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A NOTE ON THE PROBLEM OF TWO INTERACTING STOCHASTIC PARTICLES

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In the previous paper [1] we have investigated the motion of a single particle in a stochastic space and obtained equations of stochastic mechanics in both the nonrelativistic and relativistic cases. Now we shall consider the problem of two interacting particles since this is the problem of real physical interest. We shall study this question within the framework of Kershaw's stochastic model [2] which is based on the Smoluchowski equations for the probability density $\rho(\vec{x}_1, \vec{x}_2, t)$ of finding the first particle at point $\vec{x}_1$ and the second particle at $\vec{x}_2$ at time t. Let there be a potential force $U(\vec{x}_1-\vec{x}_2)$ operating between the two particles, and let $\vec{v}_1(\vec{x}_1, \vec{x}_2, t)$ and $\vec{v}_2(\vec{x}_1, \vec{x}_2, t)$ be their relative velocities.

The problem of two interacting stochastic particles was considered first by Kershaw [2] for the case of a single quantity $\rho_+(\vec{x}-\Delta\vec{x}^+, t;\ \Delta\vec{x}^+, \Delta t)$, where $\rho_+(\vec{x}-\Delta\vec{x}^+, t; \Delta\vec{x}^+, \Delta t)$ is the conditional probability density that a particle at position $\vec{x}-\Delta\vec{x}^+$ at time t will be displaced by $\Delta\vec{x}^+$ during the interval Δt, thus reaching position $\vec{x}$ at time $t+\Delta t$. He has obtained some equations describing the center-of-masses and relative motions of two particles. These equations can formally be reduced to the Schrödinger equation for a two-particle system.

Following Kershaw [2] and an idea of paper [1] we shall investigate in this paper the problem of motion of two interacting particles in the stochastic space with a small stochastic component by using two conditional probability densities ρ_+ and ρ_-, where $\rho_-(\vec{x}+\Delta\vec{x}^-, t; \Delta x^-, \Delta t)$ denote the probability density that a particle with position $\vec{x}+\Delta\vec{x}^-$ at time t has been displaced through $\Delta\vec{x}^-$ in the preceding interval Δt and thus would have been found at $\vec{x}$ at the earlier instant $t-\Delta t$.
Let

$$\rho_+^1(\Delta\vec{x}_1^+;\Delta t) = (2\pi\tau_1\Delta t/m_1)^{-3/2} \exp\left\{-m_1(\Delta\vec{x}_1^+)^2/(2\tau_1\Delta t)\right\},$$

$$\rho_-^1(\Delta\vec{x}_1^-;\Delta t) = (2\pi\tau_1\Delta t/m_1)^{-3/2} \exp\left\{-m_1(\Delta\vec{x}_1^-)^2/(2\tau_1\Delta t)\right\} \tag{1}$$

and

$$\rho_+^2(\Delta\vec{x}_2^+,\Delta t) = (2\pi\tau_2\Delta t/m_2)^{-3/2} \exp\left\{-m_2(\Delta\vec{x}_2^+)^2/(2\tau_2\Delta t)\right\}, \tag{2}$$

$$\rho_-^2(\Delta\vec{x}_2^-,\Delta t) = (2\pi\tau_2\Delta t/m_2)^{-3/2} \exp\left\{-m_2(\Delta\vec{x}_2^-)^2/(2\tau_2\Delta t)\right\}$$

be transition probability densities for the first and second particles, respectively. Here $\tau_1 = 2\mathcal{D}_1 m_1$ and $\tau_2 = 2\mathcal{D}_2 m_2$.

Assuming $\tau_1 = \tau_2 = \tau$ and at first we carry out investigation concerning the quantities $\rho_+^1(\Delta\vec{x}_1^+,\Delta t)$ and $\rho_+^2(\Delta\vec{x}_2^+,\Delta t)$.

Now let

$$\vec{r}^+ = \vec{x}_1^+ - \vec{x}_2^+ \quad \text{and} \quad \vec{R}^+ = (m_1\vec{x}_1^+ + m_2\vec{x}_2^+)/(m_1+m_2), \tag{3}$$

then

$$\rho_+(\Delta\vec{r}^+,\Delta t) = \frac{\partial(\vec{x}_2^+)}{\partial(\vec{r}^+)}\int \rho_+^1(\Delta\vec{x}_1^+,\Delta t)\cdot$$
$$\cdot\rho_+^2(\Delta\vec{x}_1^+ - \Delta\vec{r}^+,\Delta t)\, d^3(\Delta x_1^+),$$

where $\partial(\vec{x}_2^+)/\partial(\vec{r}^+)$ is the Jacobian of $\vec{x}_2^+$ with respect to $\vec{r}^+$. This implies that

$$\rho_+(\Delta\vec{r}^+,\Delta t) = \int d^3(\Delta x)\left[(2\pi\tau\Delta t)^2/m_1 m_2\right]^{-3/2}\cdot$$
$$\cdot\exp\left(-\left[m_1(\Delta\vec{x})^2 + m_2(\Delta\vec{x}-\Delta\vec{r}^+)^2\right]/(2\tau\Delta t)\right) =$$

$$= \frac{\exp(-\mu(\Delta\vec{r}^+)^2/2\tau\Delta t)}{(2\pi\tau\Delta t/\mu)^{3/2}} \int d^3(\Delta x)\left[2\pi\tau\Delta t/(m_1+m_2)\right]^{-3/2} \cdot$$

$$\exp\left\{-(m_1+m_2)\left[\Delta\vec{x} - \frac{m_2}{m_1+m_2}\Delta\vec{r}^+\right]^2 / 2\tau\Delta t\right\}.$$

From this formula we get

$$P_+(\Delta\vec{r}^+, \Delta t) = \frac{1}{(2\pi\tau\Delta t/\mu)^{3/2}} \exp\left\{-\frac{\mu(\Delta\vec{r}^+)^2}{2\tau\Delta t}\right\}, \qquad (4)$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass. Similarly, we have

$$P_+(\Delta\vec{R}^+, \Delta t) = \frac{\partial(\vec{x}_2^+)}{\partial(\vec{R}^+)} \int P_+^2\left(\frac{M}{m_2}\Delta\vec{R}^+ - \frac{m_1}{m_2}\Delta\vec{x}, \Delta t\right) \cdot$$

$$\cdot P_+^1(\Delta\vec{x}, \Delta t)\, d^3(\Delta x).$$

After some elementary calculations we obtain

$$P_+(\Delta\vec{R}^+, \Delta t) = (2\pi\tau\Delta t/M)^{-3/2} \exp\left\{-M(\Delta\vec{R}^+)^2/2\tau\Delta t\right\}, \qquad (5)$$

where $M = m_1 + m_2$ is the total mass.

Now let

$$\vec{V}^+(\vec{r}, \vec{R}, t) = (m_1\vec{v}_1^+ + m_2\vec{v}_2^+)/M,$$

$$\vec{C}^+(\vec{r}, \vec{R}, t) = \vec{v}_1^+ - \vec{v}_2^+,$$

where

$$\vec{r} = \vec{x}_1^+ + \vec{x}_1^- - \vec{x}_2^+ - \vec{x}_2^- = \vec{x}_1 - \vec{x}_2,$$

$$\vec{R} = \left[m_1(\vec{x}_1^+ + \vec{x}_1^-) + m_2(\vec{x}_2^+ + \vec{x}_2^-)\right]/M = (m_1\vec{x}_1 + m_2\vec{x}_2)/M.$$

Then the total displacement in position is given by

$$\delta\vec{R}^+ = \vec{V}^+\Delta t + \Delta\vec{R}^+$$

$$\delta\vec{r}^+ = C^+\Delta t + \Delta\vec{r}^+.$$

We can rewrite $\rho(\vec{x}_1, \vec{x}_2, t)$ as

$$\rho(\vec{r}, \vec{R}, t) = \rho(\vec{x}_1, \vec{x}_2, t)\, \partial(\vec{x}_1, \vec{x}_2)/\partial(\vec{r}, \vec{R}).$$

We have then as before (see /1/ also)

$$P_R^+(\vec{r}, \vec{R}, t; \delta\vec{R}^+, \Delta t) = (2\pi\tau\Delta t/M)^{-3/2} \exp\left\{-\frac{M(\delta\vec{R}^+ - \vec{V}^+\Delta t)^2}{2\tau\Delta t}\right\}, \quad (6)$$

$$P_r^+(\vec{r}, \vec{R}, t; \delta\vec{r}^+, \Delta t) = (2\pi\tau\Delta t/\mu)^{-3/2} \exp\left\{-\frac{\mu(\delta\vec{r}^+ - \vec{C}^+\Delta t)^2}{2\tau\Delta t}\right\}.$$

The Smoluchowski-type equation for ρ, by using quantities P_R^+ and P_r^+, acquires now the following form

$$\rho(\vec{r}, \vec{R}, t+\Delta t) = \iint \rho(\vec{r} - \delta\vec{r}^+, \vec{R} - \delta\vec{R}^+, t)\, P_r^+(\vec{r} - \delta\vec{r}^+, \vec{R} - \delta\vec{R}^+, t; \delta\vec{r}^+, \Delta t)\cdot$$

$$P_R^+(\vec{r} - \delta\vec{r}^+, \vec{R} - \delta\vec{R}^+, t; \delta\vec{R}^+, \Delta t)\, d^3(\delta r^+)\, d^3(\delta R^+) = \quad (7)$$

$$= \rho - \Delta t\left[\nabla_{r_j}(\rho C_j^+) + \nabla_{R_j}(\rho V_j^+)\right] + \Delta t \cdot \frac{\tau}{2}\left(\frac{\nabla_r^2 \rho}{\mu} + \frac{\nabla_R^2 \rho}{M}\right).$$

This implies that

$$\frac{\partial \rho}{\partial t} = -\nabla_{r_j}(\rho C_j^+) - \nabla_{R_j}(\rho V_j^+) + \mathcal{D}_\mu \nabla_r^2 \rho + \mathcal{D}_M \nabla_R^2 \rho, \quad (8)$$

where $\mathcal{D}_\mu = \tau/2\mu$ and $\mathcal{D}_M = \tau/2M$.

Similar calculations by using quantities of P_-^1 and P_-^2 make the following equation for ρ :

$$\rho(\vec{r}, \vec{R}, t-\Delta t) = \iint \rho(\vec{r} + \delta\vec{r}^-, \vec{R} + \delta\vec{R}^-, t)\, P_r^-(\vec{r} + \delta\vec{r}^-, \vec{R} + \delta\vec{R}^-, t; \delta r^-, \Delta t) \times$$

$$\times \rho_R^-(\vec{r}+\delta\vec{r}^-; \vec{R}+\delta\vec{R}^-, t; \delta\vec{R}^-, \Delta t)\, d^3(\delta r^-)\, d^3(\delta R^-), \tag{7'}$$

which implies that

$$\frac{\partial\rho}{\partial t} = -\nabla_{r_j}(\rho C_j^-) - \nabla_{R_j}(\rho V_j^-) - \mathcal{D}_\mu \nabla_r^2 \rho - \mathcal{D}_M \nabla_R^2 \rho , \tag{8'}$$

where $\vec{C}^- = \vec{v}_1^- - \vec{v}_2^-$ and $\vec{V}^- = (m_1\vec{v}_1^- + m_2\vec{v}_2^-)/M$.

From equations (8) and (8') we obtain

$$\frac{\partial\rho}{\partial t} = -\nabla_{r_j}(\rho V_{r_j}) - \nabla_{R_j}(\rho V_{c_j})$$

$$\vec{u}_c = \mathcal{D}_M \vec{\nabla}_R \ln\rho \quad \text{and} \quad \vec{u}_r = \mathcal{D}_\mu \vec{\nabla}_r \ln\rho . \tag{9}$$

Here

$$\vec{V}_c = \tfrac{1}{2}(\vec{V}^+ + \vec{V}^-) = (m_1\vec{v}_1 + m_2\vec{v}_2)/M ,$$

$$\vec{V}_r = \tfrac{1}{2}(\vec{C}^+ + \vec{C}^-) = \vec{v}_1 - \vec{v}_2 ,$$

$$\vec{u}_c = \tfrac{1}{2}(\vec{V}^+ - \vec{V}^-) = (m_1\vec{u}_1 + m_2\vec{u}_2)/M ,$$

$$\vec{u}_r = \tfrac{1}{2}(\vec{C}^+ - \vec{C}^-) = \vec{u}_1 - \vec{u}_2 .$$

The potential $U(\vec{r})$ affects only $\vec{C}^\pm$ and not $\vec{V}^\pm$. We have then

$$\vec{V}^\pm(\vec{r},\vec{R},t\pm\Delta t) = \frac{1}{N^\pm}\int \vec{V}^\pm(\vec{r}\mp\delta\vec{r}^\pm, \vec{R}\mp\delta\vec{R}^\pm, t)\,\cdot$$

$$\cdot\,\rho(\vec{r}\mp\delta\vec{r}^\pm, \vec{R}\mp\delta\vec{R}^\pm, t)\,\rho_r^\pm\,\rho_R^\pm\, d^3(\delta r^\pm)\, d^3(\delta R^\pm),$$

$$\vec{C}^\pm(\vec{r},\vec{R},t\pm\Delta t) = \frac{1}{N^\pm}\int\Big[\vec{C}^\pm(\vec{r}\mp\delta\vec{r}^\pm; \vec{R}\mp\delta\vec{R}^\pm, t) \mp \Delta t\cdot\frac{1}{\mu}\times$$

$$\times \nabla_r U(\vec{r} \mp \delta\vec{r}^{\,\pm})\Big]\rho(\vec{r} \mp \delta\vec{r}^{\,\pm}, \vec{R} \mp \delta\vec{R}^{\,\pm}, t) \times$$

$$\times P_r^{\pm} P_R^{\pm} \, d^3(\delta r^{\pm})\, d^3(\delta R^{\pm}) ,$$

where

$$N^{\pm} = \int\!\!\int \rho(\vec{r} \mp \delta\vec{r}^{\,\pm}, \vec{R} \mp \delta\vec{R}^{\,\pm}, t) P_r^{\pm} P_R^{\pm} \, d^3(\delta r^{\pm})\, d^3(\delta R^{\pm}).$$

Expanding $\vec{V}^{\pm}$, $\vec{C}^{\pm}$, $P_{r,R}^{\pm}$, ρ and U in the Taylor series, integrating, and retaining terms only of first order in Δt, we obtain

$$M\left[\frac{\partial V_j^{\pm}}{\partial t} + (V_i^{\pm}\nabla_{R_i}) V_j^{\pm} + (C_i^{\pm}\nabla_{r_i}) V_j^{\pm}\right] = M\frac{dV_j^{\pm}}{dt} = \pm M\Big\{\mathcal{D}_\mu \cdot \frac{1}{\rho} \cdot$$

$$\left(\nabla_r^2(\rho V_j^{\pm}) - V_j^{\pm}\nabla_r^2\rho\right) + \mathcal{D}_M \frac{1}{\rho}\left(\nabla_R^2(\rho V_j^{\pm}) - V_j^{\pm}\nabla_R^2\rho\right)\Big\},$$

$$\mu\left[\frac{\partial C_j^{\pm}}{\partial t} + (V_i^{\pm}\nabla_{R_i}) C_j^{\pm} + (C_i^{\pm}\nabla_{r_i}) C_j^{\pm}\right] = \mu\frac{dC_j^{\pm}}{dt} = -\nabla_{r_j} U(r) \pm$$

$$\pm\mu\Big\{\mathcal{D}_\mu \frac{1}{\rho}\left(\nabla_r^2(\rho C_j^{\pm}) - C_j^{\pm}\nabla_r^2\rho\right) + \frac{\mathcal{D}_M}{\rho}\left(\nabla_R^2(\rho C_j^{\pm}) - C_j^{\pm}\nabla_R^2\rho\right)\Big\}.$$

Summing the equations in this expression we get

$$\left.\begin{aligned} d_s' V_{c_j} - d_u' \mathcal{U}_{c_j} &= 0, \\ d_s' V_{r_j} - d_u' \mathcal{U}_{r_j} &= -\nabla_r U/\mu , \end{aligned}\right\} \qquad (10)$$

where

$$d_s' = \frac{\partial}{\partial t} + \vec{V}_c \vec{\nabla}_R + \vec{V}_r \vec{\nabla}_r ,$$

$$d'_u = \vec{u}_c \vec{\nabla}_R + \vec{u}_r \vec{\nabla}_r + \mathcal{D}_\mu \nabla_r^2 + \mathcal{D}_M \nabla_R^2 .$$

So, operators d'_s and d'_u are separated into a sum of two independent parts. Therefore, by analogy to the separation of variables in quantum mechanics, we assume that

$$\rho(\vec{r}, \vec{R}, t) = \rho_r(\vec{r}, t)\, \rho_R(\vec{R}, t) ,$$

implying that

$$\vec{V}_r(\vec{r}, \vec{R}, t) = \vec{V}_r(\vec{r}, t) , \quad \vec{u}_r(\vec{r}, \vec{R}, t) = \vec{u}_r(\vec{r}, t)$$

$$\vec{V}_c(\vec{r}, \vec{R}, t) = \vec{V}_c(\vec{R}, t) , \quad \vec{u}_c(\vec{r}, \vec{R}, t) = \vec{u}_c(\vec{R}, t) ,$$

where ρ_c , $\vec{V}_c$ and $\vec{u}_c$ describe the motion of center-of-masses (as a free motion of a particle with mass $m_1 + m_2$), but ρ_r , $\vec{V}_r$ and $\vec{u}_r$ describe the relative motion (as a motion of a particle with mass μ in a central symmetrical field $U = U(r)$).

Then equations (9) and (10) acquire the following form

$$\left.\begin{aligned} \frac{\partial \rho_R}{\partial t} &= -\operatorname{div}(\rho_R \vec{V}_c) \\ d'_R \vec{V}_c - d''_R \vec{u}_c &= 0 \end{aligned}\right\} , \tag{11}$$

and

$$\left.\begin{aligned} \frac{\partial \rho_r}{\partial t} &= -\operatorname{div}(\rho_r \vec{V}_r) \\ d'_r \vec{V}_r - d''_r \vec{u}_r &= -\nabla_r U(r)/\mu \end{aligned}\right\} . \tag{12}$$

Here

$$d'_R = \frac{\partial}{\partial t} + \vec{V}_c \vec{\nabla}_R , \quad d''_R = \vec{u}_c \vec{\nabla}_R + \mathcal{D}_M \nabla_R^2 ,$$

$$d'_r = \frac{\partial}{\partial t} + \vec{V}_r \vec{\nabla}_r , \quad d''_r = \vec{u}_r \vec{\nabla}_r + \mathcal{D}_\mu \nabla_r^2$$

By using the standard method proposed by Nelson /3/ nonlinear equations (11) and (12) may be linearized if variables $\vec{u}$, $\vec{V}$ and ρ are transformed by the following formulas:

$$\varkappa_r = \frac{1}{2} \ln \rho_r \quad , \quad \vec{u}_r = 2\mathcal{D}_\mu \vec{\nabla}_r \varkappa_r ,$$

$$\varkappa_R = \frac{1}{2} \ln \rho_R \qquad \vec{u}_c = 2\mathcal{D}_M \vec{\nabla}_R \varkappa_R ,$$

$$\psi_r = \psi_r(\vec{r},t) = exp(\varkappa_r + i\delta_r), \quad \psi_R = \psi_R(\vec{R},t) = exp(\varkappa_R + i\delta_R),$$

$$\vec{V}_r = 2\mathcal{D}_\mu \vec{\nabla}_r \delta_r \quad \text{and} \quad \vec{V}_c = 2\mathcal{D}_M \vec{\nabla}_R \delta_R .$$

From this we have the following two equations for ψ_r and ψ_R :

$$\left.\begin{aligned} \frac{\partial \psi_r}{\partial t} &= i\mathcal{D}_\mu \nabla_r^2 \psi_r - i\frac{1}{2\mu\mathcal{D}_\mu} U(r)\psi_r \\ \text{and} \qquad \frac{\partial \psi_R}{\partial t} &= i\mathcal{D}_M \nabla_R^2 \psi_R \end{aligned}\right\} \tag{13}$$

respectively.

The last two equations coincide formally with the Schrödinger equations for ψ_r and ψ_R , assuming that $\tau = \hbar$, i.e.,

$$\mathcal{D}_M = \hbar / 2M \quad \text{and} \quad \mathcal{D}_\mu = \hbar / 2\mu .$$

In conclusion we noted that the generalization of the above-mentioned results to the relativistic case needs a special investigation.

References

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