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A METHOD FOR SUMMING NONALTERNATING ASYMPTOTIC SERIES

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1. Introduction. Statement of the Problem

In recent years great progress in the perturbation theory (PT) calculations and constructive use of the functional integral representation in quantum field theory has led to a clear understanding of the asymptotic character of the PT series. Moreover, application of the steepest descent method to the functional integral has given an explicit high order asymptotic behaviour of PT coefficients directly related to the properties of the asymptotic expansion.

Because of the zero radius of convergence the PT series in quantum theory can serve as a source of the quantitative information only in a very small region of the coupling constant. This raises the problem of "summation" of such series, i.e., the problem of reconstructing a function from its asymptotic expansion.

A definite success in solving this problem has been achieved in the case of alternating asymptotic series /1-8/. Most of approaches used the Borel summation method /9/ with different modifications. However, in the case of nonalternating asymptotic series the Borel method cannot be applied because of the singularities on the integration contour and the divergence of the Laplace integral.

In the present paper a method for summation of asymptotic series is proposed which is applied both to the alternating and to the nonalternating series. In the case of alternating series it reduces to a modified Borel transform. It appears that there is a principal difference between two cases, due to the presence of an essential singularity of the sought function at the origin. It should be emphasized from the very beginning that the problem of reconstructing the function from its asymptotic expansion possesses the functional arbitrariness. A unique solution is possible only when the analytical properties are known. But usually they can only be supposed. The same problem concerns the substantiation of the Borel method for the alternating series in QFT. Our aim below is not to prove the uniqueness of the solution, but to reconstruct the function satisfying all the given properties, i.e., the given PT coefficients and asymptotics of high order behaviour. This asymptotics, obtained by the saddle-point method in a functional integral, supposes the definite form of the discontinuity of the function on the cut /11/, i.e., some analytical properties. Some other restictions are imposed also which are connected with the physical nature of the problem.

The method is illustrated by two examples: the nonalternating expansion of a model ordinary integral simulating the functional integral in a theory with degenerate minimum and the ground state energy evaluation in a double-well anharmonic oscillator. A great deal of interest to the study of nonalternating asymptotic series comes from the presence of such series in all the quantum problems with degenerate vacuum and, in particular, in the most popular today quantum field theory model - the Yang-Mills fields theory.

2. Euler's Method and the Main Difference Between the Alternating and Nonalternating Series

To demonstrate the difference between the alternating and nonalternating series originated from the presence of an essential singularity of the sought function at the origin, let us consider the Euler's method of summation /9/. This method is applied when PT series are known exactly, that is not so in real problems.

Example 1

Let the function $F(g)$ be given by the expansion

$$F(g) \sim \sum_{k=0}^{\infty} \Gamma(k+b+1) a^k g^k. \tag{1}$$

Then it satisfies the differential equation

$$a g^2 F' + a(b+1) g F - F + \Gamma(b+1) = 0 \tag{2}$$

with the boundary condition

$$F(g)\big|_{g \to +0} = \Gamma(b+1). \tag{3}$$

The solution of (2) is

$$F(g) = -\frac{\Gamma(b+1)}{a} g^{-b-1} e^{-\frac{1}{ag}} \int^{g} y^{b-1} e^{\frac{1}{ay}} dy +$$
$$+ c g^{-b-1} e^{-\frac{1}{ag}}$$

or

$$F(g) = -\Gamma(b+1) \int^{0} \frac{dt\, e^{-t}}{(1-agt)^{b+1}} + c g^{-b-1} e^{-\frac{1}{ag}} .$$

If $g \to +0$ we have

$$F(g) \to \Gamma(b+1) + \begin{cases} c \cdot \infty & , \quad a < 0 \\ c \cdot 0 & , \quad a > 0 \end{cases} .$$

Hence eq.(3) means that

i) $a < 0$ (alternating series)

$C = 0$ and $F(g)$ is defined uniquely.

ii) $a > 0$ (nonalternating series)

C is arbitrary and $F(g)$ is defined up to contributions $\sim \exp(-\frac{1}{ag})$. The arbitrariness can be fixed imposing an extra condition for $g \neq 0$, for instance, fixing the asymptotics as $g \to \infty$.

Example 2

Let the function $F(g)$ be given by the expansion

$$F(g) \sim \sum_{k=0}^{\infty} \frac{\Gamma(2k+1/2)}{\Gamma(k+1)4^k} a^k g^k . \tag{4}$$

It satisfies the equation

$$4g^2 F'' + 8gF' + \frac{3}{4} F - \frac{4}{a} F' = 0 \tag{5}$$

with the boundary condition

$$F(g)\Big|_{g \to +0} = \sqrt{\pi} . \tag{6}$$

Equation (5) is reduced to the Bessel equation with the imaginary argument /10/. One can choose two linear-independent solutions to be $I_{\pm\frac{1}{4}}\left(\frac{1}{2|a|g}\right)$. Then the general solution of eq.(5) will be

$$F(g)=\frac{1}{\sqrt{g}}\, e^{-\frac{1}{2ag}}\left[C_1 I_{\frac{1}{4}}\left(\frac{1}{2|a|g}\right)+C_2 I_{-\frac{1}{4}}\left(\frac{1}{2|a|g}\right)\right]. \quad (7)$$

The requirement of finiteness of $F(g)$ as $g\to +0$ means:

i) $a<0$ (alternating series)

Exponent $\exp(-\frac{1}{2ag})$ should be compensated, i.e., the combination $C_1 I_{\frac{1}{4}}+C_2 I_{-\frac{1}{4}}$ should contain only $\exp(-\frac{1}{2|a|g})$. This is possible if $C_1=-C_2$. Then eq.(6) uniquely fixes $C_1=-C_2=-\frac{\pi}{\sqrt{2|a|}}$.

ii) $a>0$ (nonalternating series)

Exponent $\exp(-\frac{1}{2ag})$ vanishes and gives no restriction on C_1 and C_2. From eq.(6) (as well as from any condition at $g=+0$) we get the sum $C_1+C_2=\pi/\sqrt{a}$. To fix the remaining arbitrariness (of the type $\exp(-\frac{1}{2ag})$) one should put an extra condition for $g\neq 0$ as in the previous example.

The difference between the two cases i) and ii) in both examples is due to the fact that the boundary condition was imposed as $g\to +0$. The situation will be just opposite if we impose it as $g\to -0$. The limiting values of the functions for $g\to\pm 0$ do not coincide because of the essential singularity of $F(g)$ at $g=0$ /11/. Two functions $F(g)$, real on the positive real semi-axis, defined by asymptotic expansions (1) or (4) with $a>0$ and $a<0$ have different analytical properties and cannot be deduced from each other by a simple analytical continuation $g\to -g$.

In the case of nonalternating series the sought function contains usually, apart from the PT contribution, the so-called instanton contribution $\sim e^{-\frac{1}{ag}}$ which is eliminated in the alternating case by the boundary condition as $g\to +0$.

In real problems the boundary condition (PT) at $g=+0$ means the free theory limit (the absence of interaction). The positivity of g provides the stability of the theory (stable minimum or boundedness of the potential from below). This means that the boundary condition cannot be imposed at $g=-0$. This fact leads to the arbitrariness in the sought function $F(g)$ in the nonalternating case, that we shall see in the summation procedure proposed below.

It should be stressed that our aim is to reproduce the function starting from its asymptotic expansion on the real axis. We do not pretend to reproduce all its singularities (gathering cuts, etc.) in the whole complex plane. The presence of such singularities, as we can see by the example of anharmonic oscillator with nondegenerate minimum, does not exclude the possibility to use the Borel summation method for reconstructing the ground state energy on the real semi-axis.

3. Formulation of the Method. Exactly Solvable Case

The idea of the proposed method for nonalternating asymptotic series exploits the presentation of the Euler Γ-function different from that used in the Borel transform*):

$$\Gamma(z) = \frac{1}{2i \sin \pi (z-1)} \int_C dt\, e^{-t} (-t)^{z-1}, \qquad z \text{ is not integer,} \tag{8}$$

where contour C is shown in fig. 1.

The method of summation is the following:
Let the function $F(g)$ be defined by the asymptotic expansion

$$F(g) \sim \sum_{k=k_0}^{\infty} g^k F_k, \qquad F_k \underset{k \to \infty}{\sim} k!\, k^b a^k . \tag{9}$$

Then, using (8) and proceeding in analogy with the Borel method, we put

$$F(g) = \sum_{k=k_0}^{\infty} g^k \frac{F_k}{\Gamma(k+\mu)} \frac{1}{2i \sin \pi(k+\mu-1)} \int_C dt\, e^{-t} (-t)^{k+\mu-1} = \tag{10}$$

$$= \int_C dt\, e^{-t} (-t)^{\mu-1} \frac{1}{2i \sin \pi(\mu-1)} \sum_{k=k_0}^{\infty} (gt)^k \frac{F_k}{\Gamma(k+\mu)} ,$$

where μ is noninteger.

If coefficient F_k obeys condition (9), then for k large

*) Another method using analogous Γ-function representation is discussed in paper /12/.

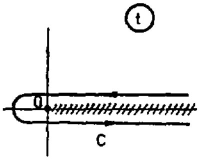

Fig. 1

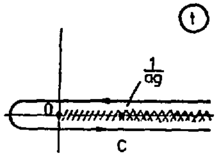

Fig 2

$$\frac{F_k}{\Gamma(k+\mu)} \sim k^{b-\mu+1} a^k$$

that means that the series

$$\Phi(t) = \sum_{k=k_0}^{\infty} (gt)^k \frac{F_k}{\Gamma(k+\mu)} \tag{11}$$

defines the function with a singularity at point $t = \frac{1}{ag}$ of the type $(1-agt)^{\mu-b-2}$ /13/.

If $a < 0$ (alternating series) this singularity lies to the left from the integrating region and integral (10) reduces to that one of discontinuity on the cut arising from the multiplier $(-t)^{\mu-1}$ and equals

$$F(g) = \int_0^{\infty} dt\, e^{-t} t^{\mu-1} \sum_{k=k_0}^{\infty} (gt)^k \frac{F_k}{\Gamma(k+\mu)}, \quad a < 0. \tag{12}$$

Hence we come to the Borel method for summation of alternating series.

If $a > 0$ (nonalternating series), the singularity lies to the right and two cuts are overlapped (see fig. 2). Thus, to find the function $F(g)$ we have to take into account the discontinuities on both the cuts.

We apply this procedure to an exactly-solvable example.

Example 3

Consider the simple integral

$$I(g) = \frac{1}{2} \int_{-\infty}^{\infty} dx\, e^{-x^2\left(1-\frac{\sqrt{g}}{2}x\right)^2}, \tag{13}$$

which is a 0-dimensional analog of the functional integral in a theory with the double-well potential. This integral has an asymptotic expansion of the form (4) with a=4. Then, according to (10):

$$F(g)=\int_C dt\, e^{-t}(-t)^{\mu-1}\frac{1}{2i\sin\pi(\mu-1)}\sum_{k=0}^{\infty}(gt)^k\frac{\Gamma(2k+1/2)}{\Gamma(k+1)\Gamma(k+\mu)}. \quad (14)$$

Using the representation $\Gamma(2k+1/2)=1/\sqrt{2\pi}\cdot 4^k\,\Gamma(k+1/4)\Gamma(k+3/4)$, we find that the series obtained can be easily summed for $\mu=1/4$ or $\mu=3/4$. After some simple calculations we have

$$F(g)=\frac{\Gamma(1-\mu)}{2i\sin\pi(\mu-1)\sqrt{2\pi}}\int_C dt\, e^{-t}(-t)^{\mu-1}(1-4gt)^{\mu-1},\quad \mu=\begin{cases}1/4\\3/4\end{cases}. \quad (15)$$

Calculating discontinuities on the cuts from the points $t=0$ and $t=1/4g$, we get

$$F(g)=\frac{\Gamma(1-\mu)}{\sqrt{2\pi}}\left\{\int_0^{\frac{1}{4g}}dt\, e^{-t}t^{\mu-1}(1-4gt)^{\mu-1}-2\cos\pi\mu\int_{1/4g}^{\infty}dt\, e^{-t}t^{\mu-1}(4gt-1)^{\mu-1}\right\}$$

$\mu=\frac{1}{4},\frac{3}{4}$

The integrals obtained can be expressed via the Bessel functions of the imaginary argument /14/

$$F(g)=\frac{\Gamma(\mu)\Gamma(1-\mu)\sqrt{\pi}}{\sqrt{2\pi}\ \sqrt{4g}}e^{-\frac{1}{8g}}\left[I_{\mu-\frac{1}{2}}\left(\frac{1}{8g}\right)-\frac{2\cos\pi\mu}{\pi}K_{\mu-\frac{1}{2}}\left(\frac{1}{8g}\right)\right]=$$

$$=\frac{\Gamma(\mu)\Gamma(1-\mu)}{\sqrt{8g}}e^{-\frac{1}{8g}}I_{-\mu+\frac{1}{2}}\left(\frac{1}{8g}\right)=\frac{\pi}{\sqrt{4g}}e^{-\frac{1}{8g}}I_{-\mu+\frac{1}{2}}\left(\frac{1}{8g}\right).$$

$\mu=\frac{1}{4},\frac{3}{4}$

So, for the function $F(g)$ given by the asymptotic expansion (4), we have

$$F(g)=\frac{\pi}{\sqrt{4g}}e^{-\frac{1}{8g}}I_{\pm\frac{1}{4}}\left(\frac{1}{8g}\right).$$

Comparing with eq.(7) we see that the whole result can be expressed by the linear combination of two Infeld functions

$$F(g)=\frac{\pi}{\sqrt{4g}}\,e^{-\frac{1}{8g}}\left[c\,I_{\frac14}\left(\frac{1}{8g}\right)+(1-c)\,I_{-\frac14}\left(\frac{1}{8g}\right)\right]. \tag{16}$$

The arbitrariness of c in (16) corresponds to the arbitrariness in the choice of parameter μ in eq.(14). Our choice $\mu=1/4$ or $3/4$ is explained only by simplicity of summation. Independently of the value of c (or μ) the function $F(g)$(16) is expanded into the asymptotic series (4) with a=4. The arbitrariness, as in the examples above, is proportional to $\exp(-1/ag)$. This is a manifestation of the ambiguities of nonalternating asymptotic series pointed out earlier. In the alternating case ($a<0$) the series (4) are summed uniquely. For eliminating this arbitrariness and providing the correspondence of $F(g)$ (16) with integral (13) we have to use an additional information which is not connected with the expansion of integral (13) in g. For example, as $g\to\infty$ $I(g)\to\frac12\Gamma(1/4)(4g)^{-1/4}$. This condition uniquely fixes the value of $c=1/2$. As a result, we get

$$I(g)=\frac{\pi}{\sqrt{4g}}\,e^{-\frac{1}{8g}}\,\frac12\left[I_{\frac14}\left(\frac{1}{8g}\right)+I_{-\frac14}\left(\frac{1}{8g}\right)\right]. \tag{17}$$

So, when the PT series are exactly known, the proposed method enables us to reconstruct the function given by the nonalternating asymptotic series up to one parameter originated from the choice of μ in eq.(10). This arbitrariness can be eliminated by some extra condition.

4. Formulation of the Method. Approximate Scheme

Consider now the case when only a limited number of coefficients and their high-order asymptotic behaviour (9) are known. This situation is typical of quantum theory. Asymptotic estimates for high-order coefficients are obtained with the use of the steepest descent method in the functional integral, where the saddle point is a solution of classical Euclidean equations of motion with finite action - the so-called instantons /15-17/.

The series (11) defines the function $\Phi(t)$ in a circle of radius $1/ag$, where it is presented by a polynomial of some degree. For its analytical continuation to the whole integration region we use the approach developed for summing alternating series by the Borel method [2,4]. Namely, we perform the conformal mapping $t \to w$ of the cut plane into the interior of the unit circle so that the interval $(0, 1/ag)$ maps into the interval (0,1) and the cut $(\frac{1}{ag}, \infty)$ maps into the boundary of this circle. We choose the mapping $w(t)$ so that its singularity at $t = 1/ag$ coincides with that of $\phi(t)$. These requirements are satisfied by the following mapping

$$t \to w(t) = \frac{1-(1-agt)^{\alpha}}{1+(1-agt)^{\alpha}} ,$$

where $\alpha = \{\mu - b - 2\}$ and $\{\ldots\}$ means the fractional value of the number.

Then, reexpanding series (11) into the series in the new variable, we have

$$\phi(t) = \sum_{k=k_0}^{N} (gt)^k \frac{F_k}{\Gamma(k+\mu)} = (gt)^{\lambda} \sum_{k=k_0}^{N} w^{k-\lambda} C_k . \qquad (18)$$

We have introduced here the new parameter λ which, as we shall see below, defines the asymptotics of the sought function $F(g)$ as $g \to \infty$. In the case when we know the whole infinite series exactly, the dependence of λ disappears.

Substituting (18) into (10), we get

$$F(g) = \int_C dt\, e^{-t} (-t)^{\mu-1} \frac{1}{2i \sin \pi(\mu-1)} (gt)^{\lambda} \sum_{k=k_0}^{N} w^{k-\lambda} C_k . \qquad (19)$$

For the evaluation of integral (19) we have to take into account the discontinuities on both cuts. For $0 < t < 1/ag$ the discontinuity is provided by the multiplier $(-t)^{\mu-1}$ only and equals $2i \cdot \sin \pi(\mu-1) t^{\mu-1}$. For $\frac{1}{ag} < t < \infty$ we have the overlapping of two cuts. In this area it is useful to represent w in the form

$$w(t) = \rho e^{i\theta} , \text{ where } \quad tg\,\theta = \frac{2 \sin \pi\alpha}{(agt-1)^{\alpha} - (agt-1)^{-\alpha}} ,$$

$$\rho = \sqrt{\frac{(agt-1)^{\alpha} + (agt-1)^{-\alpha} - 2\cos\pi\alpha}{(agt-1)^{\alpha} + (agt-1)^{-\alpha} + 2\cos\pi\alpha}} . \qquad (20)$$

Substituting (20) into (19), we find

$$F(g) = \int_0^{\frac{1}{ag}} dt\, e^{-t} t^{\mu-1} (gt)^{\lambda} \sum_{K=K_0}^{N} W^{K-\lambda} C_K +$$

$$+ \int_{\frac{1}{ag}}^{\infty} dt\, e^{-t} t^{\mu-1} (gt)^{\lambda} \sum_{K=K_0}^{N} \rho^{K-\lambda} C_K \frac{\sin[\pi(\mu-1) + \theta(K-\lambda)]}{\sin \pi(\mu-1)} . \qquad (21)$$

Performing the change of variables $t \to \frac{t}{ag}$, we finally get

$$F(g) = \frac{1}{(ag)^{\mu}} \left\{ \int_0^1 dt\, e^{-\frac{t}{ag}} t^{\mu-1} \left(\frac{t}{a}\right)^{\lambda} \sum_{K=K_0}^{N} C_K W^{K-\lambda} + \right.$$

$$\left. + \int_1^{\infty} dt\, e^{-\frac{t}{ag}} t^{\mu-1} \left(\frac{t}{a}\right)^{\lambda} \sum_{K=K_0}^{N} C_K \rho^{K-\lambda} \frac{\sin[\pi(\mu-1) + \theta(K-\lambda)]}{\sin \pi(\mu-1)} \right\}, \qquad (22)$$

where

$$W = \frac{1-(1-t)^{\alpha}}{1+(1-t)^{\alpha}} , \quad \rho = \sqrt{\frac{(t-1)^{\alpha} + (t-1)^{-\alpha} - 2\cos\pi\alpha}{(t-1)^{\alpha} + (t-1)^{-\alpha} + 2\cos\pi\alpha}} ,$$

$$\operatorname{tg} \theta = \frac{2\sin\pi\alpha}{(t-1)^{\alpha} - (t-1)^{-\alpha}} , \quad -\pi \le \theta \le 0 , \quad \alpha = \{\mu - b - 2\}.$$

Coefficients C_K can easily by obtained from (18). We present here the explicit expressions for the first four of them:

$$C_0 = \frac{F_0}{\Gamma(\mu)} \left(\frac{2}{a\alpha}\right)^{-\lambda},$$

$$C_1 = \frac{F_0}{\Gamma(\mu)} \left(\frac{2}{a\alpha}\right)^{-\lambda} \left\{ \frac{\lambda}{\alpha} + \frac{F_1}{F_0 \mu} \left(\frac{2}{a\alpha}\right) \right\},$$

$$C_2 = \frac{F_0}{\Gamma(\mu)}\left(\frac{2}{a\alpha}\right)^{-\lambda}\left\{-\frac{\lambda}{3}\left(1+\frac{2}{\alpha^2}\right)+\frac{\lambda(\lambda+1)}{2\alpha^2}+\frac{F_1}{F_0\mu}\left(\frac{2}{a\alpha}\right)\frac{\lambda-1}{\alpha}+\right.$$

$$\left.+\frac{F_2}{F_0\mu(\mu+1)}\left(\frac{2}{a\alpha}\right)^2\right\}, \qquad (23)$$

$$C_3 = \frac{F_0}{\Gamma(\mu)}\left(\frac{2}{a\alpha}\right)^{-\lambda}\left\{\frac{\lambda}{3\alpha}\left(2+\frac{1}{\alpha^2}\right)-\frac{\lambda(\lambda+1)}{3\alpha}\left(1+\frac{2}{\alpha^2}\right)+\frac{\lambda(\lambda+1)(\lambda+2)}{3!\,\alpha^3}\right.$$

$$\left.+\frac{F_1}{F_0\mu}\left(\frac{2}{a\alpha}\right)\left[\frac{\lambda(\lambda-1)}{2\alpha^2}-\frac{\lambda-1}{3}\left(1+\frac{2}{\alpha^2}\right)\right]+\frac{F_2}{F_0\mu(\mu+1)}\left(\frac{2}{a\alpha}\right)^2\frac{\lambda-2}{\alpha}+\frac{F_3\left(\frac{2}{a\alpha}\right)^3}{F_0\mu(\mu+1)(\mu+2)}\right\}.$$

Equation (22) is the main result of the present paper. For $g \to 0$ it reproduces the initial asymptotic series independently of the choice of μ and λ. For $g \to \infty$ we have

$$F(g) \xrightarrow[g\to\infty]{} g^{\lambda}\,\Gamma(\mu+\lambda)\sum_{k=k_0}^{N}(-)^k C_k\left[\cos\pi\lambda+\mathrm{ctg}\,\pi(\mu-1)\sin\pi\lambda\right]. \qquad (24)$$

We present for comparison analogous formulas for the alternating series

$$F_-(g) = \frac{1}{(|a|g)^{\mu}}\left\{\int_0^{\infty} dt\, e^{-\frac{t}{|a|g}}\, t^{\mu-1}\left(\frac{t}{|a|}\right)^{\lambda}\sum_{k=k_0}^{N} C_k(-)^k V^{k-\lambda}\right\}, \qquad (25)$$

where $V = \dfrac{(1+t)^{\alpha}-1}{(1+t)^{\alpha}+1}$ and coefficients C_k are the same as in (23). For $g \to \infty$

$$F_-(g) \xrightarrow[g\to\infty]{} g^{\lambda}\,\Gamma(\mu+\lambda)\sum_{k=k_0}^{N}(-)^k C_k\,. \qquad (26)$$

Notice that in both the cases the infinite series $\sum_{k=k_0}^{\infty}(-)^k C_k$ is convergent only if we correctly choose the asymptotics of the function as $g \to \infty$. Provided we know the whole series it will be chosen automatically. In our case we fix it either using some additional information or trying to find it by some "inner" way. One of such ways was proposed in paper /4/, where parameter λ was determined from the requirement of minimization of the modulus of the relative error

$$\Delta_N(g) = \frac{F_N(g) - F_{N-1}(g)}{F_N(g)} ,$$

where $F_N(g)$ is obtained taking into account N terms of PT. As far as Δ_N is proportional to C_N , this requirement means the minimization of $|C_N|$. In the case when the asymptotics is chosen correctly (see exactly solvable example 4 below), the coefficients C_K vanish beginning from some k . This is a starting point of the mentioned criterium for the determination of λ . The application of this criterium to the cases where asymptotics is known gives very satisfactory results /4/.

With the correct choice of λ the product

$$\Gamma(\mu+\lambda)\sum_{K=K_0}^{N}(-)^k C_K$$

is practically independent of μ and for $N\to\infty$ this weak dependence disappears. That is why in the case of alternating series eq.(25) does not contain arbitrariness. On the contrary, in the case of nonalternating series, eq.(22) still contains such an arbitrariness connected with the fractional part of μ due to the multiplier

$$[\cos\pi\lambda + ctg\,\pi(\mu-1)\sin\pi\lambda]$$

in eq.(24). This arbitrariness, as in the examples considered above, can be eliminated by fixing, for instance, the coefficient of the leading asymptotics of $F(g)$ as $g\to\infty$. Here we again can see the important difference between the alternating and nonalternating asymptotic expansions.

5. Application of the Method

Consider now application of eq.(22) for solving the problem of reconstruction of the function given by the nonalternating asymptotic expansion. In order to estimate the accuracy of the method we choose the examples admitting solutions by other methods.

Example 4

The simplest touch-stone of eq.(22) is the exactly solvable example 3 . It is of interest also because the integral (13) is a zero-dimensional analog of the functional integral in a theory with the double-well potential. Here we have a=4, b=-1. The application of the mentioned criterium for choosing parameter λ from the requirement of minimization of C_k leads to the value $\lambda=-1/4$ which is the correct number. Parameter μ can be fixed from the additional condition as $g\to\infty$, for instance, fixing the value of the coefficient of the leading asymptotics, as in example 3. This gives $\mu=3/2$, that means that $\alpha=\{\mu+1-2\}=1/2$ and coefficients C_k vanish for all k but 0. $C_0=2$.

Substituting these values of parameters into (22), we have

$$F(g)=\frac{2}{(4g)^{3/2}}\left\{\int_0^1 dt\, e^{-\frac{t}{4g}} t^{\frac14} 4^{\frac14} w^{\frac14}+\int_1^\infty dt\, e^{-\frac{t}{4g}} t^{\frac14} 4^{\frac14}\cdot \rho^{\frac14}\,\frac{\sin[\pi/2+\theta/4]}{\sin \pi/2}\right\}, \tag{27}$$

where

$$w=\frac{1-\sqrt{1-t}}{1+\sqrt{1-t}}=\frac{(1-\sqrt{1-t})^2}{t},\quad \rho=1,\quad tg\,\theta=\frac{2\sqrt{t-1}}{t-2}. \tag{28}$$

Taking into account (28) and $\cos\theta/4=\sqrt{\frac{\sqrt{t}+1}{2\sqrt{t}}}$, we finally get

$$F(g)=\frac{2\sqrt{2}}{(4g)^{3/2}}\left\{\int_0^1 dt\, e^{-\frac{t}{4g}}\sqrt{1-\sqrt{1-t}}+\int_1^\infty dt\, e^{-\frac{t}{4g}}\sqrt{\frac{\sqrt{t}+1}{2}}\right\}. \tag{29}$$

As far as by changing of variables the integral (13) can be reduced to the form (29) the obtained formula correctly reproduces the sought function.

Example 5

The problem of physical interest is the evaluation of the ground-state energy $E_0(g)$ of the one-dimensional double-well anharmonic oscillator $V(x)=\frac{x^2}{2}(1-\sqrt{g}x)^2$. $E_0(g)$ is expanded into the nonalternating asymptotic series. This problem, on the one hand, can be solved by standard quantum-mechanical tools with the use of the Schrödinger equation, and on the other hand, it can be formulated through the functional integral. The calculation of the latter by the steepest descent method gives the asymptotic behaviour of high-order coefficients of PT

$$E_0(g)\sim\sum_{k=0}^{\infty} g^k E_k,$$

where [18]

$$E_0=\frac{1}{2},\ E_1=-1\ ,\ E_2=-4{,}5\ ,\ E_3=-44{,}5\ ,\ E_4=-626{,}625,\ \text{etc.},$$

and as $k\to\infty$

$$E_K \xrightarrow[K\to\infty]{} -k!\,3^k\frac{3}{\pi}\ .$$

It follows that a=3, b=0. Using the same criterium to determine λ, we get $\lambda\simeq 0{,}30\pm0{,}04$ with the help of four terms of PT. This is in agreement with the value of $\lambda=1/3$ following from the dimensional counting.

Substituting these values of parameters into (22), we obtain

$$E_0(g)=\frac{1}{(3g)^{\mu}}\Big\{\int_0^1 dt\,e^{-\frac{t}{3g}}t^{\mu-1+1/3}3^{-1/3}\sum_{K=0}^{N}C_K W^{K-1/3}+ \tag{30}$$
$$+\int_1^\infty dt\,e^{-\frac{t}{3g}}t^{\mu-1+1/3}3^{-1/3}\sum_{K=0}^{N}C_K\rho^{K-1/3}\frac{\sin[\pi(\mu-1)+\theta(K-1/3)]}{\sin\pi(\mu-1)}\Big\},$$

where $\alpha=\{\mu-2\}$ and coefficients C_K are given by (23). To fix the parameter μ we consider the asymptotics of $E_0(g)$ as $g\to\infty$. We have

$$E_0(g)\xrightarrow[g\to\infty]{} g^{1/3}\Gamma(\mu+1/3)\frac{1+\sqrt{3}\,\mathrm{ctg}\,\pi(\mu-1)}{2}\sum_{K=0}^{N}(-)^K C_K\ . \tag{31}$$

On the other hand, it is known [19] that

$$E_0(g)\xrightarrow[g\to\infty]{} g^{1/3}/2^{1/3}\cdot 0{,}667986259\ . \tag{32}$$

This enables us to determine μ from the correspondence between (31) and (32). As is supposed, the product $\Gamma(\mu+1/3)\sum(-)^K C_K$ in (31) depends on μ very weakly and all the dependence is governed by $\mathrm{ctg}\,\pi(\mu-1)$. This means that only the fractional part of μ is significant, with an arbitrary integer part. We present the values of μ obtained from the correspondence of (31) and (32)

for $[\mu]$ =4 and 6, when several terms of PT in (31) are taken into account:

	N=2	N=3	N=4
$[\mu]$	4	4	4
$\{\mu\}$	0.404219	0.420412	0.433096
$[\mu]$	6	6	6
$\{\mu\}$	0.409556	0.418626	0.421308

Beginning from N=4 the obtained values of $\{\mu\}$ are practically independent of $[\mu]$: (N=4)

$[\mu]$	4	5	6	15	20	30
$\{\mu\}$	0.433096	0.427516	0.421308	0.426332	0.426314	0.414163

Having fixed the value of μ we can construct the function $E_0(g)$ by eq.(30). The graphs of $E_0(g)$ for N=2,3 and 4 and also the graph of the function, obtained by the numerical solution of the Schrödinger equation, are plotted in fig.3.

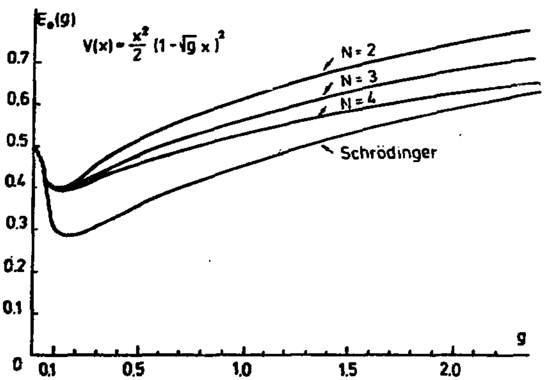

Fig.3

We see that the curves obtained are rather close to the sought function and become still closer with increasing number of the PT terms. The rate of convergence of the proposed procedure is defined by the rate of decreasing the coefficients C_k which behave like $1/k^{\mu-b-1}$ at large k.

6. Conclusion

Thus, the proposed method enables us to reconstruct the function starting from its nonalternating asymptotic series. Rather good accuracy can be reached with a relatively small number of the PT terms. This is of special importance in field theoretical applications where PT calculations are extremely difficult.

An interesting feature of eq.(24) is that for integer λ the dependence on μ practically disappears. This enables us to find the behaviour of the function as $g \to \infty$ independently of the value of μ. This may also be very useful in field-theoretical problems.

The application of the proposed method to the Yang-Mills theory for the extrapolation of the Gell-Mann-Low function $\beta(g)$ into the region of large g is the content of the nearest investigations.

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