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THE ASYMPTOTIC BEHAVIOUR OF FORM FACTORS

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Асимптотическое поведение формфакторов

С помощью метода двойной перенормировки /С.А.Аникин, О.И.Завьялов/ доказывается тождество для перенормированного оператора тока. Из этого следует нелинейное соотношение для формфактора и новое уравнение ренормгруппы для формфактора, где внешние моменты фиксированы на точке перенормировки. Метод применяется к теории $\phi^3_{(6)}$ и хромодинамике. В последнем случае однопетлевое вычисление дает для электромагнитного формфактора кварков поведение $\exp(-C \ln\ln Q^2 \ln Q^2)$. Если из этого результата выделять ведущие логарифмические члены $(g^2 \ln^2 Q^2)^n$, тогда можно получить поведение $\exp(-g^2 C' \ln^2 Q^2)$. Включение двухпетлевых вычислений показывает, что первый результат изменится, а результат, касающийся ведущих логарифмических членов, сохраняется.

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The Asymptotic Behaviour of Form Factors

Similar to the method of S.A.Anikin and O.I.Zavialov a modified renormalization procedure is applied to the already renormalized current operator. This allows the derivation of an identity

1. INTRODUCTION

The determination of the asymptotic behaviour of form factors is an old - up to now not satisfactory solved problem. There are a lot of methods to attack this problem /1/. The oldest one consists in a direct summation /2/ of the leading terms of all Feynman diagrams in each perturbation order. Another possibility is the usual renormalization group equation /3/. However this method works without difficulties only when all momenta tend to infinity. In our case two momenta are partly fixed $p^2 = \text{const}$, $p'^2 = \text{const}$, this leads to infrared difficulties in the solution of the renormalization group equation. Therefore the simplest diagrams do not determine the asymptotic behaviour, infinitely many diagrams must be taken into account. Here we apply another method which we have learned from the approximation procedure for the renormalization group equations for light-cone coefficients /4/. In principle the method works as follows. We start with the usual renormalization group equation for the form factor and add additional derivatives with respect to p^2

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \delta\frac{\partial}{\partial m^2} - \gamma_2 - \gamma_j + 2p^2\frac{\partial}{\partial p^2} - 2p^2\frac{\partial}{\partial p^2}\right] F_2(q^2, p^2, \mu^2, g, m) = 0. \tag{1.1}$$

For simplicity the squared external momenta p^2 and p'^2 are identified and fixed $p^2 = p'^2 = \mu^2$. An essential step is the treatment of this quantity as a renormalization point. Furthermore we prove

$$2p^2\frac{\partial}{\partial p^2} F_2(q^2, p^2, \mu^2, g, m)\Big|_{p^2=\mu^2} = -(\tilde{\gamma}_{(q^2)} + \gamma_2) F_2(q^2, \mu^2, \mu^2, g, m), \tag{1.2}$$

so that as final equation appears

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \delta\frac{\partial}{\partial m} - \gamma_j + \tilde{\gamma}_{(q^2)}\right) F_2(q^2, \mu^2, \mu^2, g, m) = 0. \tag{1.3}$$

This equation is the starting point for all further physical investigations. The main part of the present work is the proof of this relation. It is achieved with an unusual subtraction procedure quite analogously to the subtraction scheme introduced by S.A.Anikin and O.I.Zavialov for the proof of the light-cone expansion[5]. At first relation (1.3) is proved for scalar field theory, later on for gauge theories as QED and QCD. The calculations of the anomalous dimensions are carried out in the fourth section. For the scalar field theory $\phi^3_{(6)}$ it is shown that the anomalous dimension $\tilde{\gamma}(q^2)+\gamma_2$ vanishes at large q^2 in all orders of perturbation theory. For QCD the anomalous dimensions are calculated in fourth and second order in the coupling constant. They increase logarithmically for large momenta. In the following section the renormalization group equation is solved for both theories. In scalar field theory $\phi^3_{(6)}$ the form factor shows the same asymptotic behaviour for both limiting procedures $q^2\to-\infty$, $p^2=p'^2\to-\infty$ and $q^2\to-\infty$, $p^2=p'^2=\mu^2$. QCD behaves in both limits differently. The interpretation of the solutions of the renormalization group equation for fixed p^2 is more complicated. By taking into account the one-loop calculation for the anomalous dimension the well known result $\exp(-c\ln(-q^2)\ln\ln(-q^2))$ is obtained. However this behaviour is strongly modified if two-loop calculations are taken into account. Nevertheless it is not excluded that this result reflects the true behaviour of the form factor because the main contributions that change this behaviour come from the wrong integration boundary. On the other hand, it is possible to look at the solution of the renormalization group equation as an identity in the renormalized coupling constant. Picking out the leading terms, this means the powers of $g^2\ln^2(-q^2)$ on the right hand side, then as a result the famous Sudakov summation formula appears. This result is certainly modified by the inclusion of non leading terms $g^n\ln^{n-1}(-q^2)$, $g^n\ln^{n-2}(-q^2)$,.. as it is indicated by the behaviour $\exp(-c\ln(-q^2)\ln\ln(-q^2))$.

2. SUBTRACTION OPERATORS AND RENORMALIZATION GROUP EQUATIONS IN SCALAR FIELD THEORY

For simplicity all problems are discussed in scalar field theory at first. As current operator we choose

$$j(q)=\frac{1}{2!}\int dp\,dp'\,\delta(q+p+p'):\ \phi(p)\,\phi(p'):. \tag{2.1}$$

Perturbation theoretically calculated quantities as $TE_0(s)$, $T(jE_0(s))$; $E_0(s)=\exp s$, $E_1(s)=E_0(s)-1$, $s=i\int\mathcal{L}_{int}dx$ are renormalized with the help of the Bogolubov R operation. The symbol T of time ordered products is omitted everywhere. The renormalization procedure is formalized with the help of the technique developed by S.A.Anikin, M.C.Polivanov and O.I.Zavialov[6]. In their notation the renormalized current operator j is given by

$$R(j(q)\,E_0(s))=E_0(s_r)\frac{1}{1+ME_1(s_r)}j(q), \tag{2.2}$$

s_r renormalized action, whereby M denotes subtraction operators. The renormalization of the S-matrix graphs contained in $E_0(s_r)$ has already been performed at subtraction points μ^2 different from zero. The subtraction operator M acts on coefficient functions containing the operator vertex j

$$F(q)=\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\int dp_1\dots dp_\ell F_\ell(q,p_1,\dots p_\ell)\,\delta(q+\Sigma p_i):\phi(p_1)\dots\phi(p_\ell):$$

$$\begin{aligned} MF(q)&=\frac{1}{2!}\int dp\,dp'\,F_2^{1PI}(\mu^{\ell}_{ij})\,\delta(q+p+p'):\phi(p)\,\phi(p'):\\ &+\int dp\,(F_1^{1PI}(\mu^2)+(q^2-\mu^2)\,F_1^{1PI'}(\mu^2))\delta(q+p):\phi(p):\\ &=F_2^{1PI}(\mu_{ij})\,j(q)+(F_1(\mu^2)+(q^2-\mu^2)\,F_1^{1PI'}(\mu^2)):\phi(-q):. \end{aligned} \tag{2.3}$$

Additionally to this subtraction operator we define the subtraction operator $\mathfrak{M}$ acting on the same functional. It defines a different unusual renormalization procedure for the external current operator.

$$\begin{aligned} \mathfrak{M}F(q)&=\frac{1}{2}\int dp\,dp'\,F_2^{1PI}(q^2,\mu^2,\mu^2)\,\delta(q+p+p'):\phi(p)\,\phi(p'):\\ &+\int dp\,F_1^{1PI}(q^2)\,\delta(q+p):\phi(p):\\ &=F_2^{1PI}(q^2,\mu^2,\mu^2)\,j(q)+F_1^{1PI}(q^2):\phi(-q):. \end{aligned} \tag{2.4}$$

In these definitions we used

$$F_1'(\mu^2) = \frac{\partial}{\partial q^2} F_1(q^2)\Big|_{q^2=\mu^2}, \quad \mu_{ij}^{\ell} = \mu^2 \frac{((\ell+1)\delta_{ij}-1)}{\ell}, \quad \mu^2 < 0.$$

The subtraction operator $\mathfrak{M}$ is used to establish an identity for the renormalized current operators

$$\begin{aligned} R(j(q)\,E_0(s)) &= E_0(s_r)\frac{1}{1+ME_1(s_r)}j(q) \\ &= E_0(s_r)\frac{1}{1+\mathfrak{M}E_1(s_r)}(1+\mathfrak{M}(E_0(s_r)-1))\frac{1}{1+ME_1(s_r)}j(q) \\ &= E_0(s_r)\frac{1}{1+\mathfrak{M}E_1(s_r)}\mathfrak{M}E_0(s_r)\frac{1}{1+ME_1(s_r)}j(q) \\ &+ E_0(s_r)\frac{1}{1+\mathfrak{M}E_1(s_r)}(1-\mathfrak{M})\frac{1}{1+ME_1(s_r)}j(q)\,. \end{aligned} \quad (2.5)$$

If simple actions of the subtraction operators are taken into account

$$\begin{aligned} &\mathfrak{M}j(q) = \mathfrak{M}\int dp\, dp'\, \delta(q+p+p') : \phi(p)\,\phi(p')\frac{1}{2!} = j(q) \\ &Mj(q) = j(q), \qquad \mathfrak{M}M\,F(q) = MF(q) \end{aligned} \quad (2.6)$$

then

$$(1-\mathfrak{M})\frac{1}{1+ME_1(s_r)}j(q) = (1-\mathfrak{M})\sum_{k=0}^{\infty}(-ME_1(s_r))^k j(q) \equiv 0.$$

So, as final result the following identity is obtained

$$E_0(s_r)\frac{1}{1+ME_1(s_r)}j(q) = E_0(s_r)\frac{1}{1+\mathfrak{M}E_1(s_r)}\mathfrak{M}(E_0(s_r)\frac{1}{1+ME_1(s_r)}j(q)). \quad (2.7)$$

With the notation

$$\begin{aligned} F(q) &= R(j(q)E_0(s)) = E_0(s_r)\frac{1}{1+ME_1(s_r)}j(q) \\ \tilde F(q) &= \tilde R(j(q)E_0(s)) = E_0(s_r)\frac{1}{1+\mathfrak{M}E_1(s_r)}j(q) \end{aligned} \quad (2.8)$$

this relation takes the form

$$R(j(q)E_0(s)) = \tilde R(E_0(s)[\mathfrak{M}R(j(q)E_0(s)])\,) \quad (2.9)$$

R denotes the usual renormalization procedure. $\tilde R$ is a modified renormalization procedure which applies the operator $\mathfrak{M}$ for graphs or subgraphs containing the operator vertex j .
Now it is possible to extract relations concerning the form factor itself. Taking into account eq. (2.4) we obtain

$$R(j(q)E_0(s)) = F_2^{1PI}(q^2,\mu^2,\mu^2)\tilde R(j(q)E_0(s)) + F_1^{1PI}(q^2)\tilde R(\phi(-q)E_0(s)) \quad (2.10)$$

which means for $\ell = 2$

$$F_2^{1PI}(q^2,p^2,p'^2) = F_2^{1PI}(q^2,\mu^2,\mu^2)\tilde F_2^{1PI}(q^2,p^2,p'^2). \quad (2.11)$$

The relations for the form factor appear here as relations for coefficient functions with two external legs. F_2 is the normal in standard manner renormalized form factor. By eq. (2.11) it is connected with the form factor $\tilde F_2$ which is renormalized in a more complicated manner. From this equation it follows that also $\tilde F_2$ is a finite - therefore renormalized quantity. Of course a general proof of the renormalization properties of the $\tilde R$ operation must be considered, but this is not done here. If now the standard machinery of renormalization is applied, then we have automatically renormalization group equations for the usually renormalized form factor

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \delta\frac{\partial}{\partial m} - \gamma_2 - \gamma_j\right) F_2^{1PI}(q^2,p^2,p'^2) = 0 \quad (2.12)$$

$$\gamma_j + \gamma_2 = -2\left(q^2\frac{\partial}{\partial q^2} + p^2\frac{\partial}{\partial p^2} + p'^2\frac{\partial}{\partial p'^2}\right)\tilde F_2^{1PI}(q^2,p^2,p'^2)\Big|_{q^2=p^2=p'^2=\mu^2} \quad (2.13)$$

and for the form factor $\tilde{F}_2$

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\delta\frac{\partial}{\partial m}-\gamma_2-\tilde{\gamma}_j\right)\tilde{F}_2^{1PI}(q^2,p^2,p'^2)=0 \qquad (2.14)$$

$$\tilde{\gamma}_j+\gamma_2=-2\left(p^2\frac{\partial}{\partial p^2}+p'^2\frac{\partial}{\partial p'^2}\right)\tilde{F}_2^{1PI}(q^2,p^2,p'^2)\Big|_{p^2=p'^2=\mu^2} . \qquad (2.15)$$

Both expressions (2.13),(2.14) are special cases of generally valid expressions[7].

Let us now discuss the main problem to be solved. The main task is the determination of the asymptotic behaviour of the form factor F_2^{1PI} for $q^2\to-\infty$, $p^2=p'^2=$ fix .. . Equation (2.12) is exactly valid but in most cases useless because its solutions contain infrared divergent diagrams. So we want to derive directly a renormalization group equation for $F_2^{1PI}(q^2,\mu^2)$ that does not have these problems. The starting equation is (2.11). Taking into account eq.(2.12) and eq.(2.14) and calculating

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\delta\frac{\partial}{\partial m}-\gamma_2-\gamma_j\right)F_2^{1PI}(q^2,\mu^2,\mu^2)\,F_2^{1PI}(q^2,p^2,p'^2)=0$$

we obtain

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\delta\frac{\partial}{\partial m}-\gamma_i+\tilde{\gamma}_j\right)F_2^{1PI}(q^2,\mu^2,\mu^2)=0. \qquad (2.16)$$

This equation can be solved directly and applied for physical problems. The remaining task consists in the calculation of the modified anomalous dimension $\tilde{\gamma}_j$ and a discussion of its solutions. It should be remarked that similar equations have been written down earlier[8] without proof.

3. SUBTRACTION OPERATORS AND RENORMALIZATION GROUP EQUATIONS IN GAUGE FIELD THEORY

At thirst we consider the case of QED. As functional we introduce

$$F_\mu(q)=\sum\frac{1}{(\ell!)^2}\frac{1}{k!}\int dp_1\ldots dp_\ell\, dp'_1\ldots dp'_\ell\, dk_1\ldots dk_k\, F_{\mu\nu_1\ldots\nu_k}:\bar{\psi}(p_1)\ldots\psi(p'_\ell)A_{\nu_1}\ldots A_{\nu_k}: \qquad (3.1)$$

The operators M and $\mathfrak{M}$ must be defined on 1PI diagrams that are renormalization parts only. The operators needed for the renormalization of the S-matrix $E_0(s_r)$ must not be defined explicitly because the renormalization program done with these operators is already carried out. Let us turn to the definition of the operator M which is used to renormalize the current operator in eq.(2.2). As current operator we choose the electromagnetic current

$$j_\mu=-e:\bar{\psi}(0)\gamma\,\psi(0):=c\int dp\,dp'\,\delta(p+p'+q):\bar{\psi}(p)\gamma\,\psi(p'): \qquad (3.2)$$

(In QCD the flavour structure of the current does not change anything essentially, so it is suppressed).

Renormalization parts are diagrams with two external Fermion lines or one external photon line. In electrodynamics these diagrams are identical to the self-energy diagram and the vertex part. Graphs with two external photon line vanish because of Furry's theorem. By the consideration of the current operator as an external operator it is possible to define its renormalization independently. For simplicity we define the subtraction operators on the kinematical invariants directly. These are

$$F_{2\mu}=c\gamma_\mu F_2(q^2,p^2,p'^2)+\sum k^i_\mu(q,p,p')\,G^i(q^2,p^2,p'^2), \qquad (3.3)$$

(two Fermion lines)

$$F_{\mu\nu}=(g_{\mu\nu}q^2-q_\mu q_\nu)\,B(q^2) \qquad (3.4)$$

(one photon line)
The kinematical factors k^i_μ depend explicitly on the momenta, so they need no subtractions finally. So we define

$$\begin{aligned} MF_\mu(q)&=\int dp\,dp'\,F_2^{1PI}(q^2,p^2,p'^2)\Big|_{\substack{p^2=p'^2=\mu^2\\ q^2=0}} c:\bar{\psi}(p')\gamma_\mu\psi(p):\delta(q+p+p') \\ &\quad+(g_{\mu\nu}q^2-q_\mu q_\nu)\,B(0):A_\nu(-q): \\ &=F_2(0,\mu^2,\mu^2)\,j_\mu+(g_{\mu\nu}q^2-q_\mu q_\nu)\,B(0):A_\nu(-q): \end{aligned} \qquad (3.5)$$

and

$$\mathfrak{M} F_\mu(q) = \int dp\, dp'\, F_2^{1PI}(q^2, p^2, p'^2)\Big|_{p^2=p'^2=\mu^2} : \bar\psi(p)\, \gamma_\mu\, \psi(p'): \delta(q+p+p')\cdot c$$

$$+ (g_{\mu\nu} q^2 - q_\mu q_\nu) B(q^2) : A_\nu(-q) : \tag{3.6}$$

$$= F_2^{1PI}(q^2, \mu^2, \mu^2) j_\mu + (g_{\mu\nu} q^2 - q_\mu q_\nu) B(q^2) : A_\nu(-q) : .$$

With the given definitions all relations (2.6) are valid, so that the formal calculations of the foregoing section remain. The important relation (2.10) reads now

$$R(j_\mu(q) E_0(s)) = F_1^{1PI}(q^2, \mu^2, \mu^2) \tilde R(j_\mu(q) E_0(s)) + (g_{\mu\nu} q^2 - q_\mu q_\nu) B(q^2) \tilde R(A_\nu(-q) E_0(s)). \tag{3.7}$$

From this we extract for $\ell = 2$, $k = 0$

$$F_2^{1PI}(q^2, p^2, p'^2) = F_2^{1PI}(q^2, \mu^2, \mu^2)\, \tilde F_2^{1PI}(q^2, p^2, p'^2). \tag{3.8}$$

In similar manner also renormalization group equations can be written down.

Let us now turn to QCD. In this case the functional F_μ is more complicated because we have to take into account ghost operators too, also if they are not contained in the original operators. A renormalization expression of the bare current operators. A renormalization part is the diagram with two external Fermion lines. All other diagrams containing one electromagnetic vertex are not divergent or vanish because of the different group structure of the electromagnetic current with respect to the colour group. For example, diagrams with two external gluon lines are not allowed. According to Furry's theorem it follows that the representation of the colour group built up from the two external gluons does not contain the identity needed for the contraction with the external current operator. Thus it is possible to choose as subtraction operators

$$M F_\mu = \int dp\, dp'\, F_2^{1PI}(q^2, p^2, p'^2)\Big|_{\substack{p^2=p'^2=\mu^2 \\ q^2=0}} c : \bar\psi(p')\, \gamma_\mu \psi(p) : \delta(q+p+p')$$

$$= F_2^{1PI}(0, \mu^2, \mu^2)\, j_\mu(q)$$

$$\mathfrak{M} F_\mu = F_2^{1PI}(q^2, \mu^2, \mu^2)\, j_\mu(q).$$

Again all earlier used relations are fulfilled. The equation for the Dirac form factor takes the form (3.8). The renormalization group equations include derivatives with respect to the gauge parameter α in general.

4. CALCULATION OF THE ANOMALOUS DIMENSIONS

The anomalous dimension $\tilde\gamma_j$ must be calculated starting with its definition (2.15)

$$\tilde\gamma_j + \gamma_2 = -2\left(p^2 \frac{\partial}{\partial p^2} + p'^2 \frac{\partial}{\partial p'^2}\right) \tilde F_2^{1PI}(q^2, p^2, p'^2)\Big|_{p^2=p'^2=\mu^2}.$$

To apply standard calculations we take into account eq. (3.8). Performing the p^2 differentiation we obtain an equivalent definition

$$\tilde\gamma_j + \gamma_2 = -\frac{2\left(p^2 \frac{\partial}{\partial p^2} + p'^2 \frac{\partial}{\partial p'^2}\right) F_2^{1PI}(q^2, p^2, p'^2)}{F_2^{1PI}(q^2, p^2, p'^2)}\Bigg|_{p^2=p'^2=\mu^2}. \tag{4.1}$$

For our purpose it is sufficient to know its asymptotic behaviour for $q^2 \to -\infty$.

For the scalar theory it is possible to show $\tilde\gamma_j + \gamma_2 \to 0$ $(q^2 \to -\infty)$ for all orders of perturbation theory. It is achieved by a discussion of the α-representations of all possible diagrams /9/

$$F_2 = R \sum_{\text{Graphs}} \int \frac{d\alpha_1 \ldots d\alpha_L}{D^3(\alpha)} \exp i\left(\frac{Q_1(\alpha) q^2}{D(\alpha)} + \frac{Q_2(\alpha) p^2}{D(\alpha)} + \frac{Q_3(\alpha) p'^2}{D(\alpha)} + \sum \alpha_i(-m^2 + i\epsilon)\right) \tag{4.2}$$

As interesting quantity we have to discuss $\frac{\partial}{\partial p^2} F_2$

$$R \int \frac{d\alpha_1 \ldots d\alpha_L}{D^3(\alpha)} \frac{Q_2(\alpha)}{D(\alpha)} \exp i\left(\frac{Q(\alpha)q^2}{D(\alpha)} + \frac{Q_2(\alpha)p^2}{D(\alpha)} + \frac{Q_3(\alpha)p'^2}{D(\alpha)} + \sum \alpha_i(-m^2+i\epsilon)\right). \quad (4.3)$$

The factor $Q_2(\alpha)/D(\alpha)$ lowers the degree of the graph itself. This must be taken into account by the application of the method /10/ for an estimate of the large q^2-behaviour. The rules given there consist in counting the so-called essential subgraphs. These are graphs that are crossed by each q^2-cut (cut that separates the q^2-vertex from the p^2 and p'^2 vertices), that have maximal divergence degree and do not contain the forbidden divergent subgraphs. Then the maximal divergence degree determines the asymptotic power, the powers of logarithms are determined more complicated by the number of essential subgraphs. In our case all subgraphs of 1PI-diagrams that are crossed by q^2-cuts have the divergence index $\omega = -1$. The largest divergence index $\omega = 0$ has the diagram itself. If we remember that the p^2-differentiation has reduced also this divergence index then we conclude

$$-2\left(p^2\frac{\partial}{\partial p^2} + p'^2\frac{\partial}{\partial p'^2}\right) F_2^{1PI} = \sum_{n=1} g^{2n} P'_{2n}(\ln q^2) : \frac{1}{q^2} + O\left(\frac{1}{q^4}\right). \quad (4.4)$$

All quantities appearing in eq.(4.1) have to be considered as formal power series in the coupling constant

$$\tilde{\gamma}_j + \gamma_2 = \hat{\gamma} = \sum_{n=1} g^{2n}\hat{\gamma}_{2n}$$

$$F_2^{1PI} = 1 + \sum_{n=1} g^{2n} P_{2n}(\ln q^2) + O\left(\frac{1}{q^2}\right) \quad (4.5)$$

(the vertex itself is always included), so that the 2N-th order term of the anomalous dimension reads

$$\hat{\gamma}_{2N} = \frac{1}{q^2}P'_{2N} - \sum_{\substack{n=1 \\ n+n'=N}} \sum_{n'=1} \hat{\gamma}_{2n} P_{2n'} + O\left(\frac{1}{q^4}\right).$$

Because of eqs.(4.4),(4.5) it is now possible to see

$$(\tilde{\gamma}_j + \gamma_2)_{2n} \xrightarrow[q^2\to-\infty]{} \frac{1}{q^2}\tilde{P}_{2n}(\ln q^2). \quad (4.6)$$

For gauge theories as QCD and QED useful general calculations of the anomalous dimension $\tilde{\gamma}_j$ are not available. The simplest calculation give

$$F_2^{1PI} = 1 + A + \frac{A^2}{2} - \frac{g^4}{8\pi^2}C_N b\frac{1}{2}L_t^2 L_\mu, \quad A = -\frac{g^2}{8\pi^2}C_N\left[L_t^2 - 2L_t + \frac{\alpha}{2}L_t - \frac{\alpha}{2}L_\mu\right] \quad (4.7)$$

$L_t = \ln\frac{q^2}{p^2}$, $L_\mu = \ln\frac{\mu^2}{p^2}$ $\qquad C_N$ group theoretical factor $\qquad$ eq.(5.5)

$$\gamma_j + \gamma_2 = -\frac{g^2}{4\pi^2}\left(C_N \ln\left(\frac{\pm q^2}{\mu^2}\right) + \text{const}\right) - \frac{g^4}{8\pi^2}\left(C_N b \ln^2\frac{q^2}{\mu^2} + \ldots\right) + \ldots \quad (4.8)$$

All results are valid for general gauge parameter. The results of the two loop calculations with general gauge parameter are extracted from the corresponding calculation of the Dirac form factor /11/.

5. SOLUTION OF THE RENORMALIZATION GROUP EQUATION

The well-known standard solution of the renormalization group equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \delta\frac{\partial}{\partial m} - \gamma\right)\Gamma(p,g,m,\mu) = 0 \quad (5.1)$$

reads

$$\Gamma(\lambda p, g, m, \mu) = \lambda^d\, \Gamma\left(p, \bar{g}\left(\mu, \frac{\mu}{\lambda}, g, \frac{m}{\lambda}\right), \bar{m}(\ldots), \mu\right) \times$$

$$\times \exp\int_\mu^{\mu/\lambda} \frac{d\mu'}{\mu'}\gamma\left(p^2, \bar{g}\left(\mu', \frac{\mu}{\lambda}, g, \frac{m}{\lambda}\right), \bar{m}(\ldots), \mu'\right) \quad (5.2)$$

$$\mu \frac{\partial \bar g}{\partial \mu} = \beta(\bar g, \mu) \qquad \bar g(\mu, \mu, g) = g \tag{5.3}$$

$$\mu \frac{\partial \bar m}{\partial \mu} = \delta(\bar m, \bar g, \mu). \tag{5.4}$$

In lowest order we have

$$\beta = -\frac{b}{2} g^3 \qquad \bar g^2(\mu;\mu_0, g) = \frac{g^2}{1 + g^2 b \ln\frac{\mu}{\mu_0}}, \quad \delta = -m\frac{g^2}{16\pi^2} 6\cdot C_N = -d\cdot m g^2 \tag{5.5}$$

$$\bar m(\mu;\mu_0, g) = \frac{m}{(1+g^2 b \ln\frac{\mu}{\mu_0})^{d/b}}. \tag{5.6}$$

With the help of the new variable $\mu' = \mu \frac{e^\rho}{\lambda}$ the solution (5.2) reads

$$\Gamma(\lambda p, g, m, \mu) = \lambda^d \Gamma(p, \bar g, \bar m, \mu) \exp - \int_0^{\ln\lambda} d\rho\, \gamma(p, \bar g, \mu\frac{e^\rho}{\lambda}). \tag{5.7}$$

Some words to scalar field theory. Here we have to apply eq. (4.6)

$$\tilde\gamma_j \xrightarrow[q^2\to-\infty]{} -\gamma_2 + \frac{1}{q^2}\sum_{n=1} g^{2n} P_{2n}(\ln q^2)\, C_{2n}$$

in the renormalization group equation (2.16)

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \delta\frac{\partial}{\partial m} - \gamma_j + \tilde\gamma_j\right) F_2^{1PI}(q^2, \mu^2, \mu^2) = 0.$$

If we perform the limit $q^2 \to -\infty$ directly in the renormalization group equation, then this equation is identical to the renormalization group equation (2.12). This means the asymptotic behaviour of this form factor is the same for both limits $q^2\to-\infty$, $p^2 = p'^2 = \mu^2$, and $q^2 \to -\infty$, $p^2 = p'^2 \to -\infty$, as it is expected from already known results [3]. Of course, this

behaviour can also be checked more correctly for the solutions by careful estimates of the integrals in the exponential.

In QCD such a general statement is hardly possible. For this reason we consider the one-loop approximation at first (eq. (4.8), first term). Then the solution reads

$$F_2^{1PI}(\lambda^2 q^2, \mu^2, g, m, \mu) = F_2^{1PI}(q^2, \mu^2, \bar g, \bar m, \mu) \exp - \int_0^{\ln\lambda} d\rho\, \gamma(q^2, \bar g, \mu\frac{e^\rho}{\lambda}),$$

$$\gamma = \gamma_j - \tilde\gamma_j = \frac{g^2}{4\pi^2}\left(C_N \ln\frac{q^2\lambda^2}{\mu^2 e^{2\rho}} + \text{const}\right). \tag{5.8}$$

$$F_2^{1PI}(\lambda^2 q^2, \mu^2, g, m, \mu) = F_2^{1PI}(q^2, \mu^2, \bar g, \bar m, \mu) \exp \frac{1}{4\pi^2}\Big\{-\frac{1}{b}\ln(1+bg^2\ln\lambda)\times$$
$$\times\left(\text{const} + C_N \ln\frac{q^2}{\mu^2}\lambda^2\right) + \frac{2}{b} C_N\left(\ln\lambda - \frac{1}{g^2 b}\ln(1+g^2 b\ln\lambda)\right)\Big\}. \tag{5.9}$$

The leading behaviour for asymptotic free field theories is

$$F_2^{1PI}(\lambda^2 q^2, \mu^2, g, m, \mu) = F_2^{1PI}(q^2, \mu^2, \bar g, \bar m, \mu) \exp\left\{-\frac{C_N}{4\pi^2 b}\ln\ln\lambda \ln\lambda^2\right\}. \tag{5.10}$$

There is however another possibility to look at equation (5.9). Of course equation (5.9) expresses the invariance properties of the function F_2 under transformations of the renormalization group. This leads to special functional dependence expressed in this equation. Eq. (5.9) is therefore an identity, that holds also if both sides of this equation are expanded in powers of the coupling constant g (formal power series)

$$F_2^{1PI}(\lambda^2 q^2, \mu^2, g, m, \mu) = F_2^{1PI}(q^2, \mu^2, \bar g, \bar m, \mu) \exp\Big\{-\frac{1}{4\pi^2 b}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(bg^2\ln\lambda)^n}{n}\times$$
$$\times\left(\text{const} + C_N\ln\left(\frac{q^2}{\mu^2}\lambda^2\right)\right) + \frac{2C_N}{4\pi^2 b}\left(\ln\lambda - \frac{1}{g^2 b}\sum_{n=1}^{\infty}(-1)^{n+1}(g^2 b\ln\lambda)\right)\Big\}. \tag{5.11}$$

Thereby

$$\bar{g}^2 = g^2 \sum_{n=0} (-1)^n (g^2 b \ln\lambda)^n, \qquad \bar{m} = \frac{m}{\lambda} \sum_{n=0} \binom{-d/b}{n} (g^2 b \ln\lambda)^n \tag{5.12}$$

has to be taken into account. The result for the leading logarithmic terms $g^2 \ln^2\lambda$ is

$$F_2^{1PI}(\lambda^2 q^2, \mu^2, g, m, \mu) = F_2^{1PI}(q^2, \mu^2, g, \frac{m}{\lambda}) \exp - g^2 \frac{C_N}{8\pi^2} \ln^2\lambda^2. \tag{5.13}$$

This is the well-known Sudakov result.

The remaining question is: Are these result stable if corrections coming from the next orders are taken into account? If we neglect for the moment corrections coming from the two loop calculations of the effective coupling constant, then the exponential in eq.(5.8) has the additional term

$$-\int_0^{\ln\lambda} d\rho \left(\frac{g^2}{1+g^2 b\rho}\right)^2 \left(\ln\frac{p^2}{\mu^2} + \ln\lambda^2 - 2\rho\right)^2 \frac{C_N b}{8\pi^2}.$$

It is easily seen that the leading contribution now comes from the lower boundary 0 (where all calculations are questionable) and from the explicite λ^2-dependence in the term $\ln^2\lambda^2$.

$$\approx \frac{C_N}{8\pi^2} \frac{g^2}{1+bg^2\rho} \Big|_0^{\ln\lambda} \cdot \ln^2\lambda^2 = -\frac{C_N}{8\pi^2} \ln^2\lambda^2 g^2$$

This modifies the foregoing results strongly and more worse, every further correction of the anomalous dimension will change the result again. Modifications of the effective coupling constant (two-loop calculation), approximately expressed by

$$\bar{g}^2 = \left(a_1 + a_2 \ln\frac{\mu}{\mu_0} + a_3 \ln\ln\left(\frac{\mu}{\mu_0} + a_4\right)\right)^{-1}$$

do not change anything essentially. On the other hand, the Sudakov result remains stable if two-loop calculations are taken into account. They give no contribution to the one-loop result and we believe that this is the case for all higher

loop calculations. Up to now there is no general proof, however. What can be said is the following. Perturbative calculations of F_2 give the leading behaviour $F_2^{1PI} \approx \sum (g^2 \ln^2 q^2/p^2)^n C_n$. The calculation of the anomalous dimension with eq.(4.1) leads to $\tilde{\gamma} \approx \sum_{n=1} g^{2n} \ln^{2n-1} q^2 b_n$ . . But this is not sufficient to reproduce the Sudakov result. To see this directly we put the correction terms from higher loops

$$\tilde{\gamma} = \sum_{\substack{n, m_n \\ n=2}} g^{2n} \ln^{m_n} \frac{q^2}{\mu^2} a_{mn} \qquad m_n \le 2n-1$$

into the argument of the exponential of eq.(5.8). Thereby we have to take into account eq.(5.12) for the effective coupling constant. Higher loop calculations of the effective coupling constant change this power series by an addition of nonleading logarithms only, which are uninteresting in this connection. The additional terms in the exponential function read

$$-\int_0^{\ln\lambda} d\rho \sum_{\substack{n=2 \\ m}} \left(\sum_{k=0}^{\infty} g^2 (-g^2 b\rho)^k\right)^n \left(\ln\frac{q^2}{\mu^2}\lambda^2 - 2\rho\right)^{m_n} a_{mn}$$

$$\approx -\sum_{\substack{n, \\ m_n \max}} \frac{2^{m_n}}{m_n} (\ln\lambda)^{m_n+1} (g^2)^n a_{mn}.$$

This means there appear no further leading terms if $a_{mn} = 0$ for $m_n \geq 2n-1$.

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