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# GENERALIZED TOPOLOGICAL CHARGES: A SUGGESTION FOR A $\sigma$-MODEL

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## 1. Introduction

Let us consider a n-component field $\varphi(x,t)$ which is a map $R^s \times R \longrightarrow T \subset R^m$ ( $s$ is the dimension of the space, $R^s \times R$ is Minkowski space). If there exists a limit

$$\lim_{\nu \longrightarrow +\infty} \varphi(\nu \xi, t) = a \tag{1}$$

uniformly with respect to all unit vectors $\xi \in S^{s-1}$ and time $t \in I$ for every bounded interval $I \subset R$, the space $R^s$ can be compactified by adding a "point at infinity" and the field $\varphi$ can be continuously extended on the compactified space. The compactified space is homeomorphic to the sphere $S^s$. If the homotopy group $\pi_s(T)$ is nontrivial, the topological charges can be defined as numbers characterizing homotopy class of the map $S^s \rightarrow T$ given by the field /1/.

The class of the initial conditions for the fields leading to the uniform existence of the limit (1) have to be investigated, however. The limit (1) does not exist even for some fields of finite energy at all $\xi \in S^{s-1}$. The existence of the limit (1) was shown in /2/ for almost all $\xi \in S^{s-1}$ only if $\nabla\varphi \in L^2(R^s)$ and $s \geqslant 3$. It was also shown that the limit (1) has the same value at two times for almost all $\xi \in S^{s-1}$ for the fields of locally finite energy under very general assumptions about the interaction /2/. The limit (1) is constant in almost all $\xi \in S^{s-1}$ for $s \geqslant 3$ and in all $\xi$ at which it exists for $s=2$ at a given time /3/. The examples of smooth initial values $\varphi_0$ of fields for which $\nabla\varphi_0 \in L^2(R^s)$ (and even $\varphi_0 \in L^2(R^s)$ ) but the limit

(1) does not exist at some $\xi$ or is nonconstant in $\xi$ can be really given.

We suggest a possibility to generalize the notion of the topological charge to some field configurations which do not define a continuous map $S^s \longrightarrow T$ themselves but can be approximated by a sequence of maps which do it. If the corresponding sequence of topological charges has a limit independent of the choice of the sequence of maps, the limit can be defined as a (generalized) topological charge of the field in question.

## 2. Generalized Topological Charges for a $\sigma$-Model

Let us consider a $\sigma$-model as an example. We choose $T = S^s \subset R^{s+1}$ to have simple hombtopy properties. The number of field components $m = s+1$ and the field satisfies the constraint

$$|\varphi(x,t)| = 1 \; .$$

The Lagrangian of the model is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi)$$

and the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 + \frac{1}{2}(\nabla \varphi)^2 .$$

For the smooth field with constant uniform limit (1) the topological charge defined as the degree of mapping is expressed by the formula

$$\deg(\varphi) = \frac{1}{\omega(S^s)} \int_{R^s} \det(\varphi)\, d^s x \, , \qquad (2)$$

where $\omega(S^s)$ is the Lebesgue measure (surface) of the unit sphere $S^s$ and we denote

$$
\deg(\varphi) = \begin{vmatrix} \varphi^1 & \cdots & \varphi^{s+1} \\ \frac{\partial \varphi^1}{\partial x^1} & \cdots & \frac{\partial \varphi^{s+1}}{\partial x^1} \\ \cdot & \cdots & \cdot \\ \frac{\partial \varphi^1}{\partial x^s} & \cdots & \frac{\partial \varphi^{s+1}}{\partial x^s} \end{vmatrix} .
$$

Let us consider a field $\varphi : R^s \times R \longrightarrow S^s$ such that there is a system of open intervals in $R$ covering $R$ and for every interval $I$ from this system there exists a sequence of fields $\varphi_m : R^s \times I \to S^s$ ($m = 1, 2, \dots$) with continuous first derivatives and the following properties:

(i) the constant $\lim_{r \to +\infty} \varphi_m(r\xi, t) = a$ exists uniformly with respect to $\xi \in S^{s-1}$ and $t \in I$ for every $m = 1, 2, \dots$;

(ii) at all $t \in I$

$$
\lim_{m \to \infty} \varphi_m(x,t) = \varphi(x,t) \quad , \quad \lim_{m \to \infty} \nabla \varphi_m(x,t) = \nabla \varphi(x,t)
$$

for almost all $x \in R^s$ ;

(iii) at all $t \in I$ there exists a function $g_t \in L^s(R^s)$ such that

$$
|\nabla \varphi_m(x,t)| \leqslant g_t(x)
$$

for almost all $x \in R^s$ and all $m = 1, 2, \cdots$ .

Then there exists

$$
\lim_{m \to \infty} \deg(\varphi_m) = \deg(\varphi),
$$

where $\deg(\varphi)$ is given by equation (2). It is a conserved integer number as a limit of such numbers*, independent of the choice of the sequence $\{\varphi_m\}_{m=1}^{\infty}$ of the properties (i-iii). The number $\deg(\varphi)$ can be therefore defined as a generalized topological

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* $\deg(\varphi_m)$ are conserved in time intervals $I$ , $\deg(\varphi)$ is conserved for $t \in R$ consequently.

charge of the field $\varphi$ although it may not have a simple interpretation as a degree of mapping.

The assumptions (i-iii) can be somewhat varied, e.g. the uniformity of the limit at $n \to +\infty$ with respect to $t$ can be replaced by the independence of the majorant $g_t$ on $t$ or directly by continuity of $deg(\varphi)$ given by eq.(2) in $t$. The fields $\varphi_m$ do not need to satisfy the field equations (even they do not need to be continuous in time if $deg(\varphi)$ is continuous for other reasons) in general, although the generalized solutions of field equations $\varphi$ defined by a sequence of smooth solutions $\varphi_m$ might be of interest.

It should be stressed that the existence of the sequence $\{\varphi_m\}_{m=1}^{\infty}$ is our assumption and we did not study the conditions warranting it. The sequence can be easily constructed if $s \geq 2$, the field $\varphi$ has continuous first derivatives, $\nabla\varphi \in L^s(R^s)$ and if there exists a constant $b \in S^s$ and for any bounded interval $I \subset R$ there exist positive numbers $A_I$, $\varepsilon_I$ such that $|\varphi(x,t) - b| > \varepsilon_I$ for $|x| > A_I$ and $t \in I$. The maps $\varphi_m$ are obtained from $\varphi$ by modification of its behaviour (redefinition) at large $|x|$ to obtain a map constant outside a compact subset (ball specifically) of $R^s$.

We can define

$$\varphi_m(x,t) = \psi_m(x,t)\,|\psi_m(x,t)|^{-1} ,$$

where

$$\psi_m(x,t) = \varphi(x,t)\,w_m(|x|) - (1 - w_m(|x|))\,b ,$$

and the function $w_m$ can be chosen in the form

$$u_m(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le a_m \\ \left[1-\left(\ln\frac{r}{a_m}\right)^2\left(\ln\frac{a_{m+1}}{a_m}\right)^{-2}\right]^2 & \text{for} \quad a_m \le r \le a_{m+1} \\ 0 & \text{for} \quad a_{m+1} \le r \end{cases}.$$

The increasing sequence of positive numbers $\{a_m\}_{m=1}^{\infty}$ is taken to satisfy the condition

$$\sum_{m=1}^{\infty} \int_0^{\infty} |u_m'(r)|^s \, r^{s-1} \, dr < +\infty ,$$

i.e.,

$$\sum_{m=1}^{\infty} \ln^{1-s} \frac{a_{m+1}}{a_m} < +\infty .$$

The majorant

$$g_t(x) = \frac{4}{\varepsilon_I}\left(|\nabla\varphi(x,t)| + \sum_{m=1}^{\infty} |u_m'(|x|)|\right).$$

The existence of $b$ is perhaps superfluous assumption since the field $\varphi$ restricted to the sphere of large radius $r$ is a map $S^{s-1}(r) \to S^s$ and therefore homotopically trivial. The possibility of smooth redefinition with $\nabla\varphi_m \in L^s(R^s)$ has to be investigated, however. In $\sigma$-models with the range of field variables $S^m, m < s$, the possibility of the construction of maps $\varphi_m$ constant outside compact sets by modification of the field $\varphi$ at large values of $|x|$ requires really nontrivial assumptions.

## 3. Summary

The notion of the topological charge was generalized to some fields which have not asymptotic behaviour (at spatial infinity) necessary for the compactification of the space but can be approximated by a sequence of well behaved fields. The method is similar to the extension of functionals on the space of functions with compact support to the completed space. The method was discussed for the example of a $\sigma$-model but the general approach can be used for other models as well.

References

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