объединенный
институт
ядерных
исследований
дубна

E2-80-587

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NUMERICAL STUDY OF PROPERTIES OF MANY-DIMENSIONAL SOLITON TYPE OBJECTS

Submitted to the International Meeting
on Problems of Mathematical Simulation
in Nuclear Physics Research (Dubna, 1980).

1980

1. There are two classes of physical phenomena, which may be reduced to the equations possessing soliton and soliton-like solutions. They differ in their problems as well as in the interpretation of the results, although they come together at the classical level.

The problems of studying nonlinear wave phenomena in real continuous media are of the first class and field theoretic problems are of the second one.

In the first case the investigation used to go through the following stages: it starts at the classical or quantum discrete level, then with some degree of rigour, the study proceeds to the classical or quasi-classical continuum limit, that defines the form of the resulting equations. As the final result of such a transformation, we have the soliton or quasi-soliton classical solitons (to which through an inverse transformation may, in principle, be given a quantum meaning).

In the second case the models under investigation are constructed with the help of a Lagrangian formalism and allowance for certain requirements come from the general physical conceptions and laws (invariance under the Poincare group, global or gauge internal symmetries some of which may be broken and so on). In this case classical wave equations appear at the initial stage and their solutions become the basis for constructing "real" quantum objects including that of a soliton type (extended particles)* . Here unlike conventional atomic physics, the quantum soliton properties are determined by classical solutions as $h \to 0$. All this defines the form of the second class of equations.

These directions come together in the intermediate stage of the investigation of the localized (soliton) solutions of the finite energy to the classical wave equations.

Thus, the investigation of the general properties of the classical solitons (CS) may be carried out, to some extent, forgetting their physical interpretation. As it was shown in[1], there are possible stable solitons to exist in four-dimensional (x, y, z, t) one-field models with a so-called

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*The so-called non-perturbative approach in quantum field theory.

saturable non-linearity. Undoubtedly, being of interest within the first class, they might be rejected for the second class theories because of renormalizability conditions.

We consider below non-linear phenomena in the systems allowing stochastization, i.e., in non-integrable systems. The behaviour of such systems may be governed by the degree of their proximity to some complete integrable analogs. In this sense the integrable models can be considered as a zero order (non-linear!) approximation to the description of the real physical systems, and further study can be performed with all discretion as a perturbation series in this small deviation.

Note, that at present only complete integrable models may be strictly analytically investigated by various methods. But analytical methods are, as a rule, practically helpless (at least, at present) in studying the evolution of non-integrable systems. Therefore, with rare exceptions, all the results on the evolution of ergodic (even one-dimensional) systems were obtained by computer experiments.

2. What should primarily stress, is that it was a computer which created the Fermi-Pasta-Ulam (FPU) problem (about 24 years ago) and then discovered solitons. As a result of numerical experiments on the dynamics of KdV nonlinear waves, a concept of the solitons (Zabusky) appeared to be solitary waves which emerged from the collision without changing their shapes and velocities. Somewhat earlier Perring and Skyrme have found via a computer the analogous effects in the framework of the sine-Gordon equation, but for rather different objects.

It is interesting to note, that "two-soliton" solutions (bions) have been obtained a decade earlier (Seeger, Donth and Kochendörfer). Then Ooyama and Saito (1970) have found solitons on "Toda lattice" nearing the FPU problem. Finally, solitons in the framework of the Schrödinger equation with cubic nonlinearity (S3) were discovered by Yajima and Outi in 1971. All references may be found in the review by Scott, Chu and McLaughlin[2].

Thus, the computer creating a new branch in the theory of nonlinear partial differential equations fell behind. A boom time began of discovering and studying the completely integrable Hamiltonian systems and the related methods of the inverse scattering problem, Hirota and Bäcklund transformations. Developing and formalising these techniques being of an international competition character displayed, that

integrable equations may be generated in the unlimited amount. All this gave rise to the view, that the majority (if not all) of the Lagrangian systems are completely integrable.

The first impact on this outlook was done again by a computer. The inelastic interaction of Langmuir solitons in plasma was discovered at Dubna in 1974[3]; analogously, for solitons of the "improved" versions of Boussinesq and KdV equations, of the Higgs and Klein-Gordon (KG3) equations[4]. It turned out, that even a "small" alteration of the equation may render it non-integrable. Moreover, as it was shown, certain particular properties of the integrability disappeared under transformation from the plane (x,t) geometry to the spherically (or cyllindrically) symmetric (r,t) one[5].

The conception of near integrable systems, for which, as has been pointed out, integrable equations may serve as somewhat original zero order approximation, appeared. The role of the interaction parameter in an investigation can play the deviation from an appropriate integrable equation[6,7].

It should be emphasized that the numerical experiments are now one of the most powerful tools to investigate Hamiltonian systems, especially to answer if a given system is the complete integrable. The elastic collision of solitons can imply the positive answer (recall KdV). The inelastic interaction of solitons would make searching for the integrability consequences, in particular, many-soliton formulas to be fruitless[8].

Nevertheless, the concept of a nearness between a given system and some integrable one somewhat helps to discover pulsating (bound states) solitons, bions, both in the plane (x,t)[9] geometry and in a spherically symmetric (r,t) one[10], for the KG3 and Higgs equations. If in the plane case one can still find an approximate analytical solution for the pulsons (pulsating solitons), then, on the contrary, in the (r,t) geometry, the discovery of the pulsons as well as the investigation of their properties are only due to a computer (the conventional analytical methods turned out to be powerless because of the actual nonlinearity and the absence of a small parameter).

3. As a result of a great deal of work all over the world, the properties of solitons in the plane (x,t) world have been learned quite well. It was time to proceed to more real and intricate many-dimensional worlds. This transition as should be expected, was non-trivial. Here the stability problem went ahead, when we proceed from one to many space

dimensions (unstable solitons were found, in the plane (x,t) world, only in the framework of the KG3 equation).

The Derrick theorem states that the constant energy surface in the functional space can not be a valley. At best it is a saddle, i.e., the surface not possessing an absolute minimum. There are no absolutely stable solutions in such models. One of the ways to stabilize them is including an isotopic symmetry group into Lagrangian and the related conservation laws.

We discuss the properties of models /11/ with the most simple U(1) group that gives rise to the "isocharge" conservation law

$$Q = \frac{i}{2} \int (\phi_t^* \phi - \phi^* \phi_t) d^D x , \quad \frac{dQ}{dt} = \dot{0},$$

with ϕ and D being the field function and the space dimensions, accordingly.

The isocharge conservation implies the possible perturbations have to be restricted, namely $\delta Q[\phi] = 0$, that leads to the stability condition for SLS /4/ (i.e., solutions with a good behaviour at the origin and infinity)

$$\frac{\omega}{Q} \frac{dQ}{d\omega} < 0. \tag{1}$$

Real stationary field configurations can apparently not satisfy this condition and are therefore unstable. All these statements have been confirmed in a series of numerical experiments performed by different groups.

Here we should underline the great importance of investigating many-space-dimensional SLS, since up to now the only couple of two-space-dimensional evolution integrable equations (the Kadomtsev-Petviashvili equation and cylindric KdV) is known for which soliton solutions however behave badly at infinities in the direction perpendicular to the soliton motion (weak power law decreasing). Moreover, the decay of an initial state is not so perfect as it was in the one-dimensional case. Therefore, of especial interest is studying via computer the dynamical properties of two- then three-spacedimensional well localized solutions for various field theories. Such an investigation of the qualitative properties of SLS may prompt a way for further studying them with analytical (probably approximate) methods.

3. Consider two classical field theory models with the following interaction potential in Lagrangian*

$$U = \ln(1 + |\phi|^2)\,\phi^2 \tag{2}$$

and

$$U = \ln(|\phi|^2)\,\phi^2. \tag{3}$$

These models can be easily seen to be different in the following sense: the first one has as the $\phi \to 0$ limit a free theory since $\phi^2 \ln(1+|\phi|^2) \approx \phi^4$, the second contains constituents with infinite rest mass for $\ln(|\phi|^2) \to -\infty$ as $\phi \to 0$. This means that under certain conditions in the first case a nonlinear solution may decay into constituents thereby radiating linear plane waves. In the second model such a decay is impossible (forbidden by the conservation law) and all admissible field configurations are only nonlinear solutions, therefore such models are sometimes called "confining" models. As a result the instability of SLS gives rise to decay in the first case and to collapse in the second.

The confining model is interesting also because its SLS may be derived in explicit form for the space of any dimensions D. Finally, it follows from the Q-theorem that stable U(1) symmetrical SLS $\phi = \Phi(r)\, e^{-i\omega t}$ exist at $\omega > \omega_{cr} = 2^{-1/2}$ regardless of D. The $\ln((\phi)^2)$ model is in this sense scale invariant and qualitatively unlike the model (2), where ω_{cr} depends essentially on D. Approximate dependences $Q(\omega)$ are depicted in Figure 1 for D = 2 and D = 3.

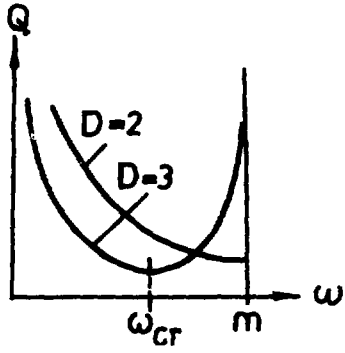

Figure 1. The function $Q(\omega)$ at D = 2 and D = 3 for models having free field theory as the $\phi \to 0$ limit.

Assumption: The character of soliton interaction at collisions is governed by "dispersion" dependence $Q(\omega)$ rather than by the model type (abstracting from the instability form, decay or collapse).

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* There exist no stable SLS at D 1 in the case of a simple ϕ^4_D theory even including isogroup.

This assumption has been verified in a series of computer experiments [12,13]. Two parameters were alterated during computations: the velocity of relative motion of quasi-solitons v and their isocharge Q. In both cases the following types of interactions have been observed:

1) elastic and quasi-elastic interaction of SLS;
2) creating a stable (long-living) bound state of two SLS, i.e., bion (breather and so on);
3) creating an unstable (short-living) bound state;
4) instability (of the decay or collapse type) of interacted quasi-solitons.

This implies in fact the model independent character of quasi-soliton interaction (at any rate for the models considered). The last two types of interactions are possible only in the vicinity of $dQ/d\omega \simeq 0$. All this makes us to suppose that the above four types of interaction will be manifested in models with higher symmetry groups, if the dependence of corresponding isocharge (isospin projection) $Q(\omega)$ resembles that in Figure 1.

More careful study of quasi-soliton imteraction processes displays their dependence on the impact parameter p (or angular momentum $\ell = pmv$) and initial phase difference $\Delta\theta$. The numerical experiments show:

a) There is a resonance region in angular momentum ℓ where inelasticity of quasi-soliton interaction increases sharply (see [14]);
b) Purely antisymmetrical initial field configuration leads to elastic quasi-soliton repulsion.

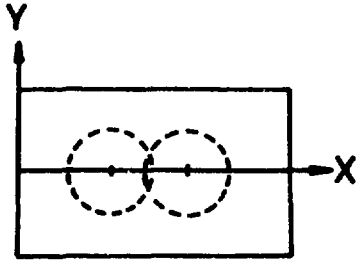

Figure 2. Initial field configuration of two real unstable quasi-solitons.

5. As it has been pointed out the stationary configurations of the real fields cannot be stable, i.e., the real quasi-solitons do not exist. Moreover, the stable quasi-solitons exist neither in every system with internal isosymmetry, nor always. Such solutions can appear in the systems having conditional (or local) minima of the constant energy surface in the functional space. Naturally, a question arises: are there any nonstationary stable configurations of real fields in such systems? Thereat, the non-stationarity works as a

stabilizing factor analogous to exp $(-i\omega t)$ for the U(1) group* .

This assumption has been verified in a series of computer experiments for the model (2) carried out by G. Kummer and the authors in Dubna [15]. The results of those experiments looked at first glance paradoxical. Placing the unstable soliton-like objects with sufficiently small opposite velocities closely enough (Figure 2), so that the kinematic time of their interaction was less than their decay time, we observed the production of a quasi-soliton bound state - two-dimensional bion. The field amplitude at the bion centre was regularly oscillating and slightly decreasing during calculation (for some oscillation periods). Further studies showed that sufficiently heavy one-soliton initial state can also produce analogous objects. The time behaviour and type of the discovered bions qualitatively coincide with those of the pulsons considered recently in the paper of I.L.Bogolubsky and one of the authors [16]. Thereby, it is shown that the existence of pulsons is not the priviledge of the systems with degenerate vacuum, such as the Higgs and sine-Gordon field equations where the field function oscillates between two adjacent vacua. Note that analogous pulsons should, naturally, also appear for the system (3). The possible explanation of pulson stability (using certain adiabatic invariant) has been proposed in refs. [1,17] .

Ultimately, we note that in nuclear physics stable bound states emerging from unstable constitutients have long been known (deuteron). This state, as in our case, bears no resemblance to the bound state of two classical objects, like Earth-Moon, double starts, etc. Constitutients in our case loose their individuality during the formation of the bound state. Therefore, it is suitable to refer to the proverb, recalled by Wheeler D.A. [18] and widespread in the middle of our century "both the nature and nonlinear equations are complicated, so the nature should be described by nonlinear equations" (double retranslation). In fact, as it was shown above, the dynamics of sufficiently simple nonlinear systems is quite rich, various and sometimes unexpected.

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*Remember the P.L.Kapitsa problem concerning the pendulum with oscillating basis.

REFERENCES

1. Makhankov V.G. Phys.Scripta, 1979, 20, p.558; JINR, P2-10362, Dubna, 1977 (in Russian).
2. Scott A., Chu F., McLaughlin D. Proc. IEEE, 1973, 61, p.1443.
3. Abdulloev Kh.O., Bogolubsky I.L., Makhankov V.G. Phys. Lett., 1974, 48A, p.161; Nuclear Fussion, 1975, 15, p.21.
4. Makhankov V.G. Phys.Reports, 1978, 35C, p 1-128.
5. Bogolubsky I.L., Makhankov V.G. Pis'ma ZhETF, 1976, 24, p.15 (JETF Lett., 1976, 24, p.12).
6. Karpman V.I., Maslov V.P. Phys.Lett., 1977, 60A, p.307; ZhETF, 1977, 73, p.537.
7. Makhankov V.G., Fedyanin V.K. Sov.Phys.Doklady, 1977, 236, p.838; JINR, E17-10507, Dubna, 1977.
8. Abdulloev Kh.O., Bogolubsky I.L., Makhankov V.G. Phys. Lett., 1976, 56A, p.427.
9. Kudryavtsev A.E. JETP Lett., 1975, 22, p.83. Getmanov B.S., ibid. 1976, 24, p.291.
10. Bogolubsky I.L., Makhankov V.G. JETP Lett., 1977, 25, p.107.
11. Makhankov V.G., Shvachka A.B. JINR, P2-10341, Dubna, 1980 (in Russian); Phys. D (in press).
12. Makhankov V.G., Kummer G., Shvachka A.B. Phys.Scripta, 1979, 20, p.454; JINR, P2-11913, Dubna, 1978 (in Russian).
13. Makhankov V.G. et al. JINR, P2-80-367, Dubna, 1980 (in Russian); Phys.Scripta (in press).
14. Devi S., Stayer M., Irvine J. J.Phys.G., Nucl.Phys., 1979, 5, p.281.
15. Makhankov V.G., Kummer G., Shvachka A.B. JINR, P2-13042, Dubna, 1980 (in Russian); Phys. D (in press).
16. Bogolubsky I.L., Makhankov V.G. JETP Lett., 1976, 14, p.12; ibid. 1977, 25, p.107.
17. Manakov S.V. JETP Lett., 1977, 25, p.589.
18. Wheeler J.A. Neutrinos, Gravitation and Geometry, Bologna, 1960.