сообщения
объединенного
института
ядерных
исследований
дубна

E2-80-40

M.M.Enikova, V.I.Karloukovski

EXACT CLASSICAL SOLUTIONS IN THE NAMBU-JONA-LASINIO SPINOR-FIELD MODEL

1980

It is difficult to overestimate the significance of the spinor fields in the theory of elementary particles. They are used to describe the really fundamental stable constituents of matter like electrons, protons, neutrinos, quarks. The primary role of the spinor fields was brought to its philosophical extreme long ago by Heisenberg and his coworkers /1-4/. The work of Finkelstein et al./5,6/ where certain soliton-type solutions were written down is also to be noted.

A new strand in this field of research was initiated by Nambu and Jona-Lasinio /7,8/. It was inspired by the works of Bardeen, Cooper, Schrieffer /9/ and Bogolubov /10,11/ on superconductivity and intimately related to the idea of chiral symmetry (to account for the particle-hole symmetry in the assumed analogy with super conductivity). A more discussion of this model (related to the extended models of elementary particles and in the framework of the relativistic Hartree-Fock-Bogolubov approximation) one can find in ref./12/, for instance. In ref./13/ it is applied to the electromagnetic mass difference of the nucleon and pion.

The general increase of interest in exact (localized) solutions to nonlinear problems in the last few years applied to spinor-field theories in particular /14-20/, mostly in 1+1-dimensional space-time solutions to certain spinor-field equations were even studied numerically /17,20/. Let us also recall that the spinor theories present a way out of the difficulties related to the Derrick theorem /21-23/.

In the present work we find exact solutions to the equations

$$i\gamma^{\mu}\partial_{\mu}\psi + 2g_0[(\tilde{\psi}\psi) - (\tilde{\psi}\gamma^5\psi)\gamma^5]\psi = 0 \qquad (1)$$

of the Nambu-Jona-Lasinio model /7,8/ (with commuting field components) in 3+1-dimensional space-time (in 1+1 space-time dimensions it is known as the (massless) Thirring model). Its Lagrangian is

$$\mathfrak{L} = i\tilde{\psi}\gamma^{\mu}\partial_{\mu}\psi + g_0[(\tilde{\psi}\psi)^2 - (\tilde{\psi}\gamma^5\psi)^2]. \qquad (2)$$

The ordinary gauge symmetry, i.e., the invariance of (2) under the transformation $\psi \to e^{i\alpha}\psi$, implies the conservation of the nucleon number

$$\partial_\mu j^\mu(x) = 0, \qquad j^\mu(x) = i\tilde{\psi}(x)\gamma^\mu\psi(x), \qquad Q = \int j^0(x)\,d^3x \tag{3}$$

common to a wide class of theories. The Lagrangian density (2), however, is also invariant under a second gauge group, $\psi \to e^{i\alpha\gamma_5}\psi$, which implies the conservation of the chirality

$$\partial_\mu j_5^\mu(x) = 0, \qquad j_5^\mu(x) = i\tilde{\psi}(x)\gamma^\mu\gamma_5\psi(x), \qquad Q_5 = \int j_5^0(x)\,d^3x, \tag{4}$$

i.e., a higher symmetry is simply realized in this model which increases a priori the chance of solving the field equations.

Following the procedure we have used in our previous work[24-29] we look first of all for plane-wave solutions of Eq. (1),

$$\psi = \psi(\tau), \qquad \tau = nx, \tag{5}$$

where n is a constant vector (for definiteness we choose n to be a unit time-like vector, $n^2 = 1$). Then Eq. (1) takes the form

$$in_\mu\gamma^\mu\dot{\psi} + 2g_0[(\tilde{\psi}\psi) - (\tilde{\psi}\gamma^5\psi)\gamma^5]\psi = 0. \tag{6}$$

We shall denote by ξ and η the real and imaginary parts of ψ. It can be demonstrated that the system of ordinary differential equations (6) is derivable from the Hamiltonian

$$H = -4g_0(J_{14} + J_{23})^2 - 4g_0(J_{12} + J_{34})^2, \tag{7}$$

where

$$J_{ab} = (\xi \wedge \eta)_{ab} = \xi_a\eta_b - \xi_b\eta_a, \tag{8}$$

and ξ and η are certain linear combinations of coordinates q and momenta p.

It is convenient to use a Majorana representation for the Dirac γ-matrices. We shall work, in particular, with

$$\gamma^0 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i\sigma_0 \\ i\sigma_0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} i\sigma_0 & 0 \\ 0 & -i\sigma_0 \end{pmatrix}, \tag{9}$$

where σ_0 and σ_n are the unit and the Pauli matrices. Then the Poisson brackets of ξ'- and η's are

$$\{\xi_a, \xi_b\} = 0 = \{\eta_a, \eta_b\}, \qquad \{\xi_a, \eta_b\} = \frac{1}{2}\Gamma^{-1}_{ab}. \tag{10}$$

We denote

$$\Gamma = \gamma^0 \gamma^\mu n_\mu, \qquad \Gamma^{-1} = n_\mu \gamma^\mu \gamma^0. \tag{11}$$

In this case the (non-unique) expression of ξ and η in terms of coordinates and momenta can be chosen to be

$$\xi_a = \frac{1}{\sqrt{2}} q_a, \qquad \eta_a = \frac{1}{\sqrt{2}} \Gamma^{-1}_{an} p_n \tag{12}$$

The quantities (8) close a Lie algebra

$$\{J_{ab}, J_{mn}\} = \frac{1}{2}(\Gamma^{-1}_{am} J_{bn} + \Gamma^{-1}_{bn} J_{am} - \Gamma^{-1}_{an} J_{bm} - \Gamma^{-1}_{bm} J_{an}). \tag{13}$$

Defining a new basis

$$\begin{aligned} \Omega_0 &= -J_{13} + J_{24} & F &= J_{12} + J_{34} \\ \Omega_1 &= J_{14} - J_{23} & G &= J_{14} + J_{23} \\ \Omega_2 &= J_{13} + J_{24} & & \\ \Omega_3 &= J_{12} - J_{34} & & \end{aligned} \tag{14}$$

so that

$$H = -4 g_0 (F^2 + G^2) \tag{15}$$

and making use of the representation (9) of the γ-matrices it can be brought into the form

$$\{\Omega_\mu, \Omega_\nu\} = \epsilon_{\mu\nu\lambda\sigma} n^\sigma \Omega^\lambda$$

$$\{\Omega_\mu, F\} = n_\mu G, \quad \{\Omega_\mu, G\} = -n_\mu F \quad . \tag{16}$$

$$\{F, G\} = n.\Omega .$$

Let e_a be three vectors satisfying

$$e_a \cdot n = 0, \qquad e_a \cdot e_b = -\delta_{ab} \tag{17}$$

and define

$$Y_0 = n \cdot \Omega = n^\mu \Omega_\mu , \qquad Y_a = e_a \cdot \Omega = e_a^\mu \Omega_\mu . \tag{18}$$

Then the Lie algebra (13) or (16) can be cast into a third form

$$\{Y_a, Y_b\} = \epsilon_{abc} Y_c$$

$$\{Y_a, Y_0\} = 0, \quad \{Y_a, F\} = 0 = \{Y_a, G\} \tag{19}$$

$$\{Y_0, F\} = G, \quad \{F, G\} = Y_0, \quad \{G, Y_0\} = F$$

which makes it explicit that it is the algebra of SU(2) x SU(2).
It follows from (15) and (16) that Ω_μ are constants of the motion

$$\{\Omega_\mu, H\} = 0 \tag{20}$$

and

$$\dot{F} = \{F, H\} = -8 g_0 (n\Omega) G , \qquad \dot{G} = \{G, H\} = 8 g_0 (n\Omega) F \tag{21}$$

so that

$$F = C\cos(\omega\tau + \chi), \quad G = C\sin(\omega\tau + \chi), \quad \omega = 8 g_0 (n.\Omega). \tag{22}$$

This allows one to find out the general solution of the system (6). The result in the rest frame, $n = (1,0,0,0)$ is

$$\xi_1 = A_1 \cos(\alpha_1 \tau + \delta_1) + A_2 \cos(\alpha_2 \tau + \delta_2)$$

$$\xi_2 = 4 g_0 C \left[\frac{A_1}{\alpha_2} \sin(-\alpha_2 \tau + \delta_1 + \chi) + \frac{A_2}{\alpha_1} \sin(-\alpha_1 \tau + \delta_2 + \chi) \right]$$

$$\xi_3 = A_1 \sin(\alpha_1 \tau + \delta_1) + A_2 \sin(\alpha_2 \tau + \delta_2)$$

$$\xi_4 = -4 g_0 C \left[\frac{A_1}{\alpha_2} \cos(-\alpha_2 \tau + \delta_1 + \chi) + \frac{A_2}{\alpha_1} \cos(-\alpha_1 \tau + \delta_2 + \chi) \right] \tag{23}$$

$$\eta_1 = B_1 \cos(\alpha_1 \tau + \epsilon_1) + B_2 \cos(\alpha_2 \tau + \epsilon_2)$$

$$\eta_2 = 4 g_0 C \left[\frac{B_1}{\alpha_2} \sin(-\alpha_2 \tau + \epsilon_1 + \chi) + \frac{B_2}{\alpha_1} \sin(-\alpha_1 \tau + \epsilon_2 + \chi) \right]$$

$$\eta_3 = B_1 \sin(\alpha_1 \tau + \epsilon_1) + B_2 \sin(\alpha_2 \tau + \epsilon_2)$$

$$\eta_4 = -4 g_0 C \left[\frac{B_1}{\alpha_2} \cos(-\alpha_2 \tau + \epsilon_1 + \chi) + \frac{B_2}{\alpha_1} \cos(-\alpha_1 \tau + \epsilon_2 + \chi) \right],$$

where

$$\alpha_{1,2} = -\frac{\omega}{2} \pm \sqrt{\left(\frac{\omega}{2}\right)^2 + (4 g_0 C)^2}$$

and A_i, B_i, δ_i, ϵ_i are integration constants. The solution in a moving frame is readily obtained by a boost transformation on the spinors.

There are three more constants of the motion

$$X_1 = \frac{1}{2}(\xi \Gamma \xi - \eta \Gamma \eta)$$

$$X_2 = \frac{1}{2}(\xi \Gamma \xi + \eta \Gamma \eta) \tag{24}$$

$$X_3 = \xi \Gamma \eta$$

which close the algebra of SU(1,1)

$$\{X_1, X_2\} = X_3\,, \quad \{X_2, X_3\} = X_1\,, \quad \{X_3, X_1\} = -X_2 \tag{25}$$

and are in involution with Y_α (or Ω_μ)

$$\{Y_\alpha, X_j\} = 0. \tag{26}$$

(Even more generaly, $\{J_{ab}, X_j\} = 0$). In this way we see that the Lie algebra of the constants of motion Y_α , X_j is U(1) x SU(2) x SU(1,1). These seven constants, however, are not independent. They are related by

$$Y_1^2 + Y_2^2 + Y_3^2 = -X_1^2 + X_2^2 - X_3^2\,, \tag{27}$$

i.e., the Casimir elements of the two algebras SU(2) and SU(1,1) should be equal. The Hamiltonian is also expressible in terms of these constants

$$H = 8 g_0 (Y_0^2 - Y_1^2 - Y_2^2 - Y_3^2). \tag{28}$$

The constants H, Y_0, Y_2, and X_2 form a set of four independent constants in involution, i.e., the dynamical system we are studying is completely integrable.

Let us note that the one-parameter subgroups generated by the constants Y_α (or Ω_μ) and X_j can easily be found solving the equations

$$\begin{aligned}
\frac{\partial\psi}{\partial\omega_\mu} &= \{\psi, \Omega_\mu\} = \frac{i}{2}\gamma^0\gamma^5\gamma^\mu\psi \\
\frac{\partial\psi}{\partial\sigma_1} &= \{\psi, X_1\} = -i\bar{\psi} \\
\frac{\partial\psi}{\partial\sigma_2} &= \{\psi, X_2\} = -i\psi \\
\frac{\partial\psi}{\partial\sigma_3} &= \{\psi, X_3\} = \bar{\psi}\,.
\end{aligned} \tag{29}$$

In particular, if $\psi(\tau)$ is any solution of Eq. (6) in the rest frame, the set of constants in involution generates the following three-parameter family of solutions to Eq. (6)

$$\psi(\omega_0, \omega_2, \sigma_2, \tau) = \exp(\frac{i}{2}\omega_0 \gamma^0 \gamma^5 \gamma^0) \exp(\frac{i}{2}\omega_2 \gamma^0 \gamma^5 \gamma^2) e^{-i\sigma_2} \psi(\tau). \quad (30)$$

We shall proceed further looking for more general solutions to the field equations (1). To do this we shall make a second step allowing the parameters τ, ω_0, ω_2, $\sigma_2 = \frac{i}{2}\sigma$ to be space-time dependent and applying a local boost

$$S(x) = A_\mu(x)\gamma^\mu \gamma^0 \quad (31)$$

to the field ψ, i.e., we suggest the ansatz

$$\psi(x) = S(x)\psi(\omega_0(x), \omega_2(x), \sigma(x), \tau(x)). \quad (32)$$

Inserting (32) into the field equation (1) yields the following two systems of equations for the unknown functions $A_\mu(x)$, $\tau(x)$, $\sigma(x)$, $\omega_0(x)$, and $\omega_2(x)$

$$\gamma^\mu (\partial_\mu \tau) S(x)\gamma^0 = A^2(x) S(x) \quad (33)$$

$$i\gamma^\mu \partial_\mu S(x) - \frac{1}{2}\gamma^\mu(\partial_\mu \omega_0) S\gamma^0\gamma^5\gamma^0 - \frac{1}{2}\gamma^\mu(\partial_\mu\omega_2) S\gamma^0\gamma^5\gamma^2 + \\ + \frac{i}{2}\gamma^\mu(\partial_\mu\sigma) S = 0. \quad (34)$$

The solution of Eq. (33) is remarkably simple

$$A^\mu(x) = \frac{\partial^\mu \tau \pm n^\mu \sqrt{(\partial\tau)^2}}{\sqrt{2}[n\partial\tau \pm \sqrt{(\partial\tau)^2}]^{1/2}}. \quad (35)$$

Then Eq. (34) can be considered as a complicated system of eight first-order partial differential equations for $\tau(x)$, $\sigma(x)$, $\omega_0(x)$, and $\omega_2(x)$. We do not know its general solution. The simplest possibility is to assume $\tau = \tau(x_0)$, $n = (1,0,0,0)$. Then it reduces to

$$\frac{\partial^2 \tau}{\partial x_0^2} + \frac{1}{2} \frac{\partial \tau}{\partial x_0} \frac{\partial \sigma}{\partial x_0} = 0 \qquad \frac{\partial \omega_0}{\partial x_0} = - \frac{\partial \omega_2}{\partial x_2}$$

$$\frac{\partial \sigma}{\partial x_1} = \frac{\partial \omega_2}{\partial x_3} \qquad \frac{\partial \omega_0}{\partial x_1} = 0$$

$$\frac{\partial \sigma}{\partial x_2} = 0 \qquad \frac{\partial \omega_0}{\partial x_2} = - \frac{\partial \omega_2}{\partial x_0} \tag{36}$$

$$\frac{\partial \sigma}{\partial x_3} = - \frac{\partial \omega_2}{\partial x_1} \qquad \frac{\partial \omega_0}{\partial x_3} = 0$$

and has the obvious solution

$$\omega_0 = f(x_0 - x_2) - g(x_0 + x_2)$$

$$\omega_2 = f(x_0 - x_2) + g(x_0 + x_2) + \kappa(x_1, x_3) \tag{37}$$

$$\sigma = - \frac{1}{2} \ln \frac{\partial \tau}{\partial x_0} + \lambda(x_1, x_3)$$

with f and g arbitrary and $\kappa(x_1, x_3)$ and $\lambda(x_1, x_3)$ the real and imaginary parts of an arbitrary analytic function. This demonstrates that the class of plane-wave solutions is essentially enlarged by our second step.

There are also other solutions of the system (34) which one could write explicitly. For instance, let us assume that ω_0 and ω_2 are constants

$$\omega_0 = \text{const}, \qquad \omega_2 = \text{const} \tag{38}$$

and that

$$\partial_\mu \tau \, \partial^\mu \tau = \phi^2(\tau), \qquad \phi \ \text{arbitrary}. \tag{39}$$

Then it turns out that four of the equations (34) are satisfied identically and the other four take the form

$$\partial_\nu \sigma = \frac{2}{A^2}(A_\mu \partial^\mu A_\nu - A_\nu \partial^\mu A_\mu) - \partial_\nu \ln A^2 . \tag{40}$$

The integrability condition for this system is

$$\Box \tau = \phi'(\tau) \phi(\tau). \tag{41}$$

Let $u(x)$ be any solution of the wave equation

$$\Box u = 0 \tag{42}$$

obeying

$$\partial_\mu u \, \partial^\mu u = 1 \tag{43}$$

and let

$$\tau(x) = F(u(x)), \qquad F^{-1}(\tau) = \int \frac{d\tau}{\phi(\tau)}. \tag{44}$$

Then, inserting $u = F^{-1}(\tau)$ in (42) and (43), one readily verifies that τ satisfies (39) and (41). That is, any solution of (42), (43) yields an infinite family of solutions to (39) and (41) for which the system (40) is integrable and in this way one obtains an infinite family of solutions to the field equation (1). Equations (42), (43) were discussed in/24-29/ .

For any such solution one can calculate explicitly the energy-momentum

$$T^{\mu\nu} = i\tilde{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\,[i\tilde{\psi}\gamma^\lambda\partial_\lambda\psi + g_0((\tilde{\psi}\psi)^2 - (\tilde{\psi}\gamma^5\psi)^2)] \tag{45}$$

and the angular momentum

$$M^\lambda_{\mu\nu} = x_\nu T^\lambda_\mu - x_\mu T^\lambda_\nu + S^\lambda_{\mu\nu}$$

$$S^\lambda_{\mu\nu} = -\frac{i}{4}\tilde{\psi}\gamma^\lambda[\gamma_\mu, \gamma_\nu]\psi. \tag{46}$$

densities as well as the nucleon number (3) and chirality (4).

One could look further for physical (finite-energy) solutions and investigate their stability and possible deviation from the Derrick theorem/21-23/ . And these are not the only problems which arise but we put off their discussion.

We thank Prof. I.T.Todorov for valuable discussion and encouragement.

REFERENCES

1. Heisenberg W. Nachr. Acad. Wiss. Göttingen, 1953, 2a, p.111.
2. Heisenberg W. Zeits. Natur., 1954, 9a, p.292.
3. Dürr H.P. et al. Nuovo Cimento, 1965, 38, p.1120.
4. Heisenberg W. Introduction to the Unified Field Theory of Elementary Particles, New York, 1966.
5. Finkelstein R.J., Lelevier R., Ruderman M. Phys. Rev., 1951, 83, p.326.
6. Finkelstein R.J., Fronsdal C., Kaus P. Phys. Rev., 1956, 103, p.1571.
7. Nambu Y., Jona-Lasinio G. Phys. Rev., 1961, 122, p.345.
8. Nambu Y., Jona-Lasinio G. Phys. Rev., 1961, 124, p.246.
9. Bardeen J., Cooper L.N., Schrieffer J.R. Phys. Rev., 1957, 106, p.162.
10. Bogolubov N.N. JETP, 1958, 34, pp.58,73.
11. Bogolubov N.N., Tolmachev V.V., Shirkov D.V. A New Method in the Theory of Superconductivity, 1958.
12. Eguchi T., Sugavara H. Phys. Rev., 1974, 10D, p.4257.
13. Hoskizaki N., Itami T., Kokubo H. Progr. Theor. Phys., 1979, 61, p.1865.
14. Chang S.-J., Ellis S.D., Lee B.W. Phys. Rev., 1975, 11D, p.3572.
15. Lee S.Y., Kuo T.K., Gavrielides. Phys. Rev., 1975, 12D, pp.2249,3880.
16. Kaus P. Phys. Rev., 1976, 14D, p.1722.
17. Rafelski J. Phys. Rev., 1976, 14D, p.2358.
18. Friedberg R., Lee T.D. Phys. Rev., 1977, 15D, p.1694; 16D, p.1096.
19. Van Der Marwe P.Du T. Nuovo Cimento, 1977, 41A, p.621.
20. Rafelski J. Phys. Lett., 1977, 66B, p.262.
21. Hobart R. Proc. Phys. Soc., 1963, 82, p.201.
22. Derrick G.H. J. Math. Phys., 1964, 5, p.1252.
23. Goldstone J., Jackiw R. Phys. Rev., 1975, 11D, p.1486.
24. Enikova M.M., Karloukovski V.I. Bulg. J. Phys., 1979, 6, p.123.
25. Enikova M.M., Karloukovski V.I. Phys. Lett., 1978, 69A, p.175.
26. Enikova M.M., Karloukovski V.I. Phys. Lett., 1979, 70A, p.67.
27. Enikova M.M., Karloukovski V.I., Velchev C.I., Nucl. Phys., 1979, 151B, p.172.
28. Enikova M.M., Karloukovski V.I. JINR, E2-12257, 1979, Dubna.
29. Enikova M.M., Karloukovski V.I. JINR, E2-12783, 1980, Dubna.

Received by Publishing Department
on January 18 1980.