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FREE MASSLESS SCALAR FIELDS IN TWO DIMENSIONS

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Свободные безмассовые скалярные поля
в двумерном пространстве

Рассматриваются двумерные безмассовые скалярные поля, играющие основную роль при конструировании решения безмассовой модели Тирринга. Оба поля реализованы как операторы в общем пространстве Фока. Показывается, что теория каждого из этих двух полей совпадает со стандартной теорией безмассового скалярного поля, и в то же время они удовлетворяют линейной дифференциальной связи. Используются методы аксиоматического подхода к теории поля с индефинитной метрикой. Заряды определены корректно как операторы в пространстве Фока и показано, что они генерируют из вакуума трансляционно-инвариантные состояния.

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Free Massless Scalar Fields
in Two Dimensions

INTRODUCTION

The theory of the free massless scalar field in two space-time dimensions is very rich and instructive. It is far from being the simplest quantum field theory, requiring indefinite metric state space or, alternatively, violating some of the other Wightman axioms [1,2]. Besides, the massless scalar fields are essentially used in building explicit solutions of nontrivial models as the massless Thirring model [1,3,4,5,24,25], the Schwinger model [6,7], the sine-Gordon model [8], the Schroer model [2], etc. It was remarked by Skyrme [9] and rigorously proved by Streater and Wilde [10] that properly defined exponents of the massless scalar field behave like fields with spin (in particular, with half-integer spin). The theory of the massless scalar field can serve as an illustration [11,12] for the methods of quantization in indefinite metric [13] and is a simpler analogue to the promising theory of the dipole field in four dimensions [14,15], see also ref.[16].

In the massless Thirring model the potentials of the conserved current and pseudocurrent play an essential role [1,4]. They satisfy a system of two linear differential equations and as a result, the D'Alembert equation. The commutation relations between the components of the currents [17,1,18] show that the potential of the current and the potential of the pseudocurrent can be defined as canonical local free zero-mass fields, but nonlocal with respect to each other. Systematically, though at a formal level, these fields were considered in refs.[19,20].

In the present paper the two scalar fields are realized in a common Fock space, equipped with indefinite metric structure (see ref.[13]). The theory is a generalization of the theory of a single free massless scalar field [1,11,12] (see also ref.[24]), each of the two fields acts conventionally in a proper Fock space, which is a subspace of the common space. The results of refs.[19,20] are essentially used and confirmed. The two charges, which correspond to the formal Noether charges [22] are correctly defined and it is shown that they generate translationally invariant states from the vacuum.

1. THE TWO FIELDS ϕ AND $\tilde{\phi}$

The peculiarities of the theory of the free scalar massless field in two dimensions are well known [1,2,4,11,12]. Its two-point function

$$\frac{1}{i} D^{-}(x) = -\frac{1}{4\pi} \ln(-\mu^2 x^2 + i0x^\circ), \quad \mu > 0 \tag{1.1}$$

is not positive definite distribution on $\mathcal{S}(R^2)$ but only on the subspace consisting of the test-functions $\hat{f}$ which satisfy the condition $\int \hat{f}(x) d^2x = 0$, i.e., whose Fourier transforms belong to

$$\mathcal{S}_0(R^2) = \{ f \in \mathcal{S}(R^2),\ f(0) = 0 \} \tag{1.2}$$

(It should be noted that defining the field ϕ on $\mathcal{S}_0(R^2)$ one automatically defines its derivatives on $\mathcal{S}(R^2)$: if $\hat{f} \in \mathcal{S}(R^2)$, the functions

$$\int e^{ipx} \partial_\nu \hat{f}(x)\, d^2x = -ip_\nu f(p), \qquad \nu = 0,1 \tag{1.3}$$

belong to $\mathcal{S}_0(R^2)$).

As it was noted in the Introduction, because of the current conservation and the specific connection between the components of the current and the pseudocurrent in the massless Thirring model their potentials ϕ and $\tilde{\phi}$ satisfy the system of linear first order differential equations

$$\partial_0 \phi(x) = \partial_1 \tilde{\phi}(x), \quad \partial_1 \phi(x) = \partial_0 \tilde{\phi}(x) \tag{1.4}$$

and as a result, the D'Alembert equation. We shall postulate that $\phi(x)$ is a free scalar field with two-point function (1.1) and shall see how the field $\tilde{\phi}(x)$ can be defined in order eq. (1.4) to be satisfied. In addition, we shall aim at defining the fields ϕ and $\tilde{\phi}$ as (operator-valued) distributions on $\mathcal{S}(R^2)$.

To do this let us first write down the Fourier transform of the two-point function (1.1):

$$2\pi w(p^0, p^1) = \left(\frac{1}{p^0+p^1}\right)_{\kappa+} \delta(p^0-p^1) + \left(\frac{1}{p^0-p^1}\right)_{\kappa+} \delta(p^0+p^1), \tag{1.5}$$

$$\kappa = 2\mu e^{\Gamma'(1)},$$

where the distribution $\left(\frac{1}{q}\right)_{\kappa+}$ is defined on $\mathcal{S}(R^1)$ as

$$\left\langle \left(\frac{1}{q}\right)_{\kappa+}, f \right\rangle = \int \frac{f(q) - f(0)\,\theta(\kappa - q)}{q} dq = -\int \ln\frac{q}{\kappa} \frac{df}{dq}(q)\, dq. \tag{1.6}$$

The RHS of (1.5) is an extension on $\mathcal{S}(R^2)$ of the functional $\theta(p^0)\delta(p^2)$ which is well-defined on $\mathcal{S}_0(R^2)$ only. To construct Fock space for the field ϕ one gives to the restrictions of the functions from $\mathcal{S}(R^2)$ on the cone C_+,

$$C_+ = \{ p = (p^0, p^1),\ p^2 = 0,\ p^0 \geq 0 \} \tag{1.7}$$

the interpretation of one-particle states. The (indefinite) metric in the one-particle space is given by the two-point function (1.5). The general form of the restriction on C_+ of a function from $\mathcal{S}(R^2)$ can be obtained in the following way. Let $f \in \mathcal{S}(R^2)$ and

$$f_\pm(p^0, p^1) = \frac{1}{2}[f(p^0, p^1) \pm f(-p^0, p^1)]. \tag{1.8}$$

It can be easily seen that $f_- \in \mathcal{S}_0(R^2)$ and that $f_+(|p|, p) = f_+(p,p) \equiv f_+(p)$, $f_-(|p|, p) = \epsilon(p) f_-(p,p) \equiv \epsilon(p) f_-(p)$. Then

$$f(|p|, p) = f_+(p) + \epsilon(p) f_-(p), \quad f_-(0) = 0, \tag{1.9}$$

where the product $\epsilon(p) f_-(p)$ is defined as the point $p = 0$ by continuity (it equals zero). In order the operator equality (1.4) to be satisfied, it is necessary that the product $-\epsilon(p) f(|p|, p)$ be a restriction on C_+ of a function from $\mathcal{S}(R^2)$ too. But this is impossible, in general, because if $f \in \mathcal{S}_0(R^2)$, there is a discontinuity at $p = 0$. Thus, to define one-particle states of the field $\tilde{\phi}$ one has either to consider the field ϕ on $\mathcal{S}_0(R^2)$ or to introduce a space, larger than $\mathcal{S}(R^2)$, in which the multiplication by $-\epsilon(p)$ has sense as an involution. We shall choose the second way. To avoid working with discontinuous functions, we shall first note that any function $f \in \mathcal{S}(R^2)$ can be presented in the form

$$f(p^0, p^1) = f_s(p^0, p^1) + f_0 S(p^0, p^1), \quad S \in \mathcal{S}(R^2), \tag{1.10}$$

where $S(0,0) = 1$, $f_0 = f(0,0)$ and $f_s = f - f_0 S$. It is convenient to fix S so that (see (1.5))

$$\int |S(p^0,p^1)|^2\, w(p^0,p^1) \frac{d^2p}{(2\pi)^2} = 0. \tag{1.11}$$

One can check that, for example,

$$S(p^0,p^1) = \exp\{-b^2[(p^0)^2+(p^1)^2]\}, \quad b = \kappa \exp \frac{\Gamma'(1)}{2} \tag{1.12}$$

satisfies all these conditions. Fixing $S \in \mathcal{S}(\boldsymbol{R}^2)$, $S(0,0) = 1$ in (1.10) decomposes $\mathcal{S}(\boldsymbol{R}^2)$ into direct sum of two linear spaces: $\mathcal{S}_0(\boldsymbol{R}^2)$ and one-dimensional space, isomorphic to $\boldsymbol{C}$. Therefore the restrictions of the functions from $\mathcal{S}(\boldsymbol{R}^2)$ on the cone C_+ can be brought to the form (see (1.8) and (1.9))

$$f(|p|,\, p) = f_{+s}(p,p) + \epsilon(p)\, f_-(p,p) + f_0\, s(p), \tag{1.13}$$

where $s(p) = S(|p|,p) = \exp(-2b^2p^2)$ and $f_{+s}(p,p) + \epsilon(p)\, f_-(p,p) = f_s(|p|,p)$ is the restriction on the cone C_+ of f_s, defined by (1.10). The decomposition (1.13) suggests how to make meaningful the multiplication by $-\epsilon(p)$. It is sufficient to add to the RHS of (1.13) a term of the type $\tilde{f}_0 \tilde{s}(p)$ where $\tilde{f}_0 \in \boldsymbol{C}$ and $\tilde{s}(p) = \epsilon(p)\, s(p)$ thus enlarging the one-particle space with one more dimension. If one puts, for definiteness, $\epsilon(0) = 1$, then $\epsilon(p)^2 = 1$ and $\epsilon(p)\, \tilde{s}(p) = s(p)$.

In other words, we shall start with the linear space

$$\mathcal{E}_1 = \mathcal{S}_0(\boldsymbol{R}^2)/C_+ \oplus \boldsymbol{C} \oplus \boldsymbol{C} \tag{1.14}$$

on which the map

$$f = (f_s(p),\, f_0,\, \tilde{f}_0) \mapsto \tilde{f} = (-\epsilon(p)\, f_s(p),\, -\tilde{f}_0,\, -f_0) \tag{1.15}$$

is defined. We shall prove that a sesquilinear nondegenerate form, which is an extension on $\mathcal{E}_1$ of the form, defined by the two-point function (1.5), is provided by

$$\langle f,g\rangle_1 = \langle f_s, g_s\rangle + \bar{f}_0 \langle s, g_s\rangle + g_0 \langle f_s, s\rangle + \bar{\tilde{f}}_0 \langle \tilde{s}, g_s\rangle + \tilde{g}_0 \langle f_s, \tilde{s}\rangle, \tag{1.16}$$

where in the RHS of (1.16)

$$\langle x,y\rangle = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\overline{x(p)}\, y(p)}{|p|}\, dp. \tag{1.17}$$

For $\tilde{f}_0 = \tilde{g}_0 = 0$, $\langle f,g\rangle_1$ coincides with

$$2\pi \int \overline{f(p^0,p^1)}\, w(p^0,p^1)\, g(p^0,p^1) \frac{d^2p}{(2\pi)^2}, \tag{1.18}$$

where $f(p^0,p^1)$ and $g(p^0,p^1)$ are the corresponding functions from $\mathcal{S}(\boldsymbol{R}^2)$ (eq. (1.11) is used).

2. THE ONE-PARTICLE SPACE

Having the linear space $\mathcal{E}_1$ and the sesquilinear form (1.16) on it (the form (1.16) is nondegenerate, as it will become clear later), one can use the standard procedure for quantization in indefinite metric (see, e.g. [13]). One can define a scalar product on $\mathcal{E}_1$:

$$(f,g)_1 = \langle f_s, g_s) + \langle f_s, s\rangle\langle s, g_s\rangle + \bar{f}_0 g_0 + \langle f_s, \tilde{s}\rangle\langle \tilde{s}, g_s\rangle + \bar{\tilde{f}}_0 \tilde{g}_0 \tag{2.1}$$

and prove that the inequality

$$|\langle f,g\rangle_1|^2 \le (f,f)_1\, (g,g)_1 \tag{2.2}$$

holds for any two vectors $f,g \in \mathcal{E}_1$. (In the case of a single free massless scalar field, without the last two terms, this scalar product is used in refs. [11,12]). The completion of $\mathcal{E}_1$ with respect to the norm $\|f\|_1 = \sqrt{(f,f)_1}$ will be denoted by $\mathcal{H}_1$ (the elements of $\mathcal{H}_1$ can be realized as equivalence classes of Cauchy sequences). The sesquilinear form (1.16) and the scalar product (2.1) can be extended on $\mathcal{H}_1$ and so can the map $f \mapsto \tilde{f}$ (see (1.15)). Thus we obtain a Hilbert space, $\mathcal{H}_1$, equipped with a sesquilinear form - the extension of (1.16). For fixed $f \in \mathcal{H}_1$, because of (2.2), $\langle f,\cdot\rangle = \ell_f$ is a continuous linear functional on $\mathcal{H}_1$. Let us denote the vectors $(0,1,0) \in \mathcal{E}_1$ and $(0,0,1) \in \mathcal{E}_1$ by Ξ_1 and $\tilde{\Xi}_1$, respectively. The following equalities hold: $\langle \Xi_1, \Xi_1\rangle = \langle \tilde{\Xi}_1, \tilde{\Xi}_1\rangle = \langle \Xi_1, \tilde{\Xi}_1\rangle = 0$, $(\Xi_1, \tilde{\Xi}_1)_1 = 0$, $\|\Xi_1\|_1 = \|\tilde{\Xi}_1\|_1 = 1$. The Riesz lemma implies the existence and uniqueness of vectors $\Psi_{01} \in \mathcal{H}_1$ and $\tilde{\Psi}_{01} \in \mathcal{H}_1$ such that

$$(\Psi_{01},f)_1=\langle\Xi_1,f\rangle_1,\quad(\tilde\Psi_{01},f)_1=\langle\tilde\Xi_1,f\rangle_1,\quad\forall f\in\mathcal H_1. \tag{2.3}$$

Obviously $(\Psi_{01},\Xi_1)_1=(\Psi_{01},\tilde\Xi_1)_1=(\tilde\Psi_{01},\Xi_1)_1=(\tilde\Psi_{01},\tilde\Xi_1)_1=0$ and $\Psi_{01}\neq 0$, $\tilde\Psi_{01}\neq 0$. If $\mathcal H_{01}$ is the subspace

$$\mathcal H_{01}=\{h\in\mathcal H_1,\ (\Psi_{01},h)_1=(\Xi_1,h)_1=(\tilde\Psi_{01},h)_1=(\tilde\Xi_1,h)_1=0\} \tag{2.4}$$

and $\{e_n\}_{n=1}^{\infty}$ is an arbitrary orthogonal basis in $\mathcal H_{01}$, then $\Psi_{01},\Xi_1,\tilde\Psi_{01},\tilde\Xi_1,e_1,e_2,\ldots$ form an orthogonal basis in $\mathcal H_1$ and in this basis the operator β defined by

$$\langle f,g\rangle_1=(f,\beta g)_1,\qquad\forall f,g\in\mathcal H_1 \tag{2.5}$$

(its existence and boundedness are guaranteed by the Riesz lemma and eq.(2.2), respectively) has a nondiagonal form. The operator β is self-adjoint and $\beta^2=1$ and therefore $\langle .,.\rangle_1$ is nondegenerate. From $\Psi_{01}=\beta\Xi_1$, $\tilde\Psi_{01}=\beta\tilde\Xi_1$ and (2.5) it follows that

$$\langle\Psi_{01},\Psi_{01}\rangle_1=\langle\tilde\Psi_{01},\tilde\Psi_{01}\rangle_1=\langle\Psi_{01},\tilde\Psi_{01}\rangle_1=0,\ (\Psi_{01},\tilde\Psi_{01})_1=0,\ \|\Psi_{01}\|_1=\|\tilde\Psi_{01}\|_1=1. \tag{2.6}$$

The standard representation of the translation group $f(p)\mapsto e^{ipa}f(p)$ induces the following transformations of the vectors from $\mathcal E_1$ with $\tilde f_0=0$:

$$U(a)(f_s,f_0,0)=(e^{ipa}f_s+f_0 s(e^{ipa}-1),f_0,0)\,. \tag{2.7}$$

The transformations of arbitrary vectors from $\mathcal E_1$ can be obtained from (2.7), assuming in addition that

$$U(a)\tilde f=\widetilde{U(a)f}\,. \tag{2.8}$$

It can be proved that, for fixed a, the operator $U(a)$ is bounded on $\mathcal E_1$ (though not uniformly with respect to a):

$$\|U(a)f\|_1\le c(|a^0|+|a^1|)\|f\|_1\qquad\text{for some } c>0\,. \tag{2.9}$$

The extension by continuity of $U(a)$ on the whole $\mathcal H_1$ is a pseudounitary, i.e., unitary with respect to $\langle .,.\rangle_1$ operator. The equalities

$$\langle\Psi_{01},U(a)h\rangle_1=h_0=\langle\Psi_{01},h\rangle_1=\langle U(a)\Psi_{01},U(a)h\rangle_1,\ \forall h\in\mathcal H_1 \tag{2.10}$$

and the nondegeneracy of $\langle .,.\rangle_1$ imply that the vector Ψ_{01} is translationally invariant, i.e., $U(a)\Psi_{01}=\Psi_{01}$. One can prove quite analogously the translational invariance of $\tilde\Psi_{01}$. The existence of translationally invariant states in the one-particle space is a nontrivial fact. The vectors Ψ_{01} and $\tilde\Psi_{01}$ do not belong to $\mathcal E_1$, rather they appear as a result of the completion of $\mathcal E_1$ (with respect to the norm $\|\cdot\|_1$). Their existence and translational invariance, however, do not depend on the special choice of the scalar product.

Since on $\mathcal E_{01}=\{f\in\mathcal E_1,\ f_0=\tilde f_0=0\}$ the form $\langle .,.\rangle_1$ is nondegenerate and positively definite, $\mathcal E_{01}$ can be completed with respect to the norm $\|f\|=\sqrt{\langle f,f\rangle_1}\le\|f\|_1$, thus obtaining a Hilbert space $\mathcal H_{phys}$. In this case $U(a)$, restricted on $\mathcal E_{01}$, can be extended on $\mathcal H_{phys}$ as a unitary operator. The Hilbert space $\mathcal H_{phys}$ can be identified with $\mathcal L^2(\mathbb R^1,\frac{dp}{|p|})$. On $\mathcal H_{phys}$ the dependence on the arbitrary parameter κ drops out (each term in the RHS of (1.16) depends in general on κ).

3. THE FOCK SPACE

The Fock space over $\mathcal H_1$

$$\mathcal F(\mathcal H_1)=\bigoplus_{n=0}^{\infty}\mathcal H_n\,, \tag{3.1}$$

where $\mathcal H_0=\mathbb C$ and $\mathcal H_n=\mathrm{sym}\otimes\mathcal H_1^n$ ($n=1,2,\ldots$) is the symmetrized tensor product of n copies of $\mathcal H_1$, is defined as the completion of the finite particle space $\mathcal F_0$ consisting of all finite linear combinations of vectors from $\mathcal H_n$, $n=0,1,2,\ldots$

We shall define on $\mathcal F_0$ a generalization (because of the indefinite metric) of the Segal quantization over $\mathcal H_1$ (see /21,13/). Let $\Phi^{\pm}(f)$, $f\in\mathcal H_1$ be the maps

$$\Phi^{+}(f)S_n(f_1\otimes f_2\otimes\ldots\otimes f_n)=\sqrt{n+1}\,S_{n+1}(f\otimes f_1\otimes f_2\otimes\ldots\otimes f_n)\qquad n=0,1,2,\ldots \tag{3.2}$$

$$\Phi^{-}(f)S_n(f_1\otimes f_2\otimes\ldots\otimes f_n)=\frac{1}{\sqrt N}\sum_{k=1}^{n}\langle f,f_k\rangle_1 S_{n-1}(f_1\otimes\ldots\otimes f_{k-1}\otimes f_{k+1}\otimes\ldots\otimes f_n),\quad n=1,2,\ldots \tag{3.3}$$

and $\Phi^-(f)\mathcal{H}_0 = 0$ (S_n means symmetrization). The maps $\Phi^\pm(f)$ can be extended by linearity on dense subspaces of $\mathcal{H}_n$. It can be easily proved that these maps are bounded and therefore can be extended by continuity on $\mathcal{H}_n$ and by linearity on $\mathcal{F}_0$. The field (analogue of the Segal field)

$$\Phi(f) = \frac{1}{\sqrt{2}}(\Phi^+(f) + \Phi^-(f)) \qquad (3.4)$$

will be the basic construction, with the help of which the scalar fields ϕ and $\tilde{\phi}$ will be defined.

A nondegenerate sesquilinear form $<.,.>$, a scalar product $(.,.)$ and an indefinite metric operator on $\mathcal{F}(\mathcal{H}_1)$ can be obtained in a standard manner from the corresponding objects in $\mathcal{H}_1$. The following general theorems can be proved (see refs. /21,18/):

1. The operator $\Phi(f)$, $f \in \mathcal{H}_1$ is symmetric with respect to the form $<.,.>$, i.e., for $\forall g,h \in \mathcal{F}_0$

$$<g, \Phi(f)h> = <\Phi(f)g, h>. \qquad (3.5)$$

2. The vacuum $\Psi_0 = (1,0,0,\ldots) \in \mathcal{H}_0 \subset \mathcal{F}_0$ is a cyclic vector.
3. The following commutation relations hold:

$$[\Phi^-(f), \Phi^+(g)]h = <f,g>_1 h, \quad \forall f,g \in \mathcal{H}_1, \quad \forall h \in \mathcal{F}_0. \qquad (3.6)$$

Eq. (3.6) implies that

$$[\Phi(f), \Phi(g)]h = i\,\mathrm{Im}<f,g>_1 h. \qquad (3.7)$$

Let us define for any function $\hat{f} \in \mathcal{S}(R^2)$ a corresponding vector $E\hat{f} \in \mathcal{E}_1 \subset \mathcal{H}_1$: if

$$f(p) = \int e^{ipx} \hat{f}(x)\, d^2x \qquad (3.8)$$

is the Fourier transform of $\hat{f}(x)$, then by $E\hat{f} \in \mathcal{E}_1$ we shall denote the vector $(f_s, f_0, 0)$ which corresponds to the restriction of f on the cone C_+.

The fields $\phi(\hat{f}), \tilde{\phi}(\hat{f})$ ($\hat{f} \in \mathcal{S}(R^2)$) are defined by

$$\phi(\hat{f}) = \phi^+(\hat{f}) + \phi^-(\hat{f}), \qquad (3.9)$$

$$\tilde{\phi}(\hat{f}) = \tilde{\phi}^+(\hat{f}) + \tilde{\phi}^-(\hat{f}), \qquad (3.10)$$

where

$$\phi^\pm(\hat{f}) = \Phi^\pm(E\,\mathrm{Re}\,\hat{f}) + i\,\Phi^\pm(E\,\mathrm{Im}\,\hat{f}), \qquad (3.11)$$

$$\tilde{\phi}^\pm(\hat{f}) = \Phi^\pm(\widetilde{E\,\mathrm{Re}\,\hat{f}}) + i\,\Phi^\pm(\widetilde{E\,\mathrm{Im}\,\hat{f}}). \qquad (3.12)$$

The maps $\hat{f} \mapsto \phi(\hat{f})$ and $\hat{f} \mapsto \tilde{\phi}(\hat{f})$ are complex linear maps ($f \mapsto \Phi(f)$ is only a real linear map). On $\mathcal{F}_0$ the usual commutation relations hold:

$$[\phi^-(x), \phi^+(y)] = \frac{1}{i}D^-(x-y) = [\tilde{\phi}^-(x), \tilde{\phi}^+(y)], \qquad (3.13)$$

and

$$[\phi^-(x), \tilde{\phi}^+(y)] = \frac{1}{i}\tilde{D}^-(x-y) = [\tilde{\phi}^-(x), \phi^+(y)], \qquad (3.14)$$

where

$$\frac{1}{i}D^-(x) = \frac{1}{4\pi}\ln\frac{x^0 - x^1 - i0}{x^0 + x^1 - i0} \qquad (3.15)$$

is the nonlocal (and Lorentz noninvariant) two-point function of ϕ and $\tilde{\phi}$ (it may be verified that

$$<E\hat{f}, E\hat{g}>_1 = \iint \hat{f}(x)\frac{1}{i}D(x-y)\hat{g}(y)d^2x d^2y, \qquad (3.16)$$

$$<\widetilde{E\hat{f}}, E\hat{g}>_1 = \iint \hat{f}(x)\frac{1}{i}\tilde{D}(x-y)\hat{g}(y)d^2x d^2y. \qquad (3.17)$$

One can construct a physical Fock space $\mathcal{F}(\mathcal{H}_{phys})$ starting with $\mathcal{H}_{phys}$ instead of $\mathcal{H}_1$, and realize the fields $\phi(\hat{f}), \tilde{\phi}(\hat{f})$ as operators in it (assuming that the Fourier transform of $\hat{f}$ belongs to $\mathcal{S}_0(R^2)$). In this case all the complications connected with the indefinite metric will drop out. One can prove that the set of finite linear combinations of vectors of the type $\phi(\hat{f}_1)\phi(\hat{f}_2)\ldots\phi(\hat{f}_n)\Psi_0$, $n = 0,1,2,\ldots$ (or of vectors of the type $\tilde{\phi}(\hat{g}_1)\tilde{\phi}(\hat{g}_2)\ldots\tilde{\phi}(\hat{g}_n)\Psi_0$ - they coincide), where the Fourier transforms of $\hat{f}_i(\hat{g}_i)$, $i = 1,2,\ldots,n$ belong to $\mathcal{S}_0(R^2)$ and the field ϕ (or $\tilde{\phi}$) is defined through (3.9) and (3.11) (or (3.10) and (3.12)), is dense in $\mathcal{F}(\mathcal{H}_{phys})$.

4. THE CHARGES

The formal charges corresponding (by the Noether's theorem) to the shifts of the fields ϕ and $\tilde{\phi}$ are equal to

$$Q = \int \partial_0 \phi(x)\, dx^1, \qquad \tilde{Q} = \int \partial_0 \tilde{\phi}(x)\, dx^1. \tag{4.1}$$

We shall try to define correctly the charges as operators in the Fock space constructed above (see refs. /22,23/).

Consider the operators

$$Q_n = -\phi(\hat{f}_n \hat{\alpha}_d'), \; n = 1,2,\dots ; \; d > 0, \tag{4.2}$$

where $\hat{f}_n(x^1)$ and $\hat{\alpha}_d(x^0)$ are real functions belonging to $\mathcal{S}(R^1)$ and $\hat{\alpha}_d$ satisfy the condition

$$\int_{-\infty}^{\infty} \hat{\alpha}_d(x^0)\, dx^0 = 1, \qquad \forall d. \tag{4.3}$$

If in some sense

$$\hat{f}_n(x^1) \xrightarrow[n\to\infty]{} 1, \qquad \hat{\alpha}_d(x^0) \xrightarrow[d\to 0]{} \delta(x^0) \tag{4.4}$$

(we shall not assume that $\hat{f}_n$ and $\hat{\alpha}_d$ have compact supports), it is intuitively clear that the corresponding limit (if it exists in some sense) of the operators Q_n is a candidate for the charge operator Q.

Owing to (4.3) and (4.4) the Fourier transforms of $\hat{f}_n$ and $\hat{\alpha}_d$ satisfy $\alpha_d(0) = 1$, $\forall d > 0$; $f_n(p^1) \xrightarrow[n\to\infty]{} 2\pi\delta(p^1)$ and $\alpha_d(p^0) \xrightarrow[d\to 0]{} 1$ and therefore the Fourier transforms of the functions $\hat{f}_n \hat{\alpha}_d'$ are of the form

$$-2\pi i p^0 \psi_n(p^1)\, \alpha_d(p^0), \qquad \mathcal{S}(R^1) \ni \psi_n(p^1) \to \delta(p^1). \tag{4.5}$$

On the cone C_+ the corresponding sequence tends to $-2\pi i |p| \delta(p)$ (after taking the limit $n \to \infty$ the dependence on d disappears; that is why we attach only one index to the operators (4.2)). These heuristic considerations suggest that

$$Q = \Phi\left(\frac{i}{\sqrt{2}} \Psi_{01}\right) \tag{4.6}$$

is an appropriate definition for the charge operator (it can be easily seen from the definition of Ψ_{01} that, roughly speaking, $\Psi_{01} \sim 4\pi |p| \delta(p)$).. We shall prove that (Ψ_0 is the vacuum vector)

$$\frac{i}{2}\Psi_{01} = Q\Psi_0 = \text{w-}\lim_{n\to\infty} Q_n \Psi_0 \in \mathcal{H}_1 \tag{4.7}$$

assuming that the functions $\hat{f}_n$ and $\hat{\alpha}_d$ in (4.2) are properly defined and w-lim means the weak limit in $\mathcal{H}_1$. It is necessary and sufficient to prove that the norms $\|Q_n \Psi_0\|_1$ are bounded and that on a dense linear subspace $L \subset \mathcal{H}_1$

$$\lim_{n\to\infty} (Q_n \Psi_0 - Q\Psi_0, f)_1 = 0, \qquad \forall f \in L. \tag{4.8}$$

The boundedness of the norms $\|Q_n \Psi_0\|_1$ becomes obvious, supposing in addition that

$$\hat{f}_n(x^1) = \hat{f}\left(\frac{x^1}{n}\right), \quad \hat{f} \in \mathcal{S}(R^1), \quad \hat{f}(0) = 1. \tag{4.9}$$

Indeed (ψ is the Fourier image of $\frac{1}{2\pi}\hat{f}$)

$$\begin{aligned} \|Q_n \Psi_0\|^2 &= \pi \int |p| n^2 |\psi(np)\, \alpha_d(|p|)|^2 dp = \\ &= \pi \int |p|\, |\psi(p)\, \alpha_d\left(\frac{|p|}{n}\right)|^2 dp \end{aligned} \tag{4.10}$$

(see (4.5)) and this expression is bounded in n . The limit (4.8) is a simple corollary of the definitions of Q_n and Q $(L = \beta \mathcal{E}_1)$.

The definitions of Q and $\phi(\hat{f})$ imply

$$[Q, \phi(\hat{f})] h = -ih \int \hat{f}(x)\, d^2x = -if_0 h, \quad \forall \hat{f} \in \mathcal{S}(R^2), \forall h \in \mathcal{F}_0. \tag{4.11}$$

For $f_0 = \int \hat{f}(x)\, d^2x = 0$ (i.e., $f(p) \in \mathcal{S}(R^2)$) the field $\phi(\hat{f})$ commutes with Q, but this does not mean that the state $\phi(\hat{f})\Psi_0$ has zero charge in the common sense. Obviously Q does not leave the n-particle sectors $\mathcal{H}_n$ invariant:

$$Q = Q^+ + Q^-, \quad Q^\pm = \pm \frac{i}{2} \Phi^\pm(\Psi_{01}) \tag{4.12}$$

and besides

$$Q\Psi_0 = \frac{i}{2}\Phi^+(\Psi_{01})\Psi_0 = \frac{i}{2}\Psi_{01} \neq 0 \qquad (4.13)$$

(the invariance $\phi \mapsto \phi + \mathrm{const}$ is spontaneously broken). The operators Q^+ and Q^- commute:

$$[Q^-, Q^+] = \frac{1}{4}[\Phi^-(\Psi_{01}), \Phi^+(\Psi_{01})] = \frac{1}{4}\langle\Psi_{01},\Psi_{01}\rangle_1 = 0. \qquad (4.14)$$

The second charge $\tilde{Q}$ can be introduced in a quite analogous manner:

$$\tilde{Q} = \Phi(\frac{i}{\sqrt{2}}\tilde{\Psi}_{01}). \qquad (4.15)$$

We shall define, at the end, representations of the Lorentz boosts and the dilatations in $\mathcal{F}(\mathcal{H}_1)$. Let us denote the corresponding parameters by χ and $\lambda > 0$, respectively. Demanding the vacuum vector Ψ_0 and the form $\langle.,.\rangle_1$ to be Lorentz- and scale invariant, one is forced to define on $\mathcal{E}_1 \subset \mathcal{H}_1$

$$U(\chi)f = f_\chi + (f_0\tilde{\Psi}_{01} + \tilde{f}_0\Psi_{01})\frac{\chi}{4\pi}, \qquad (4.16)$$

where the mapping $f \mapsto f_\chi$ is induced by the transformations (Λ_χ is the hyperbolic rotation)

$$f(p) \mapsto f_\chi(p) = f(\Lambda_\chi^{-1}p), \qquad f \in \mathcal{S}(R^2) \qquad (4.17)$$

and $(\tilde{f})_\chi = \widetilde{(f_\chi)}$. Analogously,

$$U(\lambda)f = f_\lambda - (f_0\Psi_{01} + \tilde{f}_0\tilde{\Psi}_{01})\frac{\ln\lambda}{4\pi}, \qquad (4.18)$$

where on $\mathcal{S}(R^2)$

$$f(p) \mapsto f_\lambda(p) = f(\lambda^{-1}p). \qquad (4.19)$$

These operators are bounded on $\mathcal{E}_1$ and can be extended on $\mathcal{H}_1$ by continuity. It is not surprising that the translationally invariant vectors Ψ_{01} and $\tilde{\Psi}_{01}$ are also Lorentz- and dilatation invariants.

We can construct the corresponding operators on $\mathcal{F}(\mathcal{H}_1)$, denoting them still by $U(\chi)$, $U(\lambda)$. It can bee seen [13] that for $\forall f \in \mathcal{H}_1$ and $\forall \Psi \in \mathcal{F}_0$

$$U\Phi^\pm(f)U^{-1}\Psi = \Phi^\pm(Uf)\Psi, \qquad (4.20)$$

where U is $U(\chi)$ or $U(\lambda)$. From (4.16), (4.18) and (4.20) follow the transformation properties of $\phi^\pm$ and $\tilde{\phi}^\pm$ which contain nonhomogeneous terms proportional to $Q^\pm$ and $\tilde{Q}^\pm$ (see refs. [19,20]).

The representations of the Lorentz and scale transformations on the physical space are unitary.

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