pi is)^{-n/2}\int_{\boldsymbol{R}^n}d\mu_f(y)\prod_{j=1}^{n}\int_{\boldsymbol{R}}\exp(\frac{i}{2s}x_j^2+ix_jy_j)\,dx_j\ ;$$

evaluating the last integral ([28], 3.896.4) we get

$$I'_s(f)=\int_{\boldsymbol{R}^n}\exp(-\frac{is}{2}|y|^2)\,d\mu_f(y)=I_s(f). \tag{12}$$

For real s one cannot apply the Fubini theorem directly, because $\{x,y\}\mapsto\exp(\frac{i}{2s}|x|^2+i(x,y))$ does not belong to $L(\boldsymbol{R}^n\times\boldsymbol{R}^n, m\otimes\mu_f)$. Thus we express $I'_s(f)$ as follows

$$I'_s(f)=\lim_{\alpha\to\infty}(2\pi is)^{-n/2}\int_{C_\alpha}\exp(\frac{i}{2s}|x|^2)\,f(x)\,dm(x)\,, \tag{13}$$

---

*Other $I_s(.)$, $s>0$, may be called F-integrals as well. Their properties are analogous to those of $I_1(.)$, because they can be obtained one from the other through changing the Hilbert space norm by a non-zero multiplicative constant.

where $C_\alpha = \{x : |x_j| \le \alpha\}$, $j=1,2,\dots,n$. Then

$$I'_s(f) = \lim_{\alpha\to\infty} \int_{R^n} \exp\left(-\frac{is}{2}|y|^2\right) K^n_s(y,\alpha)\, d\mu_f(y),$$

where

$$K^n_s(y,\alpha) = (2\pi i s)^{-n/2} \int_{C_\alpha} \exp\left(\frac{i}{2s}|x+sy|^2\right) dm(x).$$

Concerning the last integral the following assertion is valid (see Appendix C): there exists $K^n_s > 0$ to any non-zero s such that $|K^n_s(y,\alpha)| \le K^n_s$ for all $\alpha \ge 0$, $y \in R^n$ and $\lim_{\alpha\to\infty} K^n_s(y,\alpha) = 1$. Using it together with the dominated convergence theorem, we arrive again to the relation (12).

Remark: Considerations of Appendix C do not employ the assumed integrability of f. Thus if the relation (13) is regarded as definition of $I'_s(f)$, then (12) is valid for all $f \in \mathfrak{F}(R^n)$. This is essentially the way in which improper integrals appear in the original F-map definition/21/. Let us remind here the example quoted in the introduction, which shows how much these considerations are sensitive to the limiting prescription: if $n=2$ and f is the unit function on R^2, then $I_1(f) = 1$ as well as $I'_1(f)$ in the sense of (13). However, if the blowing-up square C_α is replaced by the circle $\{x : |x| \le \alpha\}$, then the corresponding expression equals $\lim_{\alpha\to\infty}(1-\exp(\frac{i}{2}\alpha^2))$, i.e., the principal value does not exist at all.

(iii) An assertion analogous to the Fubini theorem was deduced in ref./2/ for the F-integrals. It can be generalized for the F-maps: let $\mathcal{H}$ decompose into an orthogonal sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ so that for all $\gamma = \gamma_1 + \gamma_2 \in \mathcal{H}$ we have $\|\gamma\|^2 = \|\gamma_1\|^2 + \|\gamma_2\|^2$. If $f \in \mathfrak{F}(\mathcal{H})$, $f(\gamma) = \int_{\mathcal{H}} \exp(i(\gamma,\gamma'))\, d\mu(\gamma')$, we write $d\mu(\gamma) = d\mu(\gamma_1,\gamma_2)$ and define

$$\mu_{\gamma_2}(A_1) = \tilde{\mu}_{\gamma_2}(A_1 \times \mathcal{H}_2), \quad \tilde{\mu}_{\gamma_2}(A) = \int_A \exp(i(\gamma_2,\gamma'_2))\, d\mu(\gamma'_1,\gamma'_2)$$

for each $\gamma_2 \in \mathcal{H}_2$ and Borel $A_1 \subset \mathcal{H}_1$. The mapping $\mu_{\gamma_2}(.)$ is clearly σ-additive, $\mu_{\gamma_2}(\emptyset) = 0$, $|\mu_{\gamma_2}(A)| \le |\mu|(A \times \mathcal{H}_2) \le |\mu|(\mathcal{H})$, so $\mu_{\gamma_2} \in \mathfrak{M}(\mathcal{H}_1)$; further

$$\int_{\mathcal{H}} g(\gamma_1)\, d\mu_{\gamma_2}(\gamma_1) = \int_{\mathcal{H}} \tilde{g}(\gamma)\, d\tilde{\mu}_{\gamma_2}(\gamma), \quad \tilde{g}(\gamma_1,\gamma_2) = g(\gamma_1)$$

for any $g \in L(\mathcal{H}_1, \mu_{\gamma_2})$. The Borel measure $\tilde{\mu}_{\gamma_2}$ on $\mathcal{H}$ is absolutely continuous w.r.t. μ, and therefore the last integral can be expressed by means of μ and the Radon-Nikodym derivative

$\gamma' \mapsto \exp(i(\gamma_2,\gamma_2'))$ (cf. [26], sec. III.10, cor.6). Thus we obtain

$$\int_{\mathcal{H}_1} g(\gamma_1)\, d\mu_{\gamma_2}(\gamma_1) = \int_{\mathcal{H}} \tilde{g}(\gamma')\exp(i(\gamma_2,\gamma_2'))\, d\mu(\gamma_1',\gamma_2'). \tag{14}$$

In particular, this equality with $g(\gamma_1') = \exp(i(\gamma_1,\gamma_1'))$ shows that $f_{\gamma_2}(.) \equiv f(.,\gamma_2)$ belongs to $\mathcal{F}(\mathcal{H}_1)$ for any fixed γ_2 and

$$I_s(f_{\gamma_2}) = \int_{\mathcal{H}_1} \exp(-\frac{is}{2}\|\gamma_1\|^2)\, d\mu_{\gamma_2}(\gamma_1).$$

Applying further (14) to $g(\gamma_1) = \exp(-\frac{is}{2}\|\gamma_1\|^2)$ one obtains

$$I_s(f_{\gamma_2}) = \int_{\mathcal{H}} \exp(-\frac{is}{2}\|\gamma_1\|^2 + i(\gamma_2,\gamma_2'))\, d\mu(\gamma_1',\gamma_2').$$

This integral can be in the same way as above expressed as

$$I_s(f_{\gamma_2}) = \int_{\mathcal{H}_2} \exp(i(\gamma_2,\gamma_2'))\, d\nu_s(\gamma_2'),$$

where ν_s is the complex Borel measure on $\mathcal{H}_2$ determined by the relation

$$\nu_s(A) = \int_{\mathcal{H}_1\times A} \exp(-\frac{is}{2}\|\gamma_1\|^2)\, d\mu(\gamma_1,\gamma_2).$$

Hence the function $h_s : h_s(\gamma_2) = I_s(f_{\gamma_2})$ belongs to $\mathcal{F}(\mathcal{H}_2)$ and

$$\begin{aligned} I_s(h_s) &= \int_{\mathcal{H}_2} \exp(-\frac{is}{2}\|\gamma_2\|^2)\, d\nu_s(\gamma_2) = \\ &= \int_{\mathcal{H}} \exp(-\frac{is}{2}\|\gamma_2\|^2 - \frac{is}{2}\|\gamma_1\|^2)\, d\mu(\gamma_1,\gamma_2) = I_s(f). \end{aligned} \tag{15}$$

It is also clear that an order in which the "integrations" are performed is irrelevant.

Remark: The central argument of the presented proof (deduction of the relation (14)) is not based on the Fubini theorem as stated in ref. [2], because the measure μ is not in general a product measure on $\mathcal{H}_1\times\mathcal{H}_2$ (cf. [8] sec. I.4). In fact μ_f is a product measure iff f factorizes, $f(\gamma) = f_1(\gamma_1)f_2(\gamma_2)$ for all $\gamma_i \in \mathcal{H}_i$, as can be easily seen.

(iv) A function $f: \mathcal{H} \to \mathbf{C}$ is called cylindrical or tame if there exists a finite-dimensional projection P on $\mathcal{H}$ such that $f \circ P = f$, i.e., $f(P\gamma) = f(\gamma)$ for each $\gamma \in \mathcal{H}$. The function f is in such case said to have basis (to be based) in $P\mathcal{H}$. The subset of all cylindrical functions in $\mathcal{F}(\mathcal{H})$ is denoted $\mathcal{F}^t(\mathcal{H})$. For $f \in \mathcal{F}^t(\mathcal{H})$ the above results can be used: we decompose into orthogonal sum of $\mathcal{H}_1 = (I-P)\mathcal{H}$, $\mathcal{H}_2 = P\mathcal{H}$ and define $\tilde{f}$: $f(\gamma_2) = f(0,\gamma_2) = f(P\gamma)$, $\gamma \in \mathcal{H}$, then $f(\gamma) = \tilde{f}(\gamma_2) = e_1(\gamma_1)\tilde{f}(\gamma_2)$, where e_1 is the unit function on $\mathcal{H}_1$. Now (i) and (iii) imply $I_s(f_{\gamma_2}) = I_s(\tilde{f}(\gamma_2)e_1) = \tilde{f}(\gamma_2) I_s(e_1) = \tilde{f}(\gamma_2)$ so that $I_s(f) = I_s(\tilde{f})$. Further if $\tilde{f} \in L(P\mathcal{H}, m)$, then according to (ii) $I_s(\tilde{f})$ can be expressed in the form (11) and we obtain therefore

$$I_s(f) = (2\pi is)^{-\frac{1}{2}\dim P\mathcal{H}} \int_{P\mathcal{H}} \exp\left(\frac{i}{2s}\|P\gamma\|^2\right) f(P\gamma)\, dm(P\gamma); \qquad (16)$$

here m is again the Lebesgue measure on $P\mathcal{H}$.

(v) If $a \in \boldsymbol{R}^n$ and R is a linear orthogonal transformation on $\mathcal{H} = \boldsymbol{R}^n$, then

$$I_s(f) = I_s'(f) = (2\pi is)^{-n/2} \int_{\boldsymbol{R}^n} \exp\left(\frac{i}{2s}|Rx+a|^2\right) f(Rx+a)\, dm(x)$$

for each $f \in \mathcal{F}(\boldsymbol{R}^n) \cap L(\boldsymbol{R}^n)$, because the Lebesgue measure is Euclidean-invariant: $m(RA+a) = m(A)$. This equality can be reqritten as

$$\exp\left(\frac{i}{2s}|a|^2\right) I_s(f_{R,a}) = I_s(f), \qquad (17)$$
$$f_{R,a}(x) = \exp\left(\frac{i}{s}(x, R^{-1}a)\right) f(Rx+a),$$

if $f_{R,a} \in \mathcal{F}(\boldsymbol{R}^n) \cap L(\boldsymbol{R}^n)$ too (this is an additional assumption if s is non-real and $a \neq 0$ because of the real exponent present in this case). In the same way we obtain

$$|\det B|\; I_s(f_B) = I_s(f), \qquad (18)$$
$$f_B(x) = \exp\left(\frac{i}{2s}|Bx|^2 - \frac{i}{2s}|x|^2\right) f(Bx),$$

for any regular linear operator B on $\boldsymbol{R}^n$ assuming that both f, f_B belong to $\mathcal{F}(R^n) \cap L(\boldsymbol{R}^n)$.

<u>Remark:</u> The relation (18) shows that the second formula from (P4), sec.2 in /3/ is not valid even in the finite-dimensional case; it holds for isometric T only - cf. (20) below. There is the obvious confusion here with Proposition 4.3 of /2/: for the F-integrals w.r.t. a quadratic form the determinant is included into normalization.

(vi) We shall verify further that the property (17) is preserved <u>in the infinite-dimensional case</u> if s is real non-zero. The deduction is based again on the image measure theorem. Consider first the transitions: let $f \in \mathcal{F}(\mathcal{H})$ and define f_a: $f_a(\gamma) =$

$= \exp(\frac{i}{s}(\gamma,\alpha))f(\gamma+\alpha)$ for a given $\alpha \in \mathcal{H}$. It holds

$$f_\alpha(\gamma) = \int_{\mathcal{H}} \exp(\frac{i}{s}(\gamma,\alpha) + i(\gamma+\alpha,\gamma'))\, d\mu_f(\gamma') = \int_{\mathcal{H}} \exp(i(\gamma,\gamma'')\, d\mu_\alpha(\gamma''),$$

where $\mu_\alpha(A) = \int_{A+\alpha s^{-1}} \exp(i(\alpha,\gamma'))\, d\mu_f(\gamma')$; further $\mathcal{H}+\alpha s^{-1} = \mathcal{H}$ so

$$I_s(f_\alpha) = \int_{\mathcal{H}} \exp(-\frac{is}{2}\|\gamma'+\alpha s^{-1}\|^2 + i(\alpha,\gamma'))\, d\mu_f(\gamma') =$$

$$= \exp(-\frac{is}{2}\|\alpha\|^2)\, I_s(f),$$

i.e.,

$$\exp(\frac{i}{2s}\|\alpha\|^2)\, I_s(f_\alpha) = I_s(f), \qquad f_\alpha(\gamma) \equiv \exp(\frac{i}{s}(\gamma,\alpha))\, f(\gamma+\alpha). \tag{19}$$

Analogously, if U is a regular isometric operator on $\mathcal{H}$ and $f \in \mathcal{F}(\mathcal{H})$, then we have

$$f(U\gamma) = \int_{\mathcal{H}} \exp(i(\gamma, U^{-1}\gamma'))\, d\mu_f(\gamma') = \int_{\mathcal{H}} \exp(i(\gamma,\gamma'')\, d\mu_f(U\gamma'')$$

so

$$I_s(f) = \int_{\mathcal{H}} \exp(-\frac{is}{2}\|U^{-1}\gamma'\|^2)\, d\mu_f(\gamma') = \int_{\mathcal{H}} \exp(-\frac{is}{2}\|\gamma''\|^2)\, d\mu_f(U\gamma''),$$

i.e.,

$$I_s(f_U) = I_s(f), \quad f_U(\gamma) \equiv f(U\gamma). \tag{20}$$

The last formula, however, holds for non-real s too as the proof shows. Under some additional assumptions validity of (19) can be also extended to non-real s. Finally, the Cameron-Martin-type formula (18) generalizes to the infinite-dimensional case if a special class of operators B is considered. We postpone these matters to the next paper.

(vii) Let $\{E_n\}$ be a sequence of orthogonal projections on $\mathcal{H}$ such that $s-\lim_{n\to\infty} E_n = 1$. The restriction $f_n = f \upharpoonright E_n\mathcal{H}$ of a given $f \in \mathcal{F}(\mathcal{H})$ can be expressed as follows

$$f(E_n\gamma) = \int_{\mathcal{H}} \exp(i(E_n\gamma,\gamma')\, d\mu_f(\gamma') =$$

$$= \int_{\mathcal{H}} \exp(i(E_n\gamma, E_n\gamma'))\, d\mu_f(F_n\gamma', E_n\gamma'),$$

where $F_n = I - E_n$. In analogy with the proof of (iii) we introduce the Borel measure μ_n on $E_n\mathcal{H}$ by $\mu_n(A) = \mu_f(F_n\mathcal{H} \times A)$. Clearly $\mu_n \in \mathfrak{M}(E_n\mathcal{H})$, further the image measure theorem gives

$$\int_{E_n\mathcal{H}} g(\gamma_2')\, d\mu_n(\gamma_2') = \int_{\mathcal{H}} g(E_n\gamma')\, d\mu_f(F_n\gamma', E_n\gamma') \tag{21}$$

for any Borel $g: E_n\mathcal{H} \to \mathbf{C}$. In particular, for $g(\gamma'_2) = \exp(i(E_n\gamma, \gamma'_2))$ we get $f_n \in \mathcal{F}(E_n\mathcal{H})$. Applying further the relation (21) to $g(\gamma'_2) = \exp(-\frac{is}{2}\|\gamma'_2\|^2)$ we obtain

$$I_s(f_n) = \int_{E_n\mathcal{H}} \exp(-\frac{is}{2}\|\gamma'_2\|^2)\, d\mu_n(\gamma'_2) = \int_{\mathcal{H}} \exp(-\frac{is}{2}\|E_n\gamma'\|^2)\, d\mu_f(\gamma') .$$

Finally, $\exp(-\frac{is}{2}\|\gamma\|^2) = \lim_{n\to\infty} \exp(-\frac{is}{2}\|E_n\gamma\|^2)$ due to the assumption so the dominated convergence theorem yields

$$\lim_{n\to\infty} I_s(f_n) = I_s(f). \tag{22}$$

(viii) As far we have discussed the f-dependence of $I_s(f)$. Let now in turn $f \in \mathcal{F}(\mathcal{H})$ be fixed. The standard condition under which the integral (10) can be differentiated w.r.t. the parameter s verifies easily: it holds

$$\left|-\frac{i}{2}\|\gamma\|^2 \exp(-\frac{is}{2}\|\gamma\|^2)\right| \le \frac{1}{2}\|\gamma\|^2 \exp(\frac{1}{2}\|\gamma\|^2 \operatorname{Im} s)$$

and the rhs belongs to $L(\mathcal{H}, \mu_f)$ if $\operatorname{Im} s < 0$ so the function $s \mapsto I_s(f)$ is differentiable in each $\{s: \operatorname{Im} s < s_1 < 0\}$ and

$$\frac{d}{ds} I_s(f) = -\frac{i}{2} \int_{\mathcal{H}} \|\gamma\|^2 \exp(-\frac{is}{2}\|\gamma\|^2)\, d\mu_f(\gamma) . \tag{23}$$

Consequently, the function $s \to I_s(f)$ is single-valued analytic in the open lower halfplane $\mathbf{C}_F^\circ$. Moreover, this function is continuous in $\mathbf{C}_F$ due to the dominated convergence theorem. Finally, the relation

$$\lim_{\substack{s\to 0\\ s\in \mathbf{C}_F}} I_s(f) = f(0) \tag{24}$$

holds; one can use it to define $I_0(.)$ if necessary.

Concluding this section we bring together the obtained results. The F_s -maps defined by (10) have the following properties.

Theorem 1: (a) $I_s(.)$ is a normalized linear functional and $\|I_s\| = 1$ for each $s \in \mathbf{C}_F$.

(b) Let $\{E_n\}$ be a sequence of orthoprojections on $\mathcal{H}$ which converges strongly to the unit operator. If $f \in \mathcal{F}(\mathcal{H})$, then $\lim_{n\to\infty} I_s(f \circ E_n) = I_s(f)$, where $(f \circ E_n)(\gamma) \equiv f(E_n\gamma)$.

(c) For each $f \in \mathcal{F}(\mathcal{H})$ the function $s \mapsto I_s(f)$ is single-valued analytic in $\mathbf{C}_F^\circ$ and continuous in $\mathbf{C}_F$; moreover, the relation (24) holds.

Theorem 2: Let f be a tame function, $f \in \mathcal{F}^t(\mathcal{H})$, based in a subspace $P\mathcal{H}$, and let further $f \upharpoonright P\mathcal{H}$ belong to $L(P\mathcal{H}, m)$, m being the Lebesgue measure, then $I_s(f)$ is expressed by (16). In particular, for a finite-dimensional $\mathcal{H} = \mathbf{R}^n$ and $f \in \mathcal{F}(\mathbf{R}^n) \cap L(\mathbf{R}^n)$ the relations (11), (12) are valid.

Theorem 3: Let $\mathcal{H}$ decompose into an orthogonal sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $f \in \mathcal{F}(\mathcal{H})$. We denote $f(\gamma) = f(\gamma_1, \gamma_2)$, $\gamma_i \in \mathcal{H}_i$, then the functions $f_{\gamma_2}(.) = f(., \gamma_2)$ and $f'_{\gamma_1}(.) = f(\gamma_1, .)$ belong to $\mathcal{F}(\mathcal{H}_1)$ and $\mathcal{F}(\mathcal{H}_2)$ for all $\gamma_2 \in \mathcal{H}_2$, $\gamma_1 \in \mathcal{H}_1$, respectively, further the functions h_s: $h_s(\gamma_2) = I_s(f_{\gamma_2})$ and h'_s: $h'_s(\gamma_1) = I_s(f'_{\gamma_1})$ belong to $\mathcal{F}(\mathcal{H}_2)$ and $\mathcal{F}(\mathcal{H}_1)$ respectively, and finally $I_s(f) = I_s(h_s) = I_s(h'_s)$.

Theorem 4: (a) Let s be real non-zero, $f \in \mathcal{F}(\mathcal{H})$, then $I_s(f)$ transforms under translations of $\mathcal{H}$ according to (19). Furthermore, for $s \in \mathbf{C}_F$ and U regular isometric the relation (20) is valid. In particular, these relations express transformation properties of the F-integral w.r.t. Euclidean motions of $\mathcal{H}$.
(b) If $\mathcal{H} = \mathbf{R}^n$ and $f \in \mathcal{F}(\mathbf{R}^n) \cap L(\mathbf{R}^n)$, the formula expressing transformation properties under translations holds for $s \in \mathbf{C}_F^o$ as well - cf. (17). Moreover, if $f, f_B \in \mathcal{F}(\mathbf{R}^n) \cap L(\mathbf{R}^n)$, then $I_s(f)$ transforms under "change of variables" mediated by a regular operator B on $\mathbf{R}^n$ according to (18).

4. CONCLUSIONS

Study of the F-maps started here will be continued in our forthcoming paper. We shall specify there the path space and extend the F-maps defined above by means of a general polygonal-path approximation; further we shall discuss properties and applications of the obtained extensions.

As the last item here we shall make a comment on one more property of the F-integrals. Evaluation of an integral is often simplified if the integrated function represents a limit of some sequence of functions, the integrals of which are known. In fact, this method is one of the most used for the "usual" integrals, where powerful sufficient conditions are available for convergence of the corresponding sequence of integrals, among them especially the dominated convergence theorem. We have no such assertions for the F-integrals, though there exists, e.g., a treatment of the classical limit of quantum mechanics based on the assumption of its validity (see Introduction), to say nothing of the non-rigorous path-integral calculations.

We shall show that a dominated-convergence-type theorem is not valid for the F-integrals, even if the simplest finite-dimensional case and substantially stronger assumption about the function sequence are considered. It is clear that we must avoid situations when $\lim_{n\to\infty} \| f_n - f \|_0 = 0$ formulating the counterexample, otherwise $I_s(f) = \lim_{n\to\infty} I_s(f_n)$ would follow from Theorem 1.

Example: Let $\mathcal{H} = \mathbf{R}$ and $\mu_a : \mu_a(J) = (2a)^{-1} \int_{J_a} \exp(\frac{i}{2} x^2)\, dx$, where $J_a = J \cap (-a,a)$. Obviously $\mu_a \in \mathfrak{M}(\mathbf{R})$ for each $a > 0$ and the corresponding functions f_a are bounded by the unit function, $|f_a(x)| \le |\mu_a|(\mathbf{R}) = 1 = e(x)$, which is "integrable" (cf. Proposition 2). It holds

$$f_a(x) = \int_{\mathbf{R}} \exp(ixy)\, d\mu_a(y) = $$
$$= (2a)^{-1} \exp(-\frac{i}{2} x^2) \int_{-a}^{a} \exp(\frac{i}{2}(x+y)^2)\, dy$$

and according to the assertion proved in Appendix C there exists a constant $M = (2\pi)^{1/2} K_1^1$ such that $|f_a(x)| \le M/2a$ for all $x \in \mathbf{R}$, $a > 0$, i.e.,

$$\lim_{a\to\infty} \| f_a \|_\infty = 0. \tag{25}$$

On the other hand, $I(f_a) = \int_{\mathbf{R}} \exp(-\frac{i}{2} x^2)\, d\mu_a(x) = 1$ for each $a > 0$, and therefore

$$\lim_{a\to\infty} I(f_a) = 1 \ne 0. \tag{26}$$

Notice that the net $\{f_a\}$ converges to the zero function according to (25) not only pointwise (everywhere in $\mathbf{R}$), but even uniformly, and yet $\{ I(f_a) \}$ does not converge to $I(0)$. In the same time the relations (25), (26) show that the functional $I(.)$ is not bounded w.r.t. the uniform norm $\|.\|_\infty$ on $\mathcal{F}(\mathcal{H})$.

APPENDIX A

Let τ be the norm topology in $\mathcal{H}$, $\tau\times\tau$ the product topology in $\mathcal{H}\times\mathcal{H}$, U_τ some basis of τ, $\mathcal{R}(S)$ the σ-algebra generated by the system S, $\mathcal{B}_2$ the Borel system in $\mathcal{H}\times\mathcal{H}$ (w.r.t. $\tau\times\tau$) and $\mathcal{B} \otimes \mathcal{B} = \mathcal{R}(\mathcal{B}\times\mathcal{B})$. We shall prove that $\mathcal{B}_2 = \mathcal{B} \otimes \mathcal{B}$ if $\mathcal{H}$ is separable.

According to the definition $\mathcal{B}=\mathcal{R}(\tau)$; we shall check first $\mathcal{B}=\mathcal{R}(U_\tau)$. Clearly $\mathcal{R}(\tau)\supset\mathcal{R}(U_\tau)$. $\mathcal{H}$ is separable and therefore second countable, thus each $A\in\tau$ equals $\bigcup_{k=1}^{\infty} A_k$, $A_k\in U_\tau$, but $\mathcal{R}(U_\tau)$ is a σ-algebra so $\tau\subset\mathcal{R}(U_\tau)$ and $\mathcal{R}(\tau)\subset\mathcal{R}(\mathcal{R}(U_\tau))=\mathcal{R}(U_\tau)$. Analogously $\mathcal{B}_2=\mathcal{R}(U_{\tau\times\tau})$, especially $\mathcal{B}_2=\mathcal{R}(\tau\times\tau)$ because the product topology is generated by $\tau\times\tau$. The inclusion $\mathcal{B}\times\mathcal{B}\supset\tau\times\tau$ implies $\mathcal{B}\otimes\mathcal{B}\supset\mathcal{B}_2$. There is a natural bijection between $\mathcal{B}$ and $\mathcal{B}\times B=\{A\times B: A\in\mathcal{B}\}$ for any $B\in\tau$ with $\bigcup_k (A_k\times B)=(\bigcup_k A_k)\times B$ and $(A_1\times B)-(A_2\times B)=(A_1-A_2)\times B$ so that $A\in\mathcal{B}=\mathcal{R}(\tau)$ implies $A\times B\in\mathcal{R}(\tau\times B)\subset B_2$, i.e., $\mathcal{B}\times\tau\subset\mathcal{B}_2$. Further $A\times\mathcal{B}$ is in the same way isomorphic to $\mathcal{B}$ for any $A\in\mathcal{B}$ so that $B\in\mathcal{B}$ implies $A\times B\in\mathcal{R}(A\times\tau)$. Thus $\mathcal{B}\times\mathcal{B}\subset\mathcal{R}(\mathcal{B}\times\tau)\subset\mathcal{R}(\mathcal{B}_2)=\mathcal{B}_2$, and consequently $\mathcal{B}\otimes\mathcal{B}\subset\mathcal{B}_2$. The inverse inclusion was obtained above, hence the proof is finished.

APPENDIX B

Let μ be a Borel measure on separable $\mathcal{H}$, $|\mu|(\mathcal{H})<\infty$. We define $\mu_K(A)=\mu(A\cap K)$ for any compact $K\subset\mathcal{H}$ and Borel A. It makes sense because $\mathcal{H}$ is Hausdorff so that K is closed (/29/, sec. II.6, th.4) and therefore Borel, consequently $A\cap K$ is Borel. The set function μ_K is a Borel measure on K. On the other hand, any continuous function g on $\mathcal{H}$ is Borel measurable, further g is bounded, i.e., μ-measurable on any compact set $\subset\mathcal{H}$. The inequalities

$$\mu_K[g]=\int_K g\, d\mu_K \le \|g\|_\infty |\mu_K|(K) \le \|g\|_\infty |\mu|(\mathcal{H})$$

show that $\mu_K[.]$ is a continuous linear form on C[K], i.e., a measure in the sense of /22/, chap.III, §1.3 (Radon measure - cf. /30/, sec.I.D). The family $\{\mu_K:\ K\ \text{compact}\subset\mathcal{H}\}$ fulfils the following compatibility condition: if K,L are compact, $K\subset L$, then $(\mu_L)_K(A)=\mu_L(A\cap K)=\mu(A\cap L\cap K)=\mu_K(A)$ for any $A\in\mathcal{B}$, i.e., $(\mu_L)_K=\mu_K$. Consequently, this family represents a premeasure on $\mathcal{H}$ - /22/, chap. IX, §1.2, def.3. Moreover, according to §1.2, def.5 and §1.1, def.2 of this chapter $\{\mu_K:\ K\ \text{compact}\subset\mathcal{H}\}$ would represent a measure on $\mathcal{H}$ if $|\mu|^\bullet(\mathcal{H})<\infty$. Def.4 of §1.4 gives $|\mu|^\bullet(\mathcal{H})=\sup_{K\subset\mathcal{H}}|\mu|_K^\bullet[\chi_K]$, and since χ_K is continuous on K we obtain further $|\mu|_K^\bullet[\chi_K]=|\mu|_K^*[\chi_K]=|\mu|_K[\chi_K]$ (/22/, chap.IV, §§1.1, 1.3, chap.V, §1.1) so that $|\mu|^\bullet(\mathcal{H})=\sup_{K\subset\mathcal{H}}|\mu|_K(L)$ $=$ $=\sup_{K\subset\mathcal{H}}|\mu|(K)\le|\mu|(\mathcal{H})<\infty$. Thus the (bounded) measure $\{\mu_K:\ K\ \text{compact}\subset\mathcal{H}\}$ in the sense of ref. /22/ corresponds to every (finite complex) Borel measure on $\mathcal{H}$.

Furthermore, this correspondence is <u>injective</u>: the measures $\{\mu_K :\ K\ \text{compact} \subset \mathcal{H}\}$ form a vector space (/22/, chap. IX, §1.2) and its zero element fulfils $|\mu_0|^\bullet(\mathcal{H}) = \sup_{K\subset\mathcal{H}} |\mu_0|^\bullet_K[\chi_K] = 0$. Any Borel measure μ_0 to which this element corresponds fulfils therefore $|\mu_0|(K)=0$ for every compact $K\subset\mathcal{H}$. Since $\mathcal{H}$ is separable, every ball $B\subset\mathcal{H}$ can be embedded into a countable union $\bigcup_n K_n$ of compact sets, say Hilbert bricks, so $|\mu_0|(B) \le$ $\le |\mu_0|(\bigcup_n K_n) \le \sum_n |\mu_0|(K_n) = 0$. Further a separable $\mathcal{H}$ is second countable, then mimicking the last argument we obtain $|\mu_0|(G)=0$ for each open $G\subset\mathcal{H}$, especially $|\mu_0|(\mathcal{H})=0$, i.e., $\mu_0 = 0$.

APPENDIX C

Assume first the integral $C(a,b) = \int_a^b \exp(ibt^2)\,dt$ for $a,b>0$. Since the function $z\mapsto \exp(ibz^2)$ is entire, $C(a,b)$ can be evaluated by contour integration. A suitable closed contour follows the real axis from 0 to a, then it makes a circular arc anticlockwise and returns to the origin along the half-line $\{z : \arg z = \pi/4\}$. Consequently, we have

$$C(a,b) = -ia\int_0^{\pi/4} \exp(iba^2 e^{2i\phi})\, e^{i\phi}\, d\phi + \exp\left(\frac{\pi i}{4}\right)\int_0^a \exp(-bt^2)\,dt. \qquad (*)$$

The first integral on the rhs (call it J_1) can be estimated as follows

$$|J_1| \le a\int_0^{\pi/4} |\exp(iba^2 e^{2i\phi})|\,d\phi = \frac{1}{2}a\left(\int_0^{a^{-3/2}} + \int_{a^{-3/2}}^{\pi/2}\right)\exp(-ba^2\sin\xi)\,d\xi$$

assuming $a \ge (\pi/2)^{-2/3}$. Using further $\sin\xi \ge \frac{1}{2}\xi \ge \frac{1}{2}a^{-3/2}$ for $\xi \in (a^{-3/2}, \pi/2)$ we get

$$|J_1| \le \frac{1}{2}a^{-1/2} + \frac{\pi a}{4}\exp\left(-\frac{1}{2}ba^{1/2}\right). \qquad (**)$$

The second rhs integral in $(*)$ can be estimated easily as well as $C(a,b)$ for small a: we obtain

$$|C(a,b)| \le \begin{cases} a & \dots\ a \le (\pi/2)^{-2/3} \\ \frac{1}{2}\left(\frac{\pi}{b}\right)^{1/2} + \frac{1}{2}a^{-1/2} + \frac{\pi a}{4}\exp\left(-\frac{1}{2}ba^{1/2}\right) & \dots\ a \ge (\pi/2)^{-2/3} \end{cases} \qquad (+)$$

so $|C(a,b)|$ is for every b majorized by a constant independent of a. The relation (**) further implies

$$\lim_{a\to\infty} C(a,b) = \exp\left(\frac{\pi i}{4}\right) \int_0^\infty \exp(-bt^2)\,dt = \left(\frac{\pi i}{4b}\right)^{1/2}. \qquad (++)$$

Since $C(a,-b) = \overline{C(a,b)}$, the relations (+),(++) are valid for $b<0$ as well. Assume now

$$K_s^1(y,\alpha) = (2\pi is)^{-1/2} \int_{-\alpha}^{\alpha} \exp\left(\frac{i}{2s}(x+sy)^2\right) dx =$$

$$= (2\pi is)^{-1/2} \left(\left(C(\alpha+sy, \frac{1}{2s}\right) + C\left(\alpha-sy, \frac{1}{2s}\right)\right).$$

This expression is majorized according to (+) by a constant K_s^1 which depends on s only, further (++) implies $\lim_{\alpha\to\infty} K_s^1(y,\alpha) = 1$. Finally the Fubini theorem implies $K_s^n(y,\alpha) = \prod_{j=1}^{n} K_s^1(y_j,\alpha)$ so $K_s^n = (K_s^1)^n$ and $\lim_{\alpha\to\infty} K_s^n(y,\alpha) = 1$.

REFERENCES

1. DeWitt-Morette C., Maheswari A., Nelson B. Phys.Rep., 1979, 50C, p.255-372.
2. Albeverio S.A., Hoegh-Krohn R.J. In: Lecture Notes in Math., vol.523, Springer-Verlag, Berlin, 1976.
3. Albeverio S.A., Hoegh-Krohn R.J. In: Lecture Notes in Physics, vol.106 (ed. S.Albeverio et al.), Springer-Verlag, Berlin, 1979.
4. Truman A. In: Lecture Notes in Math., vol.644, Springer-Verlag, Berlin, 1978.
5. Truman A. In the same Proceedings as Ref.3, p.73-102.
6. Feynman R.P. Rev.Mod.Phys., 1948, 20, p.367-387.
7. Feynman R.P., Hibbs A.R. Quantum Mechanics and Path Integrals, McGraw Book Co, New York, 1965.
8. Reed M., Simon B. Methods of Modern Mathematical Physics. I. Functional Analysis. II. Fourier Analysis. Self-Adjointness, Academic Press, New York, 1972, 1975.
9. Nelson E. J.Math.Phys., 1964, 5, p.332-343.
10. Kuo H.-H. Gaussian Measures in Banach Spaces. Lecture Notes in Math., vol.463, Springer-Verlag, Berlin, 1975.
11. Gel'fand I.M., Yaglom A.M. J.Math.Phys., 1960, 1, p.48-69.
12. Cameron R.H. J.Anal.Math., 1962-63, 10, p.287-361.
13. Simon B. The $P(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton University Press, Princeton, 1974.
14. Fröhlich J. In: Many Degrees of Freedom in Field Theory (ed. L.Streit), NATO Advanced Study Institutes Series, vol.30, Plenum Press, New York, 1978.

15. Combe M. et al. Rep.Math.Phys., 1978, 13, p.279-294.
16. DeWitt-Morette C. Comm.Math.Phys., 1972, 28, p.47-67.
17. DeWitt-Morette C. Comm.Math.Phys., 1974, 37, p.68-89.
18. Truman A. J.Math.Phys., 1976, 17, p.1852-1862.
19. Ito K. In: Proc. Fifth Berkeley Symposium on Math. Statistics and Probability, Univ.California Press, vol.II, part 1, Berkeley, 1967.
20. Truman A. J.Math.Phys., 1977, 18, p.1499-1509.
21. Truman A. J.Math.Phys., 1978, 19, p.1742-1750, erratum ibidem 1979, 20, p.1832-1833.
22. Bourbaki N. Éléments de mathêmatique VI: Integration, deuxieme edition, Hermann, Paris, 1974.
23. DeWitt-Morette C. Ann.Phys. (NY), 1976, 97, p.367-399, erratum ibidem 1976, 101, p.682-683.
24. Albeverio S.A., Hoegh-Krohn R.J. Inventiones Math., 1977, 40, p.59-106.
25. Truman A. J.Math.Phys., 1977, 18, p.2308-2315.
26. Dunford N., Schwartz J.T. Linear Operators I. General Theory. Interscience Publ., New York, 1958.
27. Halmos P.R. Measure Theory, Van Nostrand, New York, 1950.
28. Gradshtein I.S., Ryzhik I.M. Tables of Integrals, Sums, Series and Products. "Nauka", M., 1971.
29. Kolmogorov A.N., Fomin S.V. Elements of Function Theory and Functional Analysis (Russian). "Nauka", M., 1972.
30. Choquet-Bruhat Y., DeWitt-Morette C., Dillard-Bleick M. Analysis, Manifolds and Physics, North-Holland, Amsterdam, 1977.

Received by Publishing Department
on September 30 1980.