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3-LOOP APPROXIMATION FOR RUNNING COUPLING CONSTANT IN QUANTUM CHROMODYNAMICS

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Recently performed [1] calculations of renormalization-group (RG) parameters for $\beta(g)$ and $\gamma_i(g)$ functions in 3-loop approximations for QCD "open the door" of systematical account of 3-loop radiative corrections for observed processes of quantum chromodynamics. For this aim it is necessary first of all to make more precise the formula for invariant (i.e. running) QCD coupling constant $\bar{g}$ and define new (enlarged) energy variable region in which the behaviour of $\bar{g}$ can be considered as established reliably.

In what follows we consider this question and also discuss the popular problem of "the choice of the mass scale" taking into account new more accurate equations.

We shall use the following notation for QCD expansion parameter

$$g = \frac{\alpha_s}{\pi} = \frac{g^2_{YM}}{4\pi^2} .$$

The RG equation for invariant coupling constant $\bar{g}$ now has three known terms in the r.h.s.:

$$\frac{d\bar{g}}{dl} = \beta(\bar{g}) \simeq -\beta_1 \bar{g}^2 - \beta_2 \bar{g}^3 - \beta_3 \bar{g}^4 \equiv -\beta \bar{g}^2 \left(1 + b\bar{g} + c\bar{g}^2\right), \quad (1)$$

where $l = \ln\left(Q^2/\mu^2\right)$, $4\beta_1 = 11 - (2/3)n$, $16\beta_2 = 102 - (38/3)n$,

$$64\beta_3 = \frac{2857}{2} - \frac{5033}{18} n + \frac{325}{54} n^2 \quad , \quad n \text{ - number of}$$

flavours.

For n=4 we have

$$\beta = 2.083 \quad , \quad b = 1.540 \quad , \quad c = 3.047 . \quad (2)$$

The quadrature of Eq. (1) written down in the form

$$\varphi(\bar{g}) = \beta L \, , \tag{3}$$

where

$$\varphi(g) = \int^{g} dx/\beta(x) \tag{4}$$

and

$$L = \ell + \varphi(g)/\beta \equiv \ln\left(Q^2/\Lambda^2(\mu,g)\right) , \tag{5}$$

represents the transcendental equation for $\bar{g}$.

In the 2-loop approximation, with a certain choice of approximation of the integrand in the r.h.s. of (4), one can obtain two versions of Eq. (3):

$$\psi(\bar{g}_{21}) = \beta L \quad , \quad \psi(g) = 1/g + b \ln g \; , \tag{6}$$

or

$$\psi(\bar{g}_{22}) - b \ln(1 + b\bar{g}_{22}) = \beta L \; . \tag{7}$$

Quite analogously at the 3-loop level we have at least two different forms, e.g.:

$$\psi(\bar{g}) + (c - b^2)\bar{g} = \beta L \; , \tag{8}$$

$$\psi(\bar{g}) + \frac{2c - b^2}{K}\left\{ \operatorname{arctg}\frac{b + 2c\bar{g}}{K} - \operatorname{arctg}\frac{b}{K} \right\} - \\ - \frac{b}{2}\ln(1 + b\bar{g} + c\bar{g}^2) = \beta L \quad , \quad K = (4c - b^2)^{1/2} . \tag{9}$$

As far as the I-loop approximation

$$\frac{1}{\bar{g}} = \beta L \tag{10}$$

turns out to be very crude, usually one uses the 2-loop approximation in the form

$$\bar{g}_{2,pop}(L) = \frac{1}{\beta L} - \frac{b \ln L}{(\beta L)^2} \quad . \tag{11}$$

This expression which we shall refer to as "popular" contains the correction $\ln L/L$ to Eq. (10). It can be considered as an approximate solution of 2-loop Eq. (6). It should be noted, however, that from the solution of Eq. (6) the popular formula differs by the shift of argument

$$\bar{g}_{21}(L-\Delta) \simeq \bar{g}_{2,pop}(L) \quad , \ \Delta = b \ln \beta / \beta \quad , \ L \gg 1 . \tag{12}$$

In Fig.1 the solutions of Eqs. (6)-(9) are drawn for the region of L where the 2-loop solutions differ from the 3-loop ones. The curve of popular solution (11) is also exposed. However, the curve for $\bar{g}_{2,pop}$ is shifted horizontally by the amount Δ which is not taken from Eq. (12) but defined from the condition of the best coincidence of $\bar{g}_{2,pop}$ with 3-loop solutions in the range $3 < L < 10$. Its numerical value is equal to 0.33. For convinience the mass scale also is given, with Q variable related to L as

$$L = \ln Q^2/\Lambda^2 \quad , \qquad \Lambda = 0.5\, GeV = \Lambda_{\overline{MS}} \quad , \tag{13}$$

the numerical Λ value being taken from the deep inelastic experiments /2/ fitted by Eq. (11) in the framework of minimal subtraction $\overline{MS}$ renormalization procedure. This numerical value has an uncertainty at least $\sim$ 10% that must be taken into account using the given mass scale.

As can be seen from the figure, the 2-loop solutions $\bar{g}_{21}(L')$ and $\bar{g}_{22}(L' = L + 0.33)$ are close to each other and to $\bar{g}_{2,pop}(L)$ in the region $\bar{g} \lesssim 0.2$, $L \gtrsim 2$. In this region the discrepancies $\Delta\bar{g}$ as well as variations of the inverse functions $\Delta Q(\bar{g})$ do not exceed 10%. The boundary of this region is marked by the zigzag line in the figure. Meanwhile 3-loop formulas have the same degree of accuracy up to the values $\bar{g} \lesssim 0.3$, $L \gtrsim 1.2$ (marked in the figure by the double zigzag

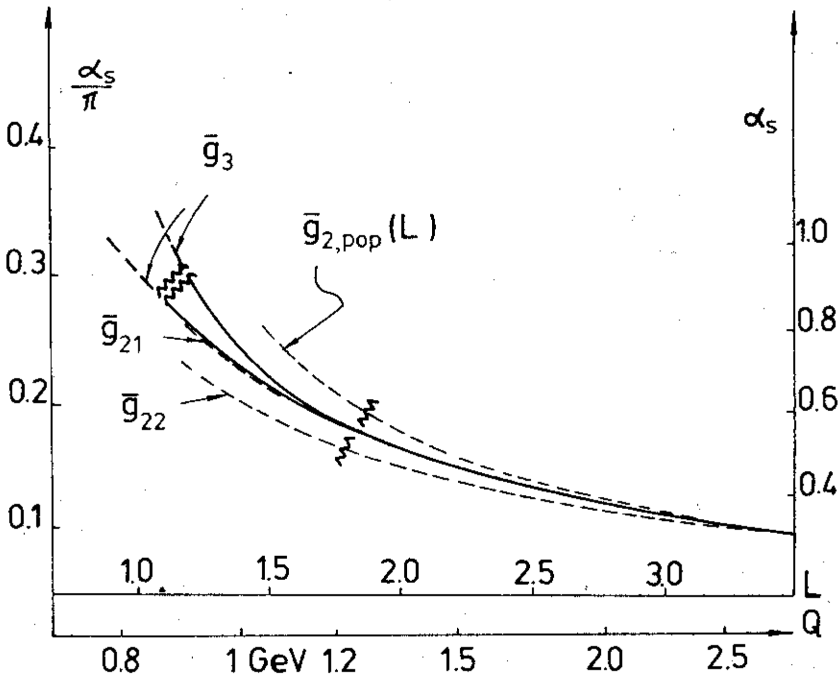

Fig.1. Behaviour of QCD running coupling constant $\bar{g} = \alpha_s/\pi$ in the "middle" energy region. Dashed curves represent various 2-loop approximations. Solid lines picture 3-loop approximations. Single zigzag line denotes the boundary of region in which 2-loop approximations contain errors less than 10% . Double zigzag line marks analogous boundary for 3-loop approximation. Horizontal energy scale for $Q = \sqrt{Q^2}$ corresponds to mass scale Λ = 0.5 GeV.

line). The estimate of 3-loop accuracy is made according to differences between curves (8) and (9) and is of a conditional nature. However, the observation that in the region of a simple zigzag 2-loop curves differ one from another by the same amount as they differ from the 3-loop curve enables us to conjecture that the estimate (15) is reliable.

Thus, the actual progress obtained by the account of 3-loop corrections consists of rising by 1.5 the upper limit of running coupling constant $\bar{g} = \bar{\alpha}_s(Q)/\pi$ from

$$\max_2\{\bar{g}\} \simeq 0.2 \qquad \text{at} \qquad \min_2\{L\} \simeq 2 \tag{14}$$

to

$$\max_3\{\bar{g}\} \simeq 0.3 \qquad \text{at} \qquad \min_3\{L\} \simeq 1.2 \tag{15}$$

and diminishing the lower boundary of energy scale from

$$\min_2\{Q\} \simeq 1.3 \text{ GeV} \qquad \text{to} \quad \min_3\{Q\} \sim 0.9 \text{ GeV} \tag{16}$$

Naturally, the corresponding enlargement of the energy region for matrix elements depends on some additional properties of perturbation expansion coefficients. The last ones depend on the type of renormalization procedure and on the choice of the mass scale Λ. As far as different renormalization schemes are equivalent to each other up to the shift of the mass scale it is reasonable to reduce the problem of "renormalization-prescription dependence" to the "mass scale dependence". In other words, instead of choosing between different renormalization schemes it is sufficient to look for the optimal mass scale unit Λ for a given physical process (or for a set of processes) in the framework of arbitrary (but fixed) renormalization procedure.

The change of mass scale corresponds to a shift of logarithmic variable L which can be described by the transformation of $\bar{g}$:

$$\bar{g}(L+\lambda) \simeq \bar{g}(L) + \lambda\frac{d\bar{g}}{dL} + \frac{\lambda^2}{2}\frac{d^2\bar{g}}{dL^2} = \bar{g} + \lambda\beta(\bar{g}) + \frac{\lambda^2}{2}\beta(\bar{g})\beta'(\bar{g}) .$$

Hence, for sufficiently small λ one has

$$\begin{aligned}\bar{g}(L+\lambda) \simeq \bar{g}(L) - \lambda\beta\,\bar{g}^2(L) + \lambda\beta\left[\lambda\beta - b\right]\bar{g}^3(L) - \\ - \lambda\beta\left[\lambda^2\beta^2 - \frac{5}{2}\lambda\beta + c\right]\bar{g}^4(L) + \dots\end{aligned} \tag{17}$$

People often use this transformation in order to reduce the coefficients at higher power of α_s/π in the perturbation expansion of matrix elements. Here one must bear in mind that in the r.h.s. of Eq. (17) the effective expansion parameter (besides the β -function parameters bg , cg^2) is the product $\lambda\beta\bar{g}$. Hence the condition for validity of transformation (17) is the following

$$\lambda\beta\bar{g}(L) << 1 \ . \tag{18}$$

Note here that the widely used in literature transition from MS scheme to $\overline{\mathrm{MS}}$ scheme corresponding to the change of the mass scale by 2.66 times (λ = 1.95) "nearly satisfies" to the criterion (18). So, for Q = 3 GeV when $\bar{g}$ = 0.1 one gets $\lambda\beta g \simeq 0.4$. This means that under the transformation $g_{MS} \rightarrow g_{\overline{MS}}$ the higher order terms being usually neglected ($\sim \bar{g}^3$ in the r.h.s. of Eq. (17)) are of relative size 0.1 – 0.2, that defines the error of running coupling transformation. Meanwhile the corresponding transformation of matrix elements under some circumstances can yield much larger errors. For illustration consider the well-known expression for the rate of paraquarkonium decay

$$\frac{\Gamma(q\bar{q}\rightarrow gg)}{\Gamma(qq\rightarrow\gamma\gamma)} = c f\left[\frac{\alpha_s(Q)}{\pi}\right]; \quad \begin{aligned} f_{MS} &= g^2(1+22.14g) \\ f_{\overline{MS}} &= g^2(1+14.0g). \end{aligned} \tag{19}$$

According to current folklore, the transition from f_{MS} to $f_{\overline{MS}}$ "improves the convergence" of perturbation expansion. However, this transition not only reduces the numerical coefficient of radiative correction but also considerably enlarges the running coupling $\bar{g}$. So, for Q=3 GeV one has $\bar{g}_{MS}$ = 0.06, $\bar{g}_{\overline{MS}}$ = 0.10 and numerical values $f_{MS}(\bar{g}_{MS})$ and $f_{\overline{MS}}(\bar{g}_{\overline{MS}})$ differ one from another almost by three (!) times, and even for Q = 9 GeV $f_{\overline{MS}}$ is half as much again as f_{MS} . In the first case the absolute and relative value of the radiative correction in $f_{\overline{MS}}$ (which "converges faster") is larger than in f_{MS} . Both effects are conditioned by neglected terms $\sim g^4$.

Thus, for a serious analysis of the possibility of convergence improving in matrix elements, it is necessary to calculate next-order terms. The success of philosophy of determining "appropriate coupling" (i.e. suitable mass scale) for a given physical process will essentially depend on the proximity of the next order coefficients to the corresponding expansion coefficients of "appropriate" running coupling (or its power). For our illustration the g^4 term in $f_{\overline{MS}}$ according to Eq. (17) should have a coefficient close to + 168. If it is really so, then $f_{\overline{MS}}$ presumably can be represented as

$$f_{\overline{MS}}\left[\bar{g}(L)\right] \simeq \left[\bar{g}(L-3.4)\right]^2,$$

i.e., $\bar{g}(L-3.4)$ will here play a role of the "appropriate coupling".

We see that the net effect of large positive coefficient at first QCD radiative correction under some conditions can produce the specific "renormalization red shift" (RRS) of the running coupling constant energy argument, i.e., "red shift enhancement" of $\bar{g}$.

Now, in turn, another trouble connected with a possible large value of the enhanced (appropriate) $\bar{g}$ could arise. In our illustration for the case ($c\bar{c}$) the logarithm-shift by 3.4 corresponds to the shift from 3 GeV to 500 MeV and the shifted $\bar{g}$ turns out to be in the region of strong coupling. It is to be noted that the transition from the total to one quark energy does not cure this trouble. Hence, the procedure of "convergence improving" under the considered conditions resembles that one of "sweeping dust under the carpet"

We can conclude now that the behaviour of the running coupling constant in the region where 3-loop contributions into $\beta(g)$ are essential (i.e. in the region $1 \lesssim L \lesssim 2$) besides the pure theoretical interest may happen to be of the physical importance. We mind here not so much matrix elements for processes at energies $Q \sim 1$ GeV (their analysis is also complicated due to mass corrections) as theoretical understanding of such processes at higher energies (3 - 20 GeV) perturbation expansion of which contain large positive coefficients. The paraquarkonium decay rate, nucleon structure functions, matrix elements of $q\bar{q}$ scattering with large transverse momenta, as well as, probably, 3-jets e^+e^- -annihilations fall under this category. For all these processes the calculation of higher-order contributions is of importance.

If it will be discovered that appropriate running couplings due to the RRS mechanism do correspond to energies close to

1 GeV then, it may happen that nonperturbative instanton contributions into the $\bar{g}$ behaviour and, as a consequence, into observed matrix elements are also important.

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References

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