сообщения
объединенного
института
ядерных
исследований
дубна

E2-80-415

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"ARBITRARINESS" IN HIGHER ORDERS OF RENORMALIZED PERTURBATION THEORY

1980

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"Неоднозначность" высших порядков перенормированной теории возмущений

Рассматривается формальная неоднозначность высших порядков перенормированной теории возмущений, обусловленная свободой в определении перенормированной константы связи. Эта неоднозначность, не отражающаяся на физических величинах, приводит к различным формальным выражениям для величин подобных бета-функции и аномальным размерностям. Исследуется возможность использования этой неоднозначности для "суммирования" асимптотических рядов, встречающихся в ренормгрупповом анализе.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1980

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"Arbitrariness" in Higher Orders of Renormalized Perturbation Theory

The formal nonuniquiness in higher orders of perturba-

The routine of renormalization of perturbation terms in quantum field theory yields the finite expression depending on renormalized masses and coupling constants. The latter can be defined in different manners in various renormalization schemes. Hence the higher-order perturbation terms can differ in various schemes.

An actual example of this kind is provided by expressions of the beta-function (GML-function) in the scalar quartic field model $\mathcal{L} = 4\pi^2 g\varphi^4/3$ calculated in the 4-loop order for three different schemes /1/ :

$$\beta_\Lambda(g) = 3g^2 - 11.33g^3 + 166.7g^4 - 3457g^5 + \ldots \tag{1}$$

in the Λ (cut-off) renormalization scheme ,

$$\beta_\lambda(g) = 3g^2 - 11.33g^3 + 154.3g^4 - 2338g^5 + \ldots \tag{2}$$

in the λ -scheme (R-operation with subtraction in λ),

$$\beta_\varepsilon(g) = 3g^2 - 11.33g^3 + 130.2g^4 - 2173g^5 + \ldots \tag{3}$$

in the dimensional renormalization scheme.

The transition from any of expressions (1), (2), (3) written down as

$$b(g) = \sum_{k\geq 2} (-g)^k b_k$$

to another

$$b(g) \to \beta(\gamma) = \sum_{k\geq 2} \beta_k (-\gamma)^k$$

can be performed by changing the coupling constant variable

$$g \to \gamma(g) = g + \gamma_2 g^2 + \gamma_3 g^3 + \dots \tag{4}$$

according to the well-known relation *)

$$\beta(\gamma) = \gamma'(g)\, b(g) . \tag{5}$$

Substituting into r.h.s. of (5) the Eq. reverse with respect to (4) and comparing the coefficients of the same powers of γ variable, we get the set of relations between the coefficients

$$\begin{aligned}
&\beta_2 = b_2 \,, \quad \beta_3 = b_3 \,, \\
&\beta_4 = b_4 + \gamma_2 b_3 + (\gamma_3 - \gamma_2^2)\, b_2 \,, \\
&\beta_5 = b_5 + 2\gamma_2 b_4 + \gamma_2^2 b_3 + 2(3\gamma_2\gamma_3 - 2\gamma_2^3 - \gamma_4)\, b_2 \,, \\
&\beta_6 = b_6 + 3\gamma_2 b_5 - \gamma_3 b_4 + 8(\gamma_2^3 + 2\gamma_2\gamma_3 - \gamma_4)\, b_3 + \\
&\qquad + (-30\gamma_2^3 - 4\gamma_3^2 + 16\gamma_2^2\gamma_3 - 12\gamma_2\gamma_4 + 3\gamma_5)\, b_2 \,, \dots
\end{aligned} \tag{6}$$

Considering this set as a system of equations for defining the coefficients $\gamma_2, \gamma_3, \gamma_4, \dots$ (all the b_i, β_k are given) we discover that this system turns out to be underdetermined. This is a consequence of a very general property that can be seen from the following. For given coefficients b_i there exists a function $\gamma(g, c; b_i)$ depending, besides on g and b_i on one numerical parameter c such that the substitution

$$g \to \gamma(g, c; b_i)$$

does not affect the beta-function. This property becomes evident if one finds the solution $\bar{g}(x, g; b_i)$ of RG (renormalization group) equations and defines

*) See, e.g. [2] Eq. (49,26) .

$$\gamma(g, \ln x; b_i) \equiv \bar{g}(x, g; b_i) \quad , c = \ln x \,. \tag{7}$$

Transformations of this type have the group structure and can be considered as proper RG transformations. These proper transformations do not change the beta-function.

All other transformations from the class (4) which cannot be reduced to proper RG transformations and change the beta-function can be considered as improper coupling constant transformations.

To fix the improper transformation uniquely it is sufficient to impose one additional condition. By choosing it in the form $\gamma_2 = 0$ we automatically eliminate proper transformations. We shall call a transformation of the type

$$g \to \tilde{g} = g + \gamma_3 g^3 + \gamma_4 g^4 + \dots = \Gamma(g\,|\,b \to \beta) \tag{8}$$

the special (improper) coupling constant transformation (ST). An arbitrary improper transformation (4) can now be represented as a composition of ST Eq. (8) and proper RG transformation(7) in which we put $c = \gamma_2 \equiv \gamma$:

$$\gamma(g) = \bar{g}\left\{ e^{\gamma}, \Gamma(g\,|\,b \to \beta); \beta \right\} = \Gamma\left\{ \bar{g}(e^{\gamma}, g; b)\,\middle|\, b \to \beta \right\}. \tag{9}$$

The coefficients of ST $\Gamma(g\,|\,b \to \beta)$ are defined by the relations

$$\begin{aligned}
&\gamma_3 = (\beta_4 - b_4)/b_2 \,, \quad \gamma_4 = (\beta_5 - b_5)/2b_2 \,, \\
&\gamma_5 = (\beta_6 - b_6)/3b_2 + (4/3)\gamma_3^2 + (\gamma_4 b_3 + \gamma_3 b_4)/3b_2 \,, \dots
\end{aligned} \tag{10}$$

The ST for the transitions between Eqs. (1), (2), (3) has the form

$$g_\lambda = g_\Lambda - 4.1\, g_\Lambda^3 + 186.5\, g_\Lambda^4 + \dots \,, \tag{11}$$

$$g_\varepsilon = g_\lambda - 8.0\, g_\lambda^3 + 27.5\, g_\lambda^4 + \dots ,$$

$$g_\Lambda = g_\varepsilon + 12.1\, g_\varepsilon^3 - 214\, g_\varepsilon^4 + \dots .$$

The temptation arises to define the ST so that:

a) to nullify all the "non-single-valued" coefficients $\beta_K (K \geq 4)$ and to get the expression for the beta-function consisting only of two terms

$$\beta_c(g_c) = 3g_c^2 - \frac{34}{3} g_c^3 . \tag{12}$$

The corresponding ST in 4-loop approximation is of the form

$$g_c = g_\varepsilon - 43.4\, g_\varepsilon^3 + 362.2\, g_\varepsilon^4 + \dots \tag{13}$$

or

$$g_\varepsilon = g_c + 43.4\, g_c^3 - 362.2\, g_c^4 + \dots . \tag{14}$$

As follows, the regions of validity of transformations (13), (14) are defined by the inequalities

$$g_c ,\ g_\varepsilon \lesssim 0.1 . \tag{15}$$

Hence the use of Eq. (12) in the region

$$g_c \quad g^* = 9/34 \sim 0.3 ,$$

where its r.h.s. has a zero, cannot be justified. For illustration we note that substituting the numerical value $g_c = 9/34$ into r.h.s. of (14) we get the expression

$$g_\varepsilon(g^*) \sim 0.26 + 0.81 - 1.79 + \dots ,$$

that looks like the convergent series outside its region of convergence or the asymptotic series in the region where it cannot serve as a source of the numerical information.

Hence the problem a) cannot be solved by the presented procedure. Therefore we state the more modest goal:

b) to take out of the higher β_K coefficients the factorially growing terms, the existence of which leads to the known fact that the series

$$\sum_K (-g)^K \beta_K \tag{16}$$

has no finite sum and represents the asymptotic series á la Poincare.

Note that attempts have already been made to use Eq. (12) with a zero at $g = 9/34$ [3,4]. In these papers, however, the problem of transition to Eq. (12) from the "standard" asymptotic series (16) essentially has not been considered.

The above stated problems a) and b) are relative as they can be reduced to the "insertion" into $\Gamma(g)$ of infinite series containing exact β_K (for the case (a)) or approximate $\tilde{\beta}_K$ (in the case (b)) coefficients of the beta-function.

Using the fact that for the case under consideration the leading asymptotics of the β_K coefficients at large k is known

$$\beta_K \to \tilde{\beta}_K = K!\, B_K , \quad B_K = 1.096\ 2^{K-1} . K^{7/2} \tag{17}$$

we state the problem:

c) to pass from the asymptotic series containing $\tilde{\beta}_K$ coefficients

$$\tilde{\beta}(g) = \sum_K (-g)^K \tilde{\beta}_K \to \tilde{\beta}_c(g) \tag{18}$$

to the truncated function $\tilde{\beta}_c$ containing only two terms

$$\tilde{\beta}_c(g) = \tilde{\beta}_2 g^2 - \tilde{\beta}_3 g^3 . \tag{19}$$

To consider this problem we have to solve the nonlinear differential Eq. (5):

$$\tilde{\beta}(g) \frac{d\Gamma(g)}{dg} = \tilde{\beta}_c(\Gamma) = \tilde{\beta}_2 \Gamma^2(g) - \tilde{\beta}_3 \Gamma^3(g) \tag{20}$$

for the singular transformation (8) . Our main goal is to find the structure of the series for the solution $\Gamma(g)$.

Eq. (20) can be integrated in an elementary way (see, e.g. /4/) that leads to the implicit transcendental Eq. for the Γ

$$1/\Gamma(g) + \ell \ln\left[g/\Gamma(g) - g\ell\right] = 1/g - \Phi(g) , \tag{21}$$

$$\Phi(g) = \int_0^g dx \left[\frac{\tilde{\beta}_2}{\tilde{\beta}(x)} - \frac{1}{x^2} - \frac{b}{x} \right], \quad b = \tilde{\beta}_3 / \tilde{\beta}_2 . \tag{22}$$

To analyse Eq. (21), we use the trick described earlier in our paper /5/ . Namely, we shall follow only the leading terms of asymptotic expansions.

For

$$\tilde{\beta}(g) \sim \sum_{k \geq 2} k!\, B_k (-g)^k \tag{23}$$

we obtain from Eq. (22)

$$\Phi(g) \sim \sum_{k \geq 1} (k+2)!\, \Phi_k (-g)^k, \quad \Phi_k = B_{k+3} / \tilde{\beta}_2 . \tag{24}$$

Now it follows from Eq. (21) that the function $1/\Gamma(g)$ has a structure similar to Eq. (24). Using the symbolical equality

$$AS_1(g) \sim \exp(AS_1(g)) \sim \ln AS_1(g) \tag{25}$$

$AS_1(g)$ being the asymptotic power series without constant term (i.e., similar to (23) or (24) , we get

$$g/\Gamma(g) \sim 1 + \sum_{k \geq 2} (k+1)!\, \Phi_k (-g)^k + \dots \tag{26}$$

and correspondingly

$$\Gamma(g) \sim \sum k!\, \Phi_{k-1} (-g)^k = \sum k!\, \frac{B_{k+2}}{\tilde{\beta}_2} (-g)^k . \tag{27}$$

Thus, we have shown that improper singular coupling constant transformation, "removing" from the beta-function the asymptotic series with factorial coefficients, or at least its "leading" component, is represented by the asymptotic series. The "leading" terms of this last series can be expressed rather simply in terms of "removed" terms

Hence for the solution of problems a), b) or c) formulated above it is necessary at least to perform summation of the asymptotic series (27) of the same structure as the initial series for the beta-function. Summation of the series for $\beta(g)$ is the direct and more simple way to the goal.

The author is indebted to Drs. A.A.Vladimirov and D.I.Kazakov for valuable discussions.

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Received by Publishing Department
on June 25 1980.

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