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ASYMPTOTIC BEHAVIOUR OF THE CLASSICAL SCALAR FIELDS

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1. Introduction

The introduction of conserved topological charges[1] widely used in the soliton and instanton physics is based substantially on the compactification of the space on which fields are defined. This requires the existence of the finite continuous limits of the field variables at the spatial infinity. The existence of the limits (at almost all spatial directions) even constant in time (dynamical charges) was shown by the extensive analysis of Parenti, Strocchi and Velo[2] under very general assumptions for the scalar fields, if they exist at some initial time. Moreover, the limits were shown to exist for all finite-energy scalar fields in the space of dimensions $s=1$ and $s \geqslant 3$ in [2]. It is noticed in the present paper, that their proof for $s=1$ can be used without substantial modifications also in the two-dimensional space ($s=2$). The existence of the limit at spatial infinity of a real scalar field with a finite energy is proved for almost all spatial directions. A simple proof is presented that the limit is constant in time if it exists at some initial time and if the field has a finite conserved energy. In fact a stronger statement that a field with finite conserved energy remains in one Hilbert space sector in the sense of[2] is proved. We offer a simple proof of this fact for those who take the energy conservation as a physical requirement and do not want to study the whole theory of[2]. However, the energy conservation is a nontrivial assumption. It can be proved for the solutions of the field equations in the completed space of the smooth functions with compact support under some conditions on the interaction part of the Lagrangian (potential)[2,3].

The main new result of the present paper is the proof that the limit at spatial infinity of a scalar field of finite energy is constant for almost all spatial directions. This is sufficient for one method of introducing a topological charge which is then trivial (zero) but insufficient for another one.

2. Limits at Spatial Infinity for Fields of Finite Energy

We consider a system of real scalar fields

$$\varphi : R^s \times R \longrightarrow R^m ,$$

$$\varphi(x,t) = \begin{pmatrix} \varphi_1(x,t) \\ \vdots \\ \varphi_m(x,t) \end{pmatrix} , \quad x = (x^1, \dots, x^s)$$

with continuous first derivatives[1]. The Lagrangian of the system is assumed to be of the form

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - U(\varphi)$$

leading to the field equations

$$\Box \varphi_j(x,t) + \frac{\partial}{\partial \varphi_j} U(\varphi(x,t)) = 0 , \quad j = 1, \dots, m$$

and energy

$$E = \int_{R^s} \left[\frac{1}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 + \frac{1}{2} (\nabla \varphi)^2 + U(\varphi) \right] d^s x .$$

The map $U : R^m \to R$ is assumed to be continuous/1/. We shall call it potential (somewhat inaccurately).

The following two propositions are the generalizations of Theorems C.1 and C.2 of/2/ for the case $s > 1$. Stronger results are known for $s \geq 3$ (Lemma 6 of/2/) and for all s if $U(z) = z^2$ (Lemma 5 of/2/).

Proposition 1. Let $m = 1$, M be the set of zero-points of continuous function U, and

[1] The second derivatives of the fields and the first derivative of the potential appear in the field equations but most of the following statements are valid for the field configurations of finite energy regardless of field equations.

$$\int_{-\infty}^{z_0} \sqrt{|U(z)|}\, dz = \int_{z_0}^{+\infty} \sqrt{|U(z)|}\, dz = +\infty \tag{1}$$

for some $z_0 \in R$.

If M is a discrete nonempty set and $\varphi : R^s \to R$ is a function with continuous first derivatives such that

$$\int_{R^s} \left[\tfrac{1}{2} (\nabla\varphi(x))^2 + |U(\varphi(x))| \right] d^s x < +\infty \quad , \tag{2}$$

then there exists

$$\lim_{r \to +\infty} \varphi(r\xi) \in M$$

for almost all unit vectors $\xi \in S^{s-1}$ (for $\xi = \pm 1$ if $s = 1$).

If $M = \emptyset$, then no function $\varphi : R^s \to R$ with continuous first derivatives satisfying (2) exists.

Proof. Let M be at most a discrete set and φ be a function satisfying (2). Then

$$\int_{1}^{+\infty} \left[\tfrac{1}{2} \left(\frac{\partial \varphi(r\xi)}{\partial r} \right)^2 + |U(\varphi(r\xi))| \right] dr < +\infty$$

for almost all $\xi \in S^{s-1}$ and we can repeat the proof of Theorem C.1 from /2/ keeping ξ fixed.

Proposition 2. Let $U : R^m \to R$ be a continuous function,

$$F(\rho) = \left[\min_{|z| = \rho} |U(z)| \right]^{\frac{1}{2}}$$

for $\rho \geq 0$ where $|z| = \left(\sum_{j=1}^{m} z_j^2 \right)^{\frac{1}{2}}$, M be the set of zero-points of the function F , and

$$\int_{0}^{+\infty} F(\rho)\, d\rho = +\infty \quad . \tag{3}$$

If M is a discrete nonempty set and $\varphi : R^s \to R^m$ is a map with continuous first derivatives satisfying the relation (2), then there exists

$$\lim_{r \to +\infty} |\varphi(r\xi)| \in M$$

for almost all $\xi \in S^{s-1}$ (for $\xi = \pm 1$ if $s = 1$).

If $M=\emptyset$, then no map $\varphi: R^s \to R^m$ with continuous first derivatives satisfying (2) exists. Proof is similar as for Proposition 1.

The next proposition is fundamental for the introduction of dynamical and topological charges of scalar fields since it gives sufficient conditions for their conservation.

Proposition 3. Let $\varphi: R^s \times R \to R^m$ be a map with continuous first derivatives and

$$\int_{R^s} \left[\left(\frac{\partial \varphi(x,t)}{\partial t} \right)^2 + \left(\nabla \varphi(x,t) \right)^2 \right] d^s x < C \tag{4}$$

for a finite constant C and all $t \in R$. Then for all $t, t_0 \in R$

$$\int_{R^s} \left[\varphi(x,t) - \varphi(x,t_0) \right]^2 d^s x < +\infty \tag{5}$$

and

$$\lim_{r \to +\infty} \left[\varphi(r\xi, t) - \varphi(r\xi, t_0) \right] = 0 \tag{6}$$

for almost all $\xi \in S^{s-1}$ (for $\xi = \pm 1$ if $s=1$). Especially if the limit $\lim_{r \to +\infty} \varphi(r\xi, t_0)$ or $\lim_{r \to +\infty} |\varphi(r\xi, t_0)|$ exists at some $t_0 \in R$ for almost all $\xi \in S^{s-1}$, then at all $t \in R$

$$\lim_{r \to +\infty} \varphi(r\xi, t) = \lim_{r \to +\infty} \varphi(r\xi, t_0)$$

or

$$\lim_{r \to +\infty} |\varphi(r\xi, t)| = \lim_{r \to +\infty} |\varphi(r\xi, t_0)|$$

for almost all $\xi \in S^{s-1}$.

Proof. By Schwarz inequality,

$$|\varphi(x,t) - \varphi(x,t_0)| = \left| \int_{t_0}^{t} \frac{\partial \varphi(x,\tau)}{\partial \tau} d\tau \right| \leq \left| \int_{t_0}^{t} \left(\frac{\partial \varphi}{\partial \tau} \right)^2 d\tau \right|^{\frac{1}{2}} |t - t_0|^{\frac{1}{2}} .$$

Now, according to (4),

$$\int_{R^s} \left[\varphi(x,t) - \varphi(x,t_0) \right]^2 d^s x \leq C (t - t_0)^2$$

and (5) is proved. Eq.(6) then follows, e.g., from Proposition 2 applied to the map $\varphi(x,t) - \varphi(x,t_0)$ and the potential $V(z) = z^2$.

Remark. In Proposition 3, we gave the condition (4) **evidently** valid for the fields of finite conserved energy if the potential is non-negative. Weaker assumptions

$$\left| \int_{t_0}^{t} \int_{R^s} \left(\frac{\partial \varphi(x,\tau)}{\partial \tau} \right)^2 d^s x \ d\tau \right| < +\infty$$

and

$$\int_{R^s} \left[\nabla \varphi(x,t) - \nabla \varphi(x,t_0) \right]^2 d^s x < +\infty$$

were used in the proof.

We gave the sufficient conditions for the existence of the limit of the field for almost all directions at spatial infinity. Now we show that this limit has the same value at almost all directions for $s \geq 2$.

Proposition 4. Let $\varphi : R^s \to R^m$ $(s \geq 2)$ be a map with continuous first derivatives, $\nabla \varphi \in L^2(R^s)$.

If $s \geq 3$, then there exists a constant $a \in R^m$ such that

$$\lim_{r \to +\infty} \varphi(r\xi) = a \tag{7}$$

for almost all $\xi \in S^{s-1}$.

If $s = 2$, then the limit (7) (or $\lim_{r \to +\infty} |\varphi(r\xi)|$) has the same (finite or infinite) value at all $\xi \in S^1$ for which it exists.

Proof. It is sufficient to give the proof for $m = 1$ and then to apply the result to each component of φ separately (the proof for $\lim_{r \to +\infty} |\varphi(r\xi)|$ in the case $s = 2$ is also similar). Let us start with the simpler case $s = 2$. If there exist $\lim_{r \to +\infty} \varphi(r\xi_1) < \lim_{r \to +\infty} \varphi(r\xi_2)$ for some $\xi_1, \xi_2 \in S^1$ with polar angles $\omega_1 \neq \omega_2$, then there exist $r_0 \geq 0$ and numbers $b, c \in R$ such that

$$\varphi(r\xi_1) < b < c < \varphi(r\xi_2)$$

for all $r > r_0$. Integrating over polar angle ω , we have now for all $r > r_0$

$$\int_0^{2\pi} (\nabla \varphi)^2 \, d\omega \geq \frac{1}{r^2} \left| \int_{\omega_1}^{\omega_2} \left(\frac{\partial \varphi}{\partial \omega} \right)^2 d\omega \right| > \frac{(c-b)^2}{r^2 |\omega_2 - \omega_1|}$$

because of Schwarz inequality

$$|\varphi(r\xi_2)-\varphi(r\xi_1)| = \left|\int_{\omega_1}^{\omega_2} \frac{\partial\varphi}{\partial\omega}\, d\omega\right| \le |\omega_2-\omega_1|^{\frac{1}{2}} \left|\int_{\omega_1}^{\omega_2} \left(\frac{\partial\varphi}{\partial\omega}\right)^2 d\omega\right|^{\frac{1}{2}} .$$

The contradiction with the assumption $\int (\nabla\varphi)^2\, r\, dr\, d\omega < +\infty$ is thus obtained.

The same argument is used for $s \geq 3$, but it is applicable after some preliminary considerations only. Let us first realize that the finite limit in the left-hand side of eq.(7) exists for almost all $\xi \in S^{s-1}$ according to Lemma 6 of /2/ and that it is a measurable function of ξ as a limit of measurable functions. It can be seen now that either the statement (7) is valid or there exist numbers $b < c$ such that the sets

$$N = \{\xi \in S^{s-1} \mid \lim_{r \to +\infty} \varphi(r\xi) < b\} ,$$

$$L = \{\xi \in S^{s-1} \mid \lim_{r \to +\infty} \varphi(r\xi) > c\}$$

have positive measures in S^{s-1}. Let us assume that the last possibility takes place. The set N can be rotated to the position $\mathcal{R}N$ where it has an intersection of nonzero measure with the set L:

$$\mu(L \cap \mathcal{R}N) > 0 ,$$

where μ is Lebesgue measure on S^{s-1} and $\mathcal{R} \in SO(s)$ is a suitable rotation. The existence of $\mathcal{R}$ is proved in Appendix. We can choose such an orthogonal system of coordinates that the matrix $\mathcal{R}$ has a canonical form /4/

$$\mathcal{R} = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & & & & & \\ \sin\alpha_1 & \cos\alpha_1 & & & & & \\ & & \ddots & & & & \\ & & & \cos\alpha_k & -\sin\alpha_k & & \\ & & & \sin\alpha_k & \cos\alpha_k & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} ,$$

where $0 < |\alpha_\ell| \leq \pi$ for $\ell = 1, \ldots, k$.

Let us introduce in R^s a system of coordinates similar to the cylindrical and spherical ones by the equations:

$$x^1 = \rho_1 \cos\omega_1 \quad , \ x^2 = \rho_1 \sin\omega_1 \ ,$$

$$\dots .$$

$$x^{2k-1} = \rho_k \cos\omega_k \ , \ x^{2k} = \rho_k \sin\omega_k \ ,$$

$$\left.\begin{aligned} \rho_1 &= r \sin\vartheta_1 \dots \sin\vartheta_{s-k-2} \sin\vartheta_{s-k-1} \ , \\ \rho_2 &= r \sin\vartheta_1 \dots \sin\vartheta_{s-k-2} \cos\vartheta_{s-k-1} \ , \\ &\dots . \\ \rho_k &= r \sin\vartheta_1 \dots \sin\vartheta_{s-2k} \cos\vartheta_{s-2k+1} \ , \end{aligned}\right\} \begin{array}{l} \text{valid for} \\ k>1 \text{ only} \end{array}$$

$$x^{2k+1} = r \sin\vartheta_1 \dots \sin\vartheta_{s-2k-1} \cos\vartheta_{s-2k} \ ,$$

$$\dots .$$

$$x^s = r \cos\vartheta_1 \ ,$$

where $r \geq 0$, $0 \leq \vartheta_i \leq \pi$ $(i=1,\dots,s-2k)$, $0 \leq \vartheta_j \leq \frac{\pi}{2}$ $(j=s-2k+1,\dots,s-k-1)$, $0 \leq \omega_\ell \leq 2\pi$ $(\ell=1,\dots,k)$. For $k=1$ (this is certainly the case for $s=3$) we obtain the usual spherical coordinates. In the following, all real values of ω_ℓ are allowed (or the formulas should be understood as valid by mod 2π). The rotation by matrix $\mathbb{R}$ is the transformation

$$\omega_\ell \mapsto \omega_\ell + \alpha_\ell \qquad (\ell = 1,\dots,k). \tag{8}$$

Let us denote

$$\sigma = \frac{1}{r} \left[\sum_{\ell=1}^{k} \rho_\ell^2 \alpha_\ell^2 \right]^{\frac{1}{2}} > 0$$

(if at least one $\rho_\ell > 0$). There exists an orthogonal matrix $\mathcal{O}$ (dependent on angles ϑ) of dimension $k \times k$ such that

$$\frac{1}{r} \begin{pmatrix} \rho_1 \alpha_1 \\ \rho_2 \alpha_2 \\ \vdots \\ \rho_k \alpha_k \end{pmatrix} = \mathcal{O} \begin{pmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

We shall use the variables ω'_ℓ defined by the equation

$$\begin{pmatrix} \omega_1' \\ \vdots \\ \omega_k' \end{pmatrix} = \frac{1}{r} \, O^{-1} \begin{pmatrix} \rho_1 \omega_1 \\ \vdots \\ \rho_k \omega_k \end{pmatrix}$$

instead of ω_ℓ. The transformation (8) becomes a translation in one variable

$$\omega'_1 \longmapsto \omega'_1 + \delta .$$

Let us denote

$$N_1 = \mathcal{R}^{-1} (L \cap \mathcal{R} N).$$

Now

$$N_1 \subset N , \ \mathcal{R} N_1 \subset L , \ \mu(N_1) > 0 .$$

There exist $\omega_{10}, \dots , \omega_{k0}$ such that $\mu(N_2) > 0$, where N_2 is the part of N_1 contained in the region described by the inequalities

$$|\omega_\ell - \omega_{\ell 0}| < \frac{\pi}{2} \qquad (\ell = 1, \dots, k).$$

There exists ω'_{10} such that

$$\mu_{s-2}(N_2^*) = \int_{N_2^*} \sigma(\vartheta) \, d^{s-k-1}\vartheta \, d\omega_2' \dots d\omega_k' > 0 ,$$

where

$$N_2^* = \left\{ (\vartheta_1, \dots, \vartheta_{s-k-1}, \omega_2', \dots, \omega_k') \,\middle|\, \xi(\vartheta_1, \dots, \vartheta_{s-k-1}, \omega_{10}', \omega_2', \dots, \omega_k') \in N_2 \right\}$$

($\xi(\vartheta, \omega')$ is the unit vector described by the variables ϑ, ω') and

$$\sigma(\vartheta) = \sin^{s-k-2}\vartheta_1 \dots \sin \vartheta_{s-k-2} .$$

Using our coordinates, we have

$$\int_{R^s} (\nabla\varphi)^2 d^s x \geq \int_{N_2^*} \left\{ \int_0^\infty \left[\int_{\omega_{10}'}^{\omega_{10}'+\delta} \left(\frac{\partial \varphi}{\partial \omega_1'}\right)^2 d\omega_1' \right] r^{s-3} \, dr \right\} \sigma(\vartheta) \, d^{s-k-1}\vartheta \, d\omega_2' \dots d\omega_k' .$$

It can be shown that the last integral is infinite similarly as in the case $s = 2$. The Proposition 4 is completely proved now.

3. Conclusions

Various assumptions of our Propositions in Sect. 2 are valid for the fields of finite conserved energy if the potential $U(z) \geq 0$

$(z \in R^m)$. The existence of a finite limit of the field at spatial infinity for almost all directions was proved. If the number of the components of the field $m > 1$ and the space dimension $s = 1, 2$, the existence of $\lim_{r \to +\infty} |\varphi(r\xi)|$ is proved only $(\xi \in S^{s-1})$. The mentioned limit is constant in time and in almost all spatial directions. Therefore the dynamical charge defined in /2/ as $\lim_{r \to +\infty} \varphi(r\xi)$ is constant in $\xi \in S^{s-1}$ for scalar fields of finite energy (this is the main new result of the present paper, formerly expected by the heuristic arguments only /1/) and can be redefined on the set of zero measure (at every time) to become a continuous (constant) function. The topological charge defined as the homotopy class of (redefined) $\lim_{r \to +\infty} \varphi(r\xi)$ is then trivial for $s \geq 2$. Our Propositions can be used also for the fields having finite energy difference from the constant field. The shift of the potential U by a constant is needed here only.

The fact that a large class of finite-energy (finite-action in Euclidean case) fields have the limit at the infinity constant in time and all directions is often used. It approves another definition of the topological charge based on the compactification of the space by adding a "point at infinity" /5/. The Euclidean space R^s becomes topologically equivalent to the sphere S^s after the compactification, and the topological charge is defined as the homotopy class (or some number characterizing it) of the whole field defined on S^s . The finiteness of energy is not sufficient for such compactification, since the uniform existence of the constant $\lim_{r \to +\infty} \varphi(r\xi)$ for all $\xi \in S^{s-1}$ is needed. The rigorous conditions sufficient for the compactification should be found. Can they be formulated as some requirements on the asymptotic behaviour of the initial conditions? A more detailed treatment of the field equations should give an answer.

The generalization of our results to gauge fields would be extremely interesting, but the Hamiltonians of gauge fields and scalar fields differ substantially.

Appendix

We shall prove two lemmas. The second lemma is needed in the proof of Proposition 4, the first one serves to the proof of the second one. The first lemma is not new, its special case we need is used, e.g., in exercise 11 to §3 of [6]. We give a simple proof here for completeness.

Lemma 1. Let G be a locally compact Lie group of differentiable transformations transitive on the finite dimensional differentiable manifold S, μ be a measure on S invariant with respect to the transformations of G, $0 < \mu(S) < +\infty$, ν be the right invariant measure on the group G. Then for every function f integrable on S and every point $\xi \in S$

$$\int_G f(R\xi)\, d\nu(R) = \frac{\nu(G)}{\mu(S)} \int_S f(\eta)\, d\mu(\eta).$$

Proof. By the substitution $\eta' = R\eta$, $R' = R$ we obtain

$$\iint_{S\times G} f(R\eta)\, d\mu(\eta)\, d\nu(R) = \nu(G) \int_S f(\eta')\, d\mu(\eta')$$

(expressing integrals as ones over the parameters of S and G, the same Jacobian appears as in the equation

$$\int_S f(R\eta)\, d\mu(\eta) = \int_S f(\eta')\, d\mu(\eta')$$

valid by the assumption; theorems on homogeneous spaces can also be used here [6]). Since G is transitive on S, to all $\xi, \eta \in S$

there exists a $R_{\xi\eta} \in G$ such that $\eta = R_{\xi\eta}\xi$. Using the fact that ν is the right invariant measure on G, we obtain

$$\iint_{S\times G} f(R\eta)\, d\omega(\eta)\, d\nu(R) = \int_S \int_G f(R R_{\xi\eta}\xi)\, d\nu(R)\, d\omega(\eta) =$$

$$= \omega(S) \int_G f(R\xi)\, d\nu(R)\,.$$

By comparison of the two expressions for the double integral, the statement of Lemma 1 follows.

Lemma 2. Let the assumptions of Lemma 1 be valid, $\nu(G) > 0$, $L \subset S$, $N \subset S$, $\omega(L) > 0$, $\omega(N) > 0$. Then there exists a transformation $R \in G$ such that

$$\omega(L \cap RN) > 0\,.$$

Proof. Let us denote by f, g, χ_T the characteristic functions of the sets $N, L, L \cap T^{-1}N$ for every $T \in G$. Then

$$\chi_T(\xi) = g(\xi) f(T\xi)$$

for $\xi \in S$ and

$$\iint_{S\times G} \chi_T(\xi)\, d\omega(\xi)\, d\nu(T) = \frac{\nu(G)}{\omega(S)}\, \omega(L)\, \omega(N) > 0$$

by Lemma 1. Therefore there exists $T \in G$ such that

$$\omega(L \cap T^{-1}N) = \int_S \chi_T(\xi)\, d\omega(\xi) > 0$$

and it is sufficient to put $R = T^{-1}$.

In the proof of Proposition 4 we use Lemma 2 for $S = S^{s-1}$ and $G = SO(s)$.

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