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THE ALTARELLI-PARISI EQUATIONS AS RENORMALIZATION GROUP EQUATIONS FOR THE COEFFICIENTS OF THE NONLOCAL LIGHT-CONE EXPANSION

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1. INTRODUCTION

The local light-cone expansion is an important tool for the treatment of deep inelastic scattering. For scalar theories it has been shown that the so-called nonlocal light-cone expansion [1] exists as an operator identity, whereas the traditional local light-cone expansion [2] exists on a dense subset of the Fock space only [3]. So it depends strongly on the properties of the target bound states if such expansions make sense. Taking into account that the nonlocal light-cone expansions can be applied in all cases, it is interesting to study the relations between both expansions. This gives an independent justification of the application of the local light-cone expansions in deep inelastic scattering finally. For practical applications both types of expansions are needed for forward scattering only. This restriction gives essential simplifications concerning all quantities as light-cone operators, light-cone coefficients, anomalous dimensions, and renormalization group equations. For example the light cone coefficients seen in forward scattering are linear combinations of the complete set of the light-cone coefficients. It is pointed out that the effective renormalization group equation for the nonlocal light-cone coefficients is equivalent to the Altarelli-Parisi equation. It is interesting that this equation was originally derived [4] on the basis of physical assumptions and ideas whereas the important non-local light-cone expansion was constructed later [1]. This work uses the results of the investigations [1,3,5] on light-cone expansions. All notation and convention used there and in the paper [6] are applied. For simplicity all calculations are done in scalar field theory. We claim that the essential features seen here turn over to gauge theories.

2. RELATIONS BETWEEN THE NONLOCAL AND LOCAL LIGHT-CONE EXPANSIONS

The basic quantity investigated in light-cone expansions is the renormalized product of two current operators $R(j(x)j(0)\,E_0(s))$ whereby $E_0(s) = \exp \int \mathcal{L}_{int}\,dx$ (The T-product symbol is omitted always). In perturbation theory it is possible to represent this operator product with the help of coefficient functions F_ℓ and normal products of the free field operator ϕ

$$R(j(x)j(0)E_0(s)) = \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \int dq_1 \ldots dq_\ell\, F_\ell(x^2, xq_i, q_i q_j, \mu^2) : \phi(q_1) \ldots \phi(q_\ell) : . \tag{2.1}$$

With these notation the nonlocal light-cone expansion reads [1]

$$R(j(x)\,j(0)\,E_0(s)) = \sum_{\ell=0}^{a} \frac{1}{\ell!} \sum_{k=0}^{\frac{a-\ell}{2}} \sum_{\substack{(m)\\ |m|\leqslant k}} \sum_{\substack{(r)\\ |r|\leqslant k-|m|}} (x^2)^{k-|m|}\, C_{k-|m|,|r|} \times \tag{2.2}$$

$$\times \int_0^1 d\kappa_1 \ldots \int_0^1 d\kappa_\ell\, F_{\ell(r)(m)}(x^2, \kappa_i, \mu^2)\, \bar{R}(O_{k\ell(r)(m)}(\kappa)E_0(s)) + Q^a ,$$

thereby the coefficient functions $F_{\ell(r)(m)}$ are given by

$$F_{\ell(r)(m)}(x^2, \kappa_i, \mu^2) = \frac{1}{(2\pi)^\ell} \int d\tilde{x}q_1 \ldots d\tilde{x}q_\ell\, e^{-i\sum \kappa_s \tilde{x} q_s}\, G_{\ell(r)(m)}(x^2, \tilde{x}q_i, \mu^2), \tag{2.3}$$

$$G_{\ell(r)(m)}(x^2, xq_i, \mu^2) = \prod_{s=1}^{\ell} \left(\frac{\partial}{\partial \tilde{x} q_s}\right)^{r_s} \prod_{\substack{s\leq t\\ s,t=1}}^{\ell} \left(\frac{\partial}{\partial q_s q_t}\right)^{m_{st}} F_\ell^{x\text{-prop}}(x^2, \tilde{x}q_i, q_i q_j, \mu^2)\Big|_{q_i q_j = \mu_{ij}}$$

and the nonlocal light-ray operators are determined by

$$O_{k\ell(r)(m)}(\kappa_i) = \int dq_1 \ldots dq_\ell \left(\frac{\sqrt{\eta^2}}{\tilde{x}\eta}\right)^{2(k-|m|)-|r|} \prod_{s=1}^{\ell} e^{i\tilde{x}q_s\kappa_s} \left(\frac{\eta}{\sqrt{\eta^2}} q_s\right)^{r_s} \cdot \frac{1}{r_s!} \times \tag{2.4}$$

$$\times \prod_{\substack{s\leq t\\ s,t=1}}^{\ell} \frac{(q_s q_t - \mu_{st})^{m_{st}}}{m_{st}!} : \phi(q_1) \ldots \phi(q_\ell) : .$$

The indices $\ell, k, \ldots$ or the multi-indices $(m), (r), (n)$ are nonnegative integers, $|m| = \Sigma m_i$, the coefficients $C_{k-|m|,|r|}$ are constants defined in [1], η denotes a constant vector $\eta^2 \neq 0$ and $\tilde{x} = \frac{1}{\eta^2}(x\eta^2 - \eta(x\eta)) + \frac{\eta}{\eta^2}((x\eta)^2 - x^2\eta^2)^{1/2}$ is a light-like vector corresponding to x. $\bar{R}$ denotes a modified R operation [1,3] whereby the graphs or subgraphs containing the operator vertex O are subtracted with the help of a special subtraction operator $\mathfrak{M}^a$ [1,5]

$$\mathfrak{M}^a F(x|\phi) = \sum_{\ell=0}^{a} \frac{1}{\ell!} M_\sigma^{a-\ell} \int dq_1 \ldots dq_\ell F_\ell^{x-\mathrm{prop}}(x^2, x_\sigma q_i, \sigma^2(q_i q_j - \mu_{ij}) + \mu_{ij}, \eta\sigma q, \eta^2) \times$$

$$\times : \phi(q_1) \ldots \phi(q_\ell) : ,$$

$$x_\sigma = \tilde{x} + \eta \frac{\tilde{x}\eta}{\eta^2}\left(\left(1 + \frac{\eta^2 x^2 \sigma^2}{(\tilde{x}\eta)^2}\right)^{1/2} - 1\right), \tag{2.4'}$$

$$M_\sigma^d f(\sigma) = \sum_{k=0}^{d} \frac{1}{k!} \left(\frac{\partial}{\partial\sigma}\right)^k f(\sigma)\Big|_{\sigma=0} ,$$

$\mu_{ij} = \mu^2(1 - (\ell + 1)\delta_{ij})(-\ell)^{-1}$ denotes subtraction points. The positive integer a can be chosen arbitrarily. It determines the smallness of the remainder Q^a near $x^2 = 0$ $(Q^a \sim (x^2)^{a-2d_j-1})$. All formulas correspond to scalar ϕ^4-theory, nearly the same formulae are true for $\phi^3_{(6)}$-theory [8]. The corresponding expressions for the local light-cone expansion are

$$R(j(x)j(0)E_0(s)) = \sum_{\ell=0}^{a} \frac{1}{\ell!} \sum_{(n)}^{\infty} \sum_{k=0}^{\frac{a-\ell}{2}} \sum_{\substack{(m) \\ |m| \le k}} \sum_{\substack{(r) \\ |r| \le k-|m|}} (x^2)^{k-|m|} C_{k-|m|,|r|} \times \tag{2.5}$$

$$\times F_{\ell(n)(r)(m)}(x^2, \mu^2) \bar{R}(O_{k\ell(n)(r)(m)} E_0(s)) + Q^a ,$$

$$F_{\ell(n)(r)(m)}(x^2, \mu^2) = \prod_{s=1}^{\ell} \left(\frac{\partial}{\partial x q_s}\right)^{r_s + n_s} \prod_{\substack{s,t=1 \\ s \le t}}^{\ell} \left(\frac{\partial}{\partial q_s q_t}\right)^{m_{st}} F_{\ell(x^2, xq_i, q_i q_j, \mu^2)}^{x-\mathrm{prop}}\Big|_{\substack{xq_i = 0 \\ q_i q_j = \mu_{ij}}} , \tag{2.6}$$

$$O_{k\ell(n)(r)(m)} = \int dq_1 \ldots dq_\ell \prod_{s=1}^{\ell} \frac{(\tilde{x}q_s)^{n_s}}{n!} \left(\frac{\eta}{\sqrt{\eta^2}} q_s\right)^{r_s} \frac{1}{r_s!} \left(\frac{\sqrt{\eta^2}}{\tilde{x}\eta}\right)^{2(k-|m|)-|r|} \times$$

$$\times \prod_{\substack{s,t=1 \\ s\leq t}}^{\ell} \frac{(q_s q_t - \mu_{st})^{m_{st}}}{m_{st}!} :\phi(q_1) \ldots \phi(q_s): . \qquad (2.7)$$

From these formulae there follow at once relations between local and nonlocal quantities.

Operators:

$$O_{k\ell(n)(r)(m)} = \prod_{s=1}^{\ell} \frac{1}{n_s!} \left(\frac{\partial}{\partial i\kappa_s}\right)^{n_s} O_{k\ell(r)(m)}(\kappa_i)|_{\kappa_i=0} , \qquad (2.8)$$

$$O_{k\ell(r)(m)}(\kappa_i) = \sum_{(n)} (i\kappa_1)^{n_1} \ldots (i\kappa_\ell)^{n_\ell} O_{k\ell(n)(r)(m)} . \qquad (2.9)$$

Coefficients functions:

$$F_{\ell(n)(r)(m)}(x^2) = \int_C^1 d\kappa_1 \ldots d\kappa_\ell (i\kappa_1)^{n_1} \ldots (i\kappa_\ell)^{n_\ell} F_{\ell(r)(m)}(\kappa_i, x^2) , \qquad (2.10)$$

$$F_{\ell(r)(m)}(\kappa_i, x^2) = $$
$$= \frac{1}{(2\pi i)^\ell} \int_{c-i\infty}^{c+i\infty} dn_1 \ldots dn_\ell \, i^{-\Sigma n_i} \kappa_1^{-n_1-1} \ldots \kappa_\ell^{-n_\ell-1} F_{\ell(n)(r)(m)}(x^2) . \qquad (2.11)$$

The exact renormalization group equations are identical for both expansions, they follow directly from the renormalization group equations for the x-proper coefficient functions $F_\ell^{x\text{-prop}}$ /1,8/

$$\left(\mu\frac{\partial}{\partial\mu} + \delta\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial g} - \ell\gamma_g + 2\gamma_j\right) F_{\ell(r)(m)}(x^2, \kappa_i) = 0 , \qquad (2.12)$$

$$\left(\mu\frac{\partial}{\partial\mu} + \delta\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial g} - \ell\gamma_g + 2\gamma_j\right) F_{\ell(n)(r)(m)}(x^2) = 0 . \qquad (2.13)$$

On the other hand the effective renormalization group equations are different. They correspond to the traditional renormalization group equations /7/ for the light-cone coefficients which are derived indirectly taking into account renormalization group equations for the light-cone operators $O_{k\ell(r)(m)}$ or $O_{k\ell(n)(r)(m)}$ /5/

$$\sum_{A'} \int_0^1 d\kappa'_1 \dots d\kappa'_\ell \{[\mu\frac{\partial}{\partial\mu} + \delta\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial g} + 2\gamma_j]\delta_{AA'} \prod_i \delta(\kappa_i - \kappa'_i) - \\ - \gamma_{A'A}(\kappa'_i, \kappa_i)\} F_{A'}(x^2, \kappa'_i) = 0 , \tag{2.14}$$

$$\sum_{A'(n')} \{[\mu\frac{\partial}{\partial\mu} + \delta\frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial g} + 2\gamma_j]\delta_{AA'}\delta_{(n)(n')} - \\ - \gamma_{(n')A'(n)A}\} F_{(n')A'}(x^2) = 0 , \qquad A = (r)(m)k,\ell . \tag{2.15}$$

Both equations can be connected by a Mellin transform. Omitting unessential details we have to transform an equation of the type $\sum_n F_n O_{nm} = 0$ into an equation of the type $\int_0^1 d\kappa F(\kappa) D(\kappa,\lambda) = 0$, whereby F_n is related to $F(\kappa)$ by eqs. (2.10), (2.11) $F_n = i^n \int_0^1 d\kappa \kappa^n F(\kappa)$, , $F(\kappa) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} dn (i\kappa)^{-n-1} F_n$. As connection between D_{nm} and $O(\kappa,\lambda)$ we receive

$$D(\kappa,\lambda) = \sum_n (i\kappa)^n \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} dm (i\lambda)^{-m-1} D_{nm} , \tag{2.16}$$

$$D_{nm} = \frac{1}{n!}\left(\frac{\partial}{\partial i\kappa}\right)^n \int_0^1 d\lambda (i\lambda)^m D(\kappa,\lambda)\Big|_{\kappa=0} . \tag{2.17}$$

Therefore the anomalous dimensions $\gamma_{(n')A'(n)A}$ and $\gamma_{A'A}(\kappa,\kappa')$ have to satisfy the relations

$$\gamma_{A'A}(\kappa',\kappa) = \sum_{(n')} \prod_{j=1}^{\ell} (i\kappa'_j)^{n'_j} \frac{1}{(2\pi)^\ell} \int_{c-i\infty}^{c+i\infty} \prod_{s=1}^{\ell} dn_s (i\kappa_s)^{-n_s-1} \gamma_{(n')A'(n)A} \tag{2.18}$$

$$\gamma_{(n')A'(n,A)} = \prod_{j=1}^{\ell} \frac{1}{n'_j!}\left(\frac{\partial}{\partial i\kappa'_j}\right)^{n'_j} \int_0^1 \prod_{s=1}^{\ell} d\kappa_s (i\kappa_s)^{n_s} \gamma_{A'A}(\kappa',\kappa). \tag{2.19}$$

To check these relations the explicit definition of the anomalous dimensions is needed. The anomalous dimensions are defined as derivatives of Z-factors whereby in our case the explicit μ^2-dependence of the operators O has to be taken into account. To get more explicit expressions we start with the renormalization condition for the operators O contained in the definition of the subtraction operator $\mathfrak{M}^a$ /6,8/

$$\hat{Z}\ \hat{O}(x, Z_2^{-1/2}\phi) = \mathfrak{M}^a \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int dq_1 \ldots dq_\ell H_\ell^{un\,1PI}(x, q_1, \ldots, q_\ell|\hat{O}) : \phi(q_1)\ldots\phi(q_\ell): \tag{2.20}$$

To simplify the notation, the complete set of operators $\hat{O}$ is ordered to a row and $\hat{Z}$ acts as a matrix in this space, H_ℓ^{1PI} and $H_\ell^{un\,1PI}$ are the 1PI-parts of the renormalized and unrenormalized coefficient functions of the operators $\hat{O}$

$$\bar{R}(\hat{O}E_0(s)) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int dq_1 \ldots dq_\ell H_\ell(x, q_1, \ldots, q_\ell|\hat{O}) : \phi(q_1)\ldots\phi(q_\ell):$$

$$\hat{O}E_0(s) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int dq_1 \ldots dq_\ell H_\ell^{un}(x, q_1, \ldots, q_\ell|\hat{O}) : \phi(q_1)\ldots\phi(q_\ell): \tag{2.21}$$

$$H_\ell^{1PI}(x, q_1, \ldots, q_\ell|\hat{O}) = \hat{Z}^{-1} Z_2^{\ell/2} H_\ell^{un\,1PI}(x, q_1, \ldots, q_\ell|\hat{O}). \tag{2.22}$$

Differentiation of eq. (2.20) with respect to μ^2 gives

$$\hat{Z}\left(\hat{Z}^{-1}\mu^2\frac{\partial}{\partial\mu^2}\hat{Z} + \left(\mu^2\frac{\partial}{\partial\mu^2}\right)_{expl.} + Z_2^{1/2}\mu^2\frac{\partial}{\partial\mu^2}Z_2^{-1/2}\frac{\partial}{\partial a}\right)\hat{O}(x|\phi Z_2^{-1/2}a)\Big|_{a=1} =$$

$$= \left[\left(\mu^2\frac{\partial}{\partial\mu^2}\mathfrak{M}^a\right) + \mathfrak{M}^a\left(\mu^2\frac{\partial}{\partial\mu^2}\right)_{expl}\right] \times$$

$$\times \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int dq_1 \ldots dq_\ell H_\ell^{un}(x, q_1, \ldots, q_\ell|\hat{O}) : \phi(q_1)\ldots\phi(q_\ell): \ .$$

The derivative $\mu^2\frac{\partial}{\partial\mu^2}$ of $\left(\mu^2\frac{\partial}{\partial\mu^2}\mathfrak{M}^a\right)$ acts on the μ^2-dependence introduced by $\mathfrak{M}^a$ itself only. Using again eq. (2.20) and the definitions

$$\frac{\partial}{\partial a}\hat{O}(x|\phi\cdot a)\Big|_{a=1} = \hat{n}_L \hat{O}(x|\phi) \qquad \gamma_2 = Z_2^{-1}\mu^2\frac{\partial}{\partial\mu^2}Z_2$$

$$\hat{\gamma} = \hat{Z}^{-1}2\mu^2\frac{\partial}{\partial\mu^2}\hat{Z} - \hat{Z}^{-1}\hat{A}\hat{Z}\ , \qquad 2\mu^2\frac{\partial}{\partial\mu^2}\hat{O} = \hat{A}\hat{O} \tag{2.23}$$

we obtain

$$\hat{\gamma}\hat{O} = -\gamma_2 \hat{n}_L \hat{O} - 2\mu^2 \frac{\partial}{\partial\mu^2}\hat{O} + (2\mu^2 \frac{\partial}{\partial\mu^2}\mathfrak{M}^a)\bar{R}(\hat{O}E_0(s)). \quad (2.24)$$

This relation can be written for the local (discrete) and nonlocal (continuous) case explicitly

$$\sum_{A'(n')} \gamma_{A(n)A'(n')} O_{A'(n')} = -\gamma_2 \sum_{A'(n')} n_{A(n)A'(n')} O_{A'(n')} - 2\mu^2 \frac{\partial}{\partial\mu^2} O_{A(n)} + (2\mu^2 \frac{\partial}{\partial\mu^2}\mathfrak{M}^a)\bar{R}(\hat{O}E_0(s))_{A(n)}, \quad (2.25)$$

$$\sum_{A'} \int d\kappa'_1 \dots d\kappa'_t \, \gamma_{AA'}(\kappa,\kappa')O(\kappa') = -\gamma_2 \sum_{A'} \int d\kappa'_1 \dots d\kappa'_t \, n_{AA'}(\kappa,\kappa')O_{A'}(\kappa') - 2\mu^2 \frac{\partial}{\partial\mu^2} O_A(\kappa) + (2\mu^2 \frac{\partial}{\partial\mu^2}\mathfrak{M}^a)\bar{R}(\hat{O}E_0(s))_A(\kappa). \quad (2.26)$$

Quite similar calculations leading to eqs. (2.16), (2.17) show that both expressions (2.24), (2.25) are connected by eqs. (2.18), (2.19). Thereby it must be taken into account that $\bar{R}(\hat{O}E_0(s))_{A(n)}$ and $\bar{R}(\hat{O}E_0(s))_A(\kappa)$ satisfy eqs. (2.8), (2.9) too. This is guaranteed because of the property of the applied transformations.

3. MINIMAL LIGHT-CONE EXPANSION IN THE CASE OF FORWARD SCATTERING

For the first investigations of deep inelastic scattering the most important operators of the light-cone expansion have been taken into account only /7/. The minimal non-trivial choice of the light-cone expansion for the ϕ^4 theory corresponds to $a = 2$. In this case we have

Local case:

$$R(j(x)\,j(0)\,E_0(s)) \underset{x^2\to 0}{\approx} \sum_{n_1,n_2} F_{n_1 n_2}(x^2,\mu^2)\,\bar{R}(O_{n_1 n_2}\,E_0(s)). \quad (3.1)$$

$$O_{n_1 n_2} = \int dq_1 dq_2 \frac{(\tilde{x}q_1)^{n_1}}{n_1!} \frac{(\tilde{x}q_2)^{n_2}}{n_2!} : \phi(q_1) \quad \phi(q_2) : \tag{3.2}$$

$$F_{n_1 n_2} = \frac{1}{2!} \left(\frac{\partial}{\partial xq_1}\right)^{n_1} \left(\frac{\partial}{\partial xq_2}\right)^{n_2} F_2^{x-prop}(x^2, xq_i, q_i q_j) \Big|_{\substack{xq_i = 0 \\ q_i q_j = \mu_{ij}}} \tag{3.3}$$

Nonlocal case:

$$R(j(x)\, j(0)\, E_0(s)) \underset{x^2 \to 0}{\approx} \int_0^1 d\kappa_1 \int_0^1 d\kappa_2 F(x^2, \kappa_1, \kappa_2) \bar{R}(O(\kappa_1, \kappa_2) E_0(s)), \tag{3.4}$$

$$O(\kappa_1, \kappa_2) = \int dq_1 dq_2\, e^{i\kappa_1 \tilde{x} q_1 + i\kappa_2 \tilde{x} q_2} : \phi(q_1)\, \phi(q_2) : , \tag{3.5}$$

$$F(x^2, \kappa_1, \kappa_2) = \frac{1}{8\pi^2} \int d\tilde{x}q_1 d\tilde{x}q_2\, e^{-(i\kappa_1 \tilde{x}q_1 + i\kappa_2 \tilde{x}q_2)} \times \tag{3.6}$$

$$\times F_2^{x-prop}(x^2, \tilde{x}q_1, \tilde{x}q_2, q_i q_j) \Big|_{q_i q_j = \mu_{ij}} .$$

There appear only terms which contain at most two field operators. Complicated $q_i q_j$ and η-dependence has been dropped out.

At this place we have to add some general remarks. The crucial point for the proof of the light-cone expansion is the choice of the subtraction operators $\mathfrak{M}$ which enable the construction of the light-cone expansion and allow a convergence proof. The first subtraction operators of this type have been given by S.A.Anikin and O.I.Zavialov [1]. Slightly changed operators have been considered later [5]. It is very important that the choice of a subtraction operator leads to a special form of the light-cone expansion and determines finally the renormalization procedure for the light-cone operators. Up to now we have chosen the subtraction operator (2.4'). With regard to the application of the light-cone expansion to forward scattering processes we choose as subtraction operator now (2.4') with $a = 2$ and different subtraction points, however,

$$\ell = 1 \quad q^2 = 0, \quad \ell = 2 \quad q_1^2 = q_2^2 = \mu^2 < 0, \quad q_1 q_2 = -\mu^2. \tag{3.7}$$

This leads to a similar minimal light-cone expansion. The difference to the expressions (3.1)-(3.6) consists in another definition of the light-cone coefficients

$$F_{n_1 n_2}(x^2) = \frac{1}{2!}\left(\frac{\partial}{\partial x q_1}\right)^{n_1}\left(\frac{\partial}{\partial x q_2}\right)^{n_2} F_2^{x\text{-prop}}(x^2, xq_i, q_i q_j)\Big|_{\substack{xq_1 = xq_2 = 0 \\ q_1^2 = q_2^2 = \mu^2 \\ q_1 q_2 = -\mu^2}} , \tag{3.8}$$

$$F(x^2, \kappa_1, \kappa_2) = \frac{1}{8\pi^2}\int d\tilde{x}q_1 d\tilde{x}q_2 e^{-(i\kappa_1 \tilde{x}q_1 + i\kappa_2 \tilde{x}q_2)} \times \\ \times F_2^{x\text{-prop}}(x^2, \tilde{x}q_i, q_i q_j)\Big|_{q_1^2 = q_2^2 = -q_1 q_2 = \mu^2} . \tag{3.9}$$

Let us turn to the case of forward scattering. Then the matrix elements $\langle p|Rj(x)\,j(0)\,E_0(s)|p\rangle$ of eqs. (3.1) and (3.4) have to be formed.

We consider the nonlocal light-cone expansion at first. If we look at eq. (3.4) then it is obvious that the matrix elements $\langle p|\bar{R}O(\kappa_i)E_0(s)|p\rangle$ have to be investigated. For free field theory such matrix elements

$$\langle p|O(\kappa_1, \kappa_2)|p\rangle = \int dq_1 dq_2 e^{(i\kappa_+(q_1+q_2)\frac{1}{2} + i\kappa_-(q_1-q_2)\frac{1}{2})\tilde{x}} \langle p|:\phi(q_1)\phi(q_2):|p\rangle$$

$$\kappa_+ = \kappa_1 + \kappa_2 \qquad \kappa_- = \kappa_1 - \kappa_2$$

do not depend on the variable κ_+ because of momentum conservation $q_1 + q_2 = 0$. This result remains true for interacting fields also. For this reason the coefficient functions of the renormalized operator

$$\bar{R}(O(\kappa_1, \kappa_2)E_0(s)) = \sum_{s=0}^{\infty} \frac{1}{s!}\int dq_1 \dots dq_s H_s(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_1, \dots, q_s) :\phi(q_1)\dots\phi(q_s): \tag{3.10}$$

must be studied for $\Sigma q_i = 0$. These functions are Green functions amputated with bare propagators at the external legs $H(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_1, \dots, q_s) = \langle \tilde{\phi}(\tilde{x}\kappa_1)\tilde{\phi}(\tilde{x}\kappa_2)\phi(q_1)\dots\phi(q_s)\rangle^{q\text{-amp}}$.
Translation invariance of these functions reads
$H_s(\tilde{x}\kappa_1 + y, \tilde{x}\kappa_2 + y, q_1, \dots, q_s) = H(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_1, \dots, q_s)\exp(-iy\Sigma q_i)$.
Setting $y = -\frac{1}{2}\tilde{x}(\kappa_1 + \kappa_2)$ we get

$$H_s(\tfrac{1}{2}\tilde{x}(\kappa_1 - \kappa_2), \tfrac{1}{2}\tilde{x}(\kappa_2 - \kappa_1), q_1, \dots, q_s) = H_s(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_1, \dots, q_s)e^{i\frac{\kappa_1+\kappa_2}{2}\tilde{x}\Sigma q_i} .$$

Here it is seen, that for $\Sigma q_i = 0$ the functions H_s do not depend on κ_+. This result can be used to simplify the non-local light-cone expansion for forward scattering. Performing the κ_+ integration in eq. (3.4) and denoting

$$C(x^2, \kappa_-) = 2\int d\kappa_+ \, F(x^2, \kappa_1, \kappa_2) \tag{3.11}$$

as new light-cone coefficient for forward scattering, then the expansion (3.4) takes the simple form

$$<p|Rj(x)j(0)E_0(s)|p> = \int d\kappa_- \, C(x^2, \kappa_-) <p|\bar{R}O(\kappa_1, \kappa_2)E_0(s)|p>. \tag{3.12}$$

It is remarkable that we are left with one integration parameter only.

For the local light-cone expansion similar statements are true. Quite analogously the forward matrix elements of the free field operator

$$(-1)^{n_2} \binom{n_1+n_2}{n_1}^{-1} <p|O_{n_1 n_2}|p> = (-1)^{n_2} \int dq_1 dq_2 \frac{(\tilde{x}q_1)^{n_1}(\tilde{x}q_2)^{n_2}}{(n_1+n_2)!} <p|:\phi(q_1)\phi(q_2):|p>$$

$$= \int dq_1 dq_2 \frac{((q_1-q_2)\tilde{x})^{n_1+n_2}}{(n_1+n_2)!} <p|:\phi(q_1)\phi(q_2):|p> \cdot 2^{-(n_1+n_2)}$$

depend on n_1+n_2 only because of $q_1+q_2=0$. In general the coefficient functions of the renormalized operator

$$\bar{R}(O_{n_1 n_2} E_0(s)) = \sum_{s=0}^{\infty} \frac{1}{s!} \int dq_1 \ldots dq_s H_{s\, n_1 n_2}(\tilde{x}, q_1, \ldots, q_s) :\phi(q_1)\ldots\phi(q_s): \tag{3.13}$$

multiplied by $(-1)^{n_2} \binom{n_1+n_2}{n_1}^{-1}$ depend only on n_1+n_2 for $\Sigma q_i = 0$. This can be seen from the connections between local and nonlocal. light-cone expansions. From eq. (2.8) follows the relation for the coefficient functions

$$H_{s\, n_1 n_2}(\tilde{x}, q_i) = \frac{1}{n_1!}\left(\frac{\partial}{\partial i\kappa_1}\right)^{n_1} \frac{1}{n_2!}\left(\frac{\partial}{\partial i\kappa_2}\right)^{n_2} H_s(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_i)\Big|_{\kappa_i=0}. \tag{3.14}$$

Because of the κ_+ independence of the nonlocal coefficient functions at $\Sigma q_i = 0$ we have $\frac{\partial}{\partial i\kappa_1} = -\frac{\partial}{\partial i\kappa_2} = \frac{\partial}{\partial i\kappa_-}$ so that

$$(-1)^{n_2}\binom{n_1+n_2}{n_1}^{-1} H_{s\, n_1 n_2}(\tilde{x}, q_i)\Big|_{\Sigma q_i=0} = \frac{1}{(n_1+n_2)!}\left(\frac{\partial}{\partial i\kappa_-}\right)^{n_1+n_2} H_s(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_i)\Big|_{\Sigma q_i=0,\kappa_j=0}.$$

This allows us to rewrite the local light-cone expansion (3.1) for the case of forward scattering in the form

$$\langle p|\, R j(x)\, j(0)\, E_0(s)|p\rangle \approx \sum_{N=0}^{\infty} C_N(x^2)\{(-1)^{n_2}\binom{n_1+n_2}{n_1}^{-1} \langle p|\bar{R}\, O_{n_1 n_2} E_0(s)|p\rangle\}\delta_{n_1+n_2,N}. \tag{3.15}$$

The new coefficient functions are

$$C_N(x^2) = \sum_{n_1=0}^{N} (-1)^{n_2}\binom{n_1+n_2}{n_1} F_{n_1 n_2}(x^2)\, \delta_{n_1+n_2,N}. \tag{3.16}$$

Again we are left with one summation index only as it is known from the application in deep inelastic scattering.

4. RENORMALIZATION GROUP EQUATIONS FOR FORWARD SCATTERING

As we have seen the restriction to forward scattering leads to a lot of simplifications. The light-cone expansion has only one summation (integration) index, the light-cone coefficients are sums (integrals) over the original quantities, the matrix elements of series of operators are identical (independence of certain variables). Such simplifications have been used from the very beginning by the treatment of deep inelastic scattering [7]. For example in the light-cone expansion there are used the operators $\bar{\psi}\gamma_{\mu_1}\vec{D}_{\mu_2}\vec{D}_{\mu_3}\psi$ but the operators $\bar{\psi}\gamma_{\mu_1}\overleftarrow{D}_{\mu_2}\vec{D}_{\mu_3}\psi, \ldots, \bar{\psi}\gamma_{\mu_1}\overleftarrow{D}_{\mu_2}\overleftarrow{D}_{\mu_3}\psi$ need not be considered separately. However, for non-forward scattering also these operators have to be taken into account and the renormalization procedure becomes more complicated. Because of the close connection between renormalization procedure and anomalous dimensions this is also true for the anomalous dimensions.

If we confine ourselves strictly to forward scattering, i.e., all operators O have no momentum flow through the operator vertex, then instead of the operators $O(\kappa_1,\kappa_2)$ and $O(n_1,n_2)$ it is possible to use

$$O(\kappa_-) = \int dq_1\, dq_2\, e^{+\frac{i}{2}\kappa_-(q_1-q_2)\tilde{x}} :\phi(q_1)\,\phi(q_2):\delta(q_1+q_2)\,,$$

$$O_N = \int dq_1 dq_2 \frac{((q_1 - q_2)\tilde{x})^N}{N!\, \lambda^N} : \phi(q_1)\phi(q_2) : \delta(q_1 + q_2) \tag{4.1}$$

from the beginning. Performing the complete renormalization procedure under this restriction then quite similar calculations (to eqs.(2.20)-(2.26)) lead to the anomalous dimensions seen in forward scattering

$$\int d\kappa_- \tilde{\gamma}(\kappa_-, \kappa_-', O(\kappa_-')) = -2\gamma_2 O(\kappa_-) + (2\mu^2 \frac{\partial}{\partial \mu^2} \mathfrak{M}^2) \bar{R}(O E_0(s))(\kappa_-), \tag{4.2}$$

$$\sum_{N'} \tilde{\gamma}_{NN'} O_{N'} = -2\gamma_2 O_N + (2\mu^2 \frac{\partial}{\partial \mu^2} \mathfrak{M}^2) \bar{R}(O E_0(s))_N . \tag{4.3}$$

In general the anomalous dimensions for the minimal light-cone expansion are defined by the equations

$$\int d\kappa_1' d\kappa_2' \gamma(\kappa, \kappa') O(\kappa_1', \kappa_2') = -2\gamma_2 O(\kappa_1, \kappa_2) + (2\mu^2 \frac{\partial}{\partial \mu^2} \mathfrak{M}^2) \bar{R}(O E_0(s))(\kappa_1, \kappa_2), \tag{4.4}$$

$$\sum_{n_1' n_2'} \gamma_{n_1 n_2 n_1' n_2'} O_{n_1' n_2'} = -2\gamma_2 O_{n_1 n_2} + (2\mu^2 \frac{\partial}{\partial \mu^2} \mathfrak{M}^2) \bar{R}(O E_0(s))_{n_1 n_2} . \tag{4.5}$$

In an Appendix we will show that these anomalous dimensions are simply connected with the anomalous dimensions seen in forward scattering

$$\binom{N}{n_1}^{-1} (-1)^{N-n_1} \sum_{n_1'=0}^{N'} (-1)^{n_1' - N'} \binom{N'}{n_1'} \gamma_{(n_1, N-n_1),(n_1', N'-n_1')} = \tilde{\gamma}_N \delta_{NN'}, \tag{4.6}$$

$$\int d\kappa_+' \gamma(\kappa_1, \kappa_2, \kappa_1', \kappa_2') = 2\tilde{\gamma}(\kappa_-, \kappa_-'). \tag{4.7}$$

These relations allow important simplifications of the renormalization group equations.

Let us consider the nonlocal light-cone expansion at first. The renormalization group equation (2.14) reads

$$\int_0^1 d\kappa_1 d\kappa_2 \{ \mu \frac{\partial}{\partial \mu} + \delta \frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial g} + 2\gamma_j] \delta(\kappa_1 - \kappa_1') \delta(\kappa_2 - \kappa_2') - \\ -\gamma(\kappa, \kappa') \} F(x^2, \kappa_1, \kappa_2) = 0 \tag{4.8}$$

for the minimal light cone expansion. Because of eq.(4.7) it is possible to integrate this equation over κ'_+ for forward scattering

$$\int d\kappa_- \{[\mu\frac{\partial}{\partial\mu}+\delta\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}+2\gamma_j]\delta(\kappa_- -\kappa'_-) - \tilde{\gamma}(\kappa_-,\kappa'_-)\} C(x^2,\kappa_-) = 0. \quad (4.9)$$

The resulting equation has just the form of the Altarelli-Parisi equation. In other words: the Altarelli-Parisi equation is a renormalization group equation for the light-cone coefficients of the nonlocal light-cone expansion specialized to forward scattering.

For completeness we consider the discrete case too. The complete renormalization group equation (2.15) reads here

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \{(\mu\frac{\partial}{\partial\mu}+\delta\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}+2\gamma_j)\delta_{(n)(n')}-\gamma_{(n)(n')}\} F_{n_1 n_2}(x^2) = 0. \quad (4.10)$$

The restriction to forward scattering allows the summation over n'_1 with the weight $\binom{N'}{n'_1}(-1)^{N'-n'_1}$. Together with eqs.(3.16),(4.6) it is obtained

$$(\mu\frac{\partial}{\partial\mu}+\delta\frac{\partial}{\partial m^2}+\beta\frac{\partial}{\partial g}+2\gamma_j-\tilde{\gamma}_N) C_N(x^2) = 0. \quad (4.11)$$

This is the usually applied renormalization group equation for the light-cone coefficients in the case of forward scattering. Of course it can be checked that both equations (4.9), (4.11) are connected by simple Mellin transforms (see also eq.(a.5)) as it was used by Altarelli and Parisi.

APPENDIX

Relations Between Anomalous Dimensions

At first we will show relations between anomalous dimensions in general and anomalous dimensions seen in the forward scattering. In the nonlocal case the anomalous dimensions are given by eqs.(4.4),(4.2). Using the subtraction operator (2.4),(3.7) explicitly we get

$$\int d\kappa'_1 d\kappa'_2 \gamma(\kappa,\kappa')O(\kappa'_1,\kappa'_2) = -2\gamma_2 O(\kappa_1,\kappa_2) + \quad (a.1)$$
$$+2\mu^2\frac{\partial}{\partial\mu^2}\int dq_1 dq_2 H_2(\tilde{x}\cdot\kappa_1,\tilde{x}\cdot\kappa_2,q_1 q_j)\Big|_{q_1^2=q_2^2=-q_1q_2=\mu^2} :\phi(q_1)\phi(q_2).$$

With the help of the identity

$$H_2(\tilde{x}\cdot\kappa_1, \tilde{x}\cdot\kappa_2, q_i \quad) = \frac{1}{(2\pi)^2}\int dz_1\, dz_2 \int d\kappa_1' d\kappa_2'\, e^{-iz_1\kappa_1' - iz_2\kappa_2'} \quad \times$$

$$\times H_2(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_i \quad)|_{\tilde{x}q_1 = z_1} \quad e^{i\kappa_1'(\tilde{x}q_1) + i\kappa_2'\tilde{x}q_2}$$

the last term of eq.(a.1) takes the form

$$2\mu^2\frac{\partial}{\partial\mu^2}\int dz_1 dz_2 \frac{1}{(2\pi)} H_2(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_i \quad)|_{q_1^2=q_2^2=-q_1q_2=\mu^2,\ \tilde{x}q_i=z_i} \int d\kappa_1' d\kappa_2' e^{-iz_1\kappa_1' - iz_2\kappa_2'} O(\kappa_1',,$$

This leads to an explicit expression for the anomalous dimension

$$\gamma(\kappa,\kappa') = -2\gamma_2\delta(\kappa_1-\kappa_1')\delta(\kappa_2-\kappa_2') + \tag{a.2}$$
$$+ 2\mu^2\frac{\partial}{\partial\mu^2}\int\frac{dz_1 dz_2}{(2\pi)^2} e^{-iz_1\kappa_1' - iz_2\kappa_2'} H_2(\tilde{x}\kappa_1, \tilde{x}\kappa_2, q_i \quad)|_{\tilde{x}q_i=z_i,\ q_1^2=q_2^2=-q_1q_2=\mu^2}.$$

A similar calculation using the operators $O(\kappa_-)$ (4.1) leads to a formula for the anomalous dimensions seen in forward scattering

$$\tilde{\gamma}(\kappa_-,\kappa_-') = -2\gamma_2\delta(\kappa_- - \kappa_-') + \tag{a.3}$$
$$+ 2\mu^2\frac{\partial}{\partial\mu^2}\int\frac{dz}{(2\pi)} e^{-iz\kappa_-} H_2(\kappa_-\frac{\tilde{x}}{2}, -\kappa_-\frac{\tilde{x}}{2}, q_i' \ )|_{q_1^2=q_2^2=-q_1q_2=\mu^2,\ \tilde{x}q_1=z}$$

To show the connection between both equations (a.2),(a.3) we introduce the variables κ_+, κ_- and integrate over κ_+'

$$\int d\kappa_+'\gamma(\kappa,\kappa') = -4\gamma_2\delta(\kappa_- - \kappa_-') + 2\mu^2\frac{\partial}{\partial\mu^2}\int\frac{dz_1dz_2}{(2\pi)^2}\delta\left(\frac{z_1}{2}+\frac{z_2}{2}\right) e^{-\frac{1}{2}(z_1-z_2)\kappa_-'} \times$$

$$\times H_2\left(\frac{1}{2}\tilde{x}(\kappa_+ + \kappa_-), \frac{1}{2}(\tilde{x}(\kappa_+ - \kappa_-), q_i\right)|_{\tilde{x}q_i=z_i,\ q_i^2=-q_1q_2=\mu^2}.$$

The function $\delta(z_1+z_2)$ enables the application of translation invariance

$$H_2\left(\frac{1}{2}\tilde{x}(\kappa_+ + \kappa_-), \frac{1}{2}\tilde{x}(\kappa_+ - \kappa_-), q_1\right) =$$
$$= e^{\frac{1}{2}(\tilde{x}q_1 + \tilde{x}q_2)\kappa_+} H_2\left(\frac{1}{2}\tilde{x}\kappa_-, -\frac{1}{2}\tilde{x}\kappa_-, q_1\right)|_{\tilde{x}q_1=z_1,\ z_1=-z_2,\ q_1^2=q_2^2=-q_1q_2=\mu^2}$$

(entire functions in $\tilde{x}q_i$). After the z_2-integration we get finally

$$\int d\kappa'_+ \gamma(\kappa,\kappa') = -4\gamma_2 \delta(\kappa_- - \kappa'_-) +$$

$$+ 4\mu^2 \frac{\partial}{\partial\mu^2} \int \frac{dz_1}{2\pi} e^{-iz_1\kappa'_-} H_2(\frac{1}{2}\tilde{x}\kappa_-, -\frac{1}{2}\tilde{x}\kappa_-, q_i)|_{\substack{\tilde{x}q_1 = -\tilde{x}q_2 = z_1 \\ (q_1+q_2)=0,\ q_1^2=\mu^2}} \quad \text{(a.4)}$$

$$= 2\tilde{\gamma}(\kappa_-,\kappa'_-).$$

Similarly to eqs.(2.18),(2.19) it can be proved that the anomalous dimensions $\tilde{\gamma}_{NN'}$ and $\tilde{\gamma}(\kappa_-,\kappa'_-)$ are connected by standard Mellin transform

$$\tilde{\gamma}_{NN'} = \frac{1}{N!}(\frac{\partial}{\partial i\kappa_-})^N \int d\kappa'_- (i\kappa'_-)^{N'} \tilde{\gamma}(\kappa_-,\kappa'_-)\ |_{\kappa'_-=0}$$

$$\text{(a.5)}$$

$$\tilde{\gamma}(\kappa_-,\kappa'_-) = \Sigma\,(i\kappa_-)^N \frac{1}{2\pi} \int dN'(i\kappa'_-)^{-N-1}\, \tilde{\gamma}_{NN'}\ .$$

With $\tilde{\gamma}(\kappa_-,\kappa'_-) = \frac{1}{\kappa_-}\tilde{\gamma}(\frac{\kappa'_-}{\kappa_-})$ we have

$$\tilde{\gamma}_{NN'} = \tilde{\gamma}_N \delta_{NN'} = \int dr\, r^N \tilde{\gamma}(r)\delta_{NN'}\ . \quad \text{(a.6)}$$

On this basis we show the relations between γ_N and $\gamma_{(n)(n')}$ in the following way. The starting point is the relations

$$\gamma_{(n)(n')} = \frac{1}{n_1!}(\frac{\partial}{\partial i\kappa_1})^{n_1} \frac{1}{n_2!}(\frac{\partial}{\partial i\kappa_2})^{n_2} \int d\kappa'_1 d\kappa'_2 (i\kappa'_1)^{n'_1}(i\kappa'_2)^{n'_2}\gamma(\underline{\kappa},\underline{\kappa}')|_{\underline{\kappa}=0}\ .$$

Then we calculate:

$$\sum_{n'_1=0}^{N'} (-1)^{n'_1-N'} \binom{N'}{n'_1} \gamma_{n_1 n_2, n'_1 N'-n'_1} =$$

$$= \sum_{n'_1=0}^{N'} (-1)^{n'_1-N'} \binom{N'}{n'_1} \frac{1}{n_1!}(\frac{\partial}{\partial i\kappa_1})^{n_1} \frac{1}{n_2!}(\frac{\partial}{\partial i\kappa_2})^{n_2} \times$$

$$\times \int d\kappa'_1 d\kappa'_2 (i\kappa'_1)^{n'_1}(i\kappa'_2)^{n'_2}\, \gamma(\underline{\kappa},\underline{\kappa}')|_{\underline{\kappa}=0} =$$

$$= \frac{1}{n_1!}(\frac{\partial}{\partial i\kappa_1})^{n_1} \frac{1}{n_2!}(\frac{\partial}{\partial i\kappa_2})^{n_2} \int d\kappa'_-(i\kappa'_-)^{N'} \int \frac{d\kappa'_+}{2}\gamma(\underline{\kappa},\underline{\kappa}')|_{\underline{\kappa}=0} =$$

$$= \frac{1}{n_1!} \frac{1}{n_2!} (-1)^{n_2} \left(\frac{\partial}{\partial i\kappa_-}\right)^N \int d\kappa'_- (i\kappa'_-)^{N'} \tilde{\gamma}(\kappa_-, \kappa'_-)|_{\kappa_-=0} =$$

$$= (-1)^{n_2} \binom{N}{n_1} \delta_{NN'} \gamma_N .$$

So we obtain finally

$$\gamma_N \delta_{NN'} = \binom{N}{n_1}^{-1} (-1)^{N-n_1} \sum_{n'_1=0}^{N'} (-1)^{n'_1-N'} \binom{N'}{n'_1} \gamma_{n_1 N-n_1, n'_1 N'-n'_1} . \qquad (a.7)$$

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