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INTEGRABLE TWO-DIMENSIONAL LORENTZ-INVARIANT NONLINEAR MODEL OF COMPLEX SCALAR FIELD (Complex Sine-Gordon-II)

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1. Introduction

In the author's paper [1], stimulated by the urgent problem of classification and enumeration of all exactly soluable two-dimensional models of field theory and devoted to the search and enumeration of two-dimensional Lorentz-invariant Lagrange equations for one complex scalar field which possess higher polynomial local conserved currents (PLCD), a new model has been found with the Lagrangian $(f_\mu = \partial_\mu f\ ,\ \ \mu = 0,1)$

$$\mathcal{L} = \frac{1}{2}\left(\varphi_\mu^2 + tg^2\frac{\varphi}{2}\beta_\mu^2 - m^2\sin^2\varphi\right), \tag{1.1a}$$

or in the variables $\Psi = \sin\varphi/2\ e^{i\beta}$

$$\mathcal{L} = \frac{|\Psi_\mu|^2}{1-|\Psi|^2} - m^2|\Psi|^2(1-|\Psi|^2). \tag{1.1b}$$

This model is a generalization ("complexification") of the model "sine-Gordon" (S.G.) integrable by the inverse scattering method (ISM) and it differs from the earlier found and studied [2-4] integrable complexification with the Lagrangian $(\Psi = \sin\varphi\, e^{i\beta})$

$$\mathcal{L} = \frac{1}{2}\left(\varphi_\mu^2 + tg^2\varphi\,\beta_\mu^2 - m^2\sin^2\varphi\right) = \frac{1}{2}\left[\frac{|\Psi_\mu|^2}{1-|\Psi|^2} - m^2|\Psi|^2\right]. \tag{1.2}$$

The model (1.1) may appropriately be called the "complex S. G.-II".

In view of recent results by M.Wadati [5], Ibragimov, Zhiber and Shabat [6,7] obtained in studying equations with a nontrivial group (or, the same, with an infinity of local conservation lows) the possibility of integrability of the model (1.1) with an infinity of PLCD by ISM becomes almost obvious.

In this paper the ISM integrability of the model (1.1) is established and some its properties are studied. In Sec.2 by the method the author has proposed in [8] the corresponding linear spectral problem ("Lax representation") is obtained in terms of matrices of the SU(3)-group algebra. The model (1.1) has soliton solutions with zero and nonzero asymptotics at infinity; in Secs.3,4 these solutions are searched for. Also, a theorem is proved which is useful in calculating one-soliton solutions in models with Lagrangians of the type (1.1), (1.2). In Sec.5 equations of the Hirota-type are found which allow a standard derivation of multi-soliton solutions. And finally, in Sec.6 the equation under consideration is deduced in a different way as the first equation (after reducing to the Lagrange form) of the obtained infinite series of integrable Lorentz-invariant systems generalizing the S.G.equation for which the problem of reduction to the Lagrange or Hamilton form is nontrivial and unsolved in other cases.

2. Lax Representation

The modern formulation of ISM [9-10] is based on the possibility to represent the studied nonlinear equation as the condition of compatibility of two linear differential operators that are rational functions of the spectral parameter.

Consider [8] the overdetermined system of linear equations $(\mu = 0,1)$

$$\partial_\mu \chi = \frac{i}{2}\,\omega_\mu \chi \; . \tag{2.1}$$

Here χ belongs to the NxN-matrix group G, ω_μ is the matrix Lorentz vector lying, obviously, in algebra AG: $\omega_\mu = -2i\,\partial_\mu \chi \cdot \chi^{-1}$ and dependent on field functions $\varphi_i(x)$ and spectral parameter K_μ, $K_\mu^2 = m^2$, that is a Lorentz vector in the case of Lorentz-invariant scalar equations. The condition of compatibility of (2.1) in the covariant form $\varepsilon_{\mu\nu}\partial_\mu\partial_\nu\chi = 0$ $(\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \varepsilon_{01} = 1)$ reads:

$$2\,\tilde\omega_{\mu,\mu} + i\,\omega_\mu\,\tilde\omega_\mu = 0 \; . \tag{2.2}$$

Hereafter $\tilde a_\mu$ is a vector dual to vector a_μ: $\tilde a_\mu = \varepsilon_{\mu\nu} a_\nu$, $\tilde{\tilde a}_\mu = a_\mu$ . .

Expand ω_μ over the complete set of matrices $\lambda_0 = I$, λ_i $(i = 1, 2, \ldots, N^2-1)$ (with complex, in general, C-number coefficients):

$$\omega_\mu = \omega_\mu^0\,\lambda_0 + \omega_\mu^i\,\lambda_i \; , \tag{2.3}$$

where λ_i are Hermitean generators of group SU(N).

Inserting (2.3) in (2.2) we arrive at the system of equations (f_{ijk} are structure constants of SU(N))

$$\tilde\omega^i_{\mu,\mu} + \frac{1}{2} f_{ijk}\,\omega^j_\mu\,\tilde\omega^k_\mu = 0$$

$$\tilde\omega^0_{\mu,\mu} = 0 \tag{2.4}$$

whence it follows that ω_μ can be regarded as a traceless matrix $(\omega_\mu^0 = 0)$.

Fixing now the rank of group N, properly choosing vectors ω_μ^i and requiring (2.4) to be identically fulfilled with respect to parameter K_μ, we can obtain nontrivial Lorentz-invariant systems of equations integrable by ISM with the use of the linear problem (2.2).

The case of SU(2) (i=1,2,3) has been carefully analysed in [8]; the S.G.equation $\partial_\mu^2\varphi + \sin\varphi\cos\varphi = 0$ is given by (2.4) at

$$\omega^1_\mu = \tilde\varphi_\mu\,;\quad \omega^2_\mu = -\tilde K_\mu \sin\varphi\,;\quad \omega^3_\mu = K_\mu\cos\varphi \quad (\varphi_\mu = \partial_\mu\varphi), \tag{2.5}$$

and "complexification" of (2.5): $(\beta_\mu = \partial_\mu\beta)$

$$\omega^2_\mu = -\tilde K_\mu \sin\varphi \longrightarrow (-\tilde K_\mu\sin\varphi + \tilde\beta_\mu\, g_1(\varphi))$$

$$\omega^3_\mu = K_\mu\cos\varphi \longrightarrow (K_\mu\cos\varphi + \beta_\mu\, g_2(\varphi))$$

appears to be consistent at $g_1 = \sec\varphi\,\mathrm{tg}\,\varphi$, $g_2 = \sec\varphi$ and results in the equation "complex S.G.-I" following from the Lagrangian (1.2).

In the framework of SU(2) there are no other complexifications of the S.G.equation, therefore, it is natural to consider the larger algebra of SU(3) with dimension 8 into which the SU(2) algebra is enclosed as a subalgebra in different ways given, e.g., by the explicit form of nonzero structure constants of SU(3):

$$f_{123} = 1\,; \quad f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}$$
$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}\,. \tag{2.6}$$

To obtain the system following from the Lagrangian (1.1):

$$\begin{cases} \partial_\mu^2 \varphi - \frac{1}{2}\, tg\frac{\varphi}{2} \sec^2\frac{\varphi}{2}\, \beta_\mu^2 + m^2 \sin\varphi \cos\varphi = 0 \\ (tg^2 \frac{\varphi}{2} \beta_\mu)_{,\mu} = 0 \end{cases} \tag{2.7}$$

we take first $\omega_\mu^1, \omega_\mu^2, \omega_\mu^3$ in the form (2.5); then eq. (2.4) gives again the S.G. equation since $\lambda_1, \lambda_2, \lambda_3$ form a subalgebra of SU(3). By analysing (2.6) it can be verified that the system (2.4) is not obviously contradictory if we put

$$\omega_\mu^4 = g_1(\varphi)\beta_\mu \;; \qquad \omega_\mu^5 = \omega_\mu^6 = 0$$
$$\omega_\mu^7 = g_2(\varphi)\tilde{\beta}_\mu \;; \qquad \omega_\mu^8 = \frac{C}{\sqrt{3}} K_\mu \;, \tag{2.8}$$

where g_1, g_2 are unknown functions and C=const. Inserting (2.5) and (2.8) into eq.(2.4) we observe that the second, third, and eighth equations become identities; the first and seventh equations give "the equations of motion" $(g' = dg/d\varphi)$

$$\begin{cases} \partial_\mu^2 \varphi + g_1 g_2 \beta_\mu^2 + m^2 \sin\varphi \cos\varphi = 0 \\ g_2 \partial_\mu^2 \beta + g_2' \varphi_\mu \beta_\mu - g_1 \varphi_\mu \beta_\mu = 0 \end{cases} \tag{2.9}$$

the fourth, fifth, and sixth equations (by equating the coefficients of equal tensor structures) lead to the "equations of constraint"

$$\begin{cases} g_1' + \frac{1}{2} g_2 = 0 \\ g_2 \sin\varphi - g_1 \cos\varphi - C g_1 = 0 \\ g_1 \sin\varphi + g_2 \cos\varphi - C g_2 = 0 \end{cases} \tag{2.10}$$

The overdetermined system (2.10) is consistent at $C^2=1$ and at C=-1 has the solution

$$g_1 = \sec\frac{\varphi}{2} \;; \qquad g_2 = -\sec\frac{\varphi}{2}\, tg\frac{\varphi}{2} \;.$$

Inserting these solutions into the system (2.9) we find that it concides with (2.7).

So, the system (2.7) is the condition of compatibility of the linear equations (2.1) where

$$\omega_\mu = \omega_\mu^i \lambda_i \;; \quad i = 1, \ldots, 8 \;; \quad \lambda_i \in ASU(3) \;;$$
$$\omega_\mu^1 = \tilde{\varphi}_\mu \;; \quad \omega_\mu^2 = -\tilde{K}_\mu \sin\varphi \;; \quad \omega_\mu^3 = K_\mu \cos\varphi \;; \quad \omega_\mu^4 = \sec\frac{\varphi}{2} \beta_\mu \;;$$
$$\omega_\mu^5 = \omega_\mu^6 = 0 \;; \quad \omega_\mu^7 = -\sec\frac{\varphi}{2}\, tg\frac{\varphi}{2}\, \tilde{\beta}_\mu \;; \quad \omega_\mu^8 = -\frac{1}{\sqrt{3}} K_\mu \;. \tag{2.11}$$

Now, by using the well-developed ISM one can prove the complete integrability of (2.7), find N-soliton solutions, deduce the recurrence formula for higher integrals of motion. In this paper our consideration will be restricted to one-soliton solutions which can most easily be obtained by the direct integration of (2.7).

3. Soliton Solutions with Zero Asymptotics

From the "potential" shape in the Lagrangian (1.1b) it can be observed that the corresponding equation can possess solutions both with zero and nonzero asymptotics as $|x| \to \infty$. To find soliton solutions, it is more convenient to work with variables $\Psi = \sin^2\frac{\varphi}{2} e^{i\beta}$ in which the Lagrangian (1.1a) looks as follows:

$$\mathcal{L} = \frac{|\Psi_\mu|^2}{|\Psi|(1-|\Psi|)} - 4m^2 |\Psi| (1 - |\Psi|)\,. \tag{3.1}$$

The corresponding equation of motion has the form

$$\partial_\mu^2 \Psi - \frac{\bar{\Psi} \Psi_\mu^2 (1 - 2|\Psi|)}{2|\Psi|^2 (1 - |\Psi|)} + 4m^2 \Psi (1 - |\Psi|)(1 - 2|\Psi|) = 0. \tag{3.2}$$

We will look for the localized solutions of (3.2) of the form standard for complex equations (in the soliton rest frame):

$$\Psi = f(x)\, e^{i\omega t} , \tag{3.3}$$

$f(x) \to 0$, $f_x = \partial_x f \to 0$ as $|x| \to \infty$. Direct integration of equations of the type (3.2) that reduce, with solution of the form (3.3), to the ordinary differential equations of second order is a practically nontrivial problem. This procedure may be considerably simplified and reduced to one quadrature through the following

Theorem. Equations with the Lagrangians

$$\mathcal{L}_1 = |\Psi_\mu|^2 - V(|\Psi|), \tag{3.4}$$

and

$$\mathcal{L}_2 = A(|\Psi|)\, \mathcal{L}_1 = A(|\Psi|)(|\Psi_\mu|^2 - V(|\Psi|)), \tag{3.5}$$

where $A(|\Psi|)$ is an arbitrary function, have coincident localized solutions of the form $\Psi = f(x)\, e^{i\omega t}$ (3.3) under the conditions:

1) $f(x) \to 0$; 2) $f_x \to 0$; 3) $V(|\Psi|) = V(f) \to 0$ as $|x| \to \infty$.

For the real field $\bar\Psi = \Psi$ the condition 1) is unnecessary.

These solutions are also solutions to the first-order equation $T_{11}(f(x), \omega) = 0$, where T_{11} is a component of the energy-momentum tensor $T_{\mu\nu}$ for (3.4).

Proof. Solutions of the equation following from (3.5) $(A' = dA/d|\Psi|)$

$$\partial_\mu (A \Psi_\mu) - \frac{A'\Psi}{2|\Psi|} |\Psi_\mu|^2 + (A'V + AV') \frac{\Psi}{2|\Psi|} = 0 \tag{3.6}$$

satisfy the equation

$$\partial_\mu^2 \Psi + V' \frac{\Psi}{2|\Psi|} = 0 , \tag{3.7}$$

that follows from (3.4) under the condition

$$A' (\bar\Psi \Psi_\mu^2 + V\Psi) = 0 . \tag{3.8}$$

With solutions of the form (3.3) eq. (3.8) takes the form (the trivial case $A' = 0$ is neglected)

$$\omega^2 f^2 + f_x^2 - V(f) = 0 \tag{3.9}$$

whereas equation (3.7) with the same solutions looks:

$$-\omega^2 f - f_{xx} + \frac{1}{2} V'(f) = 0 .$$

The result of its integration coincides with (3.9)

$$\omega^2 f^2 + f_x^2 - V = C$$

(Here C=0 in virtue of the boundary conditions).

The solution (3.9) is given by the quadrature

$$\int \frac{df}{\sqrt{V(f) - \omega^2 f^2}} = x + C_1 . \tag{3.10}$$

And finally, equation $T_{11} = 0$ with solutions (3.3) for

$$T_{\mu\nu} = \Psi_\mu \frac{\delta \mathcal{L}_1}{\delta \Psi_\nu} + \bar\Psi_\mu \frac{\delta \mathcal{L}_1}{\delta \bar\Psi_\nu} - g_{\mu\nu} \mathcal{L}_1$$

has the form (3.9)

$$T_{11} = 2|\Psi_x|^2 + (|\Psi_t|^2 - |\Psi_x|^2 - V) =$$

$$= \omega^2 f^2 + f_x^2 - V(f) = 0 \quad \blacksquare .$$

The Lagrangian (3.1) can be represented as (3.5) where

$$A = (|\Psi|(1 - |\Psi|))^{-1} , \quad V = 4m^2 |\Psi|^2 (1 - |\Psi|)^2 .$$

Then a solution (3.3) to eq. (3.2) is given by quadrature (3.10)

$$\int \frac{df}{\sqrt{4m^2 f^2 (1-f)^2 - \omega^2 f^2}} = x + C_1 . \tag{3.11}$$

By integrating (3.11) we obtain finally

$$\Psi_1 = f e^{i\omega t} = \frac{2\cos^2\alpha\, e^{2m[\cos\alpha (x + C_1) + i \sin\alpha t)}}{1 + 2e^{2m\cos\alpha (x + C_1)} + \sin^2\alpha\, e^{4m\cos\alpha (x + C_1)}} \tag{3.12}$$

Here we set $\omega = 2m\sin\alpha$ as the localized solutions exist for $|\omega| \leq 2m$. only.

For a compact representation, in an arbitrary reference frame, of the solution obtained from (3.12) through the Lorentz transformation it is convenient to use the formalism of complex Lorentz-vector proposed in [3]. So, introduce the complex variable

$$z = z' + i z'' = K_\mu (x_\mu - x_\mu^{(0)}), \tag{3.13}$$

where K_μ is the spacelike complex vector

$$K_\mu^2 = -m^2 ; \quad K_\mu = m[sh(\beta + i\alpha);\ ch(\beta + i\alpha)] ; \quad (K_\mu \bar{K}_\mu) = -m^2 \cos 2\alpha , \tag{3.14}$$

$x_\mu^{(0)}$ is an arbitrary constant vector fixing the soliton position in space-time. Finally, we arrive at the most general form of the one-soliton solution dependent on four parameters $V = th\beta,\ \alpha,\ x_\mu^{(0)}$:

$$\Psi_1 = \frac{2\cos^2\alpha\, e^{2z}}{1 + 2e^{z+\bar{z}} + \sin^2\alpha\, e^{2(z+\bar{z})}} . \tag{3.15}$$

The solution (3.15) is the soliton moving with velocity V, the amplitude and oscillation frequency of which in the rest frame are defined by the parameter α . At $\alpha = 0$, $\bar{z} = z$,

$$\varphi = 2\arcsin\sqrt{\Psi_1} = 2\,\mathrm{arc\,tg}\, e^{z}$$

is the one-soliton solution to the S. G. equation.

Note that at $\alpha = 0$ the solution (3.15) "jumps" to the solution with nonzero asymptotics as $x \to \infty$. This fact does not destroy the validity of the above theorem in integration.

4. Soliton Solutions with Nonzero Asymptotics

We shall search for solutions to eq. (3.2) with the asymptotics $|\Psi| \to 1$ when $|x| \to \infty$. In this case the theorem is inapplicable; however, this difficulty can be surmounted [12,3] by reducing eq. (3.2) to the first-order equation through the evaluation from below of the energy functional (in the soliton rest frame) via the topological charge that always exists for solutions with the nontrivial asymptotics. Let us rewrite the energy functional of eq. (3.2) (assuming $\Psi_t = 0$)

$$H = \int_{-\infty}^{\infty} dx \left[\frac{|\Psi_x|^2}{|\Psi|(1-|\Psi|)} + 4|\Psi|(1-|\Psi|) \right] \tag{4.1}$$

identically as follows

$$H = \int_{-\infty}^{\infty} dx \left| \frac{\Psi_x}{\sqrt{|\Psi|(1-|\Psi|)}} - 2\sqrt{|\Psi|(1-|\Psi|)} \right|^2 + 2\int_{-\infty}^{\infty} dx(\Psi_x + \bar{\Psi}_x) \geq 4Q ,$$

where

$$Q = \Psi'(\infty) - \Psi'(-\infty) \qquad (\Psi = \Psi' + i\Psi'') .$$

It is obvious that the function $\Psi(x)$ realizes minimum of the functional (4.1) provided it obeys the equation

$$\Psi_x = 2|\Psi|(1-|\Psi|) . \tag{4.2}$$

From (4.2) it follows immediately that $\Psi'' = c = const$. Then for Ψ' we get the equation

$$\Psi'_x = 2\left(\sqrt{\Psi'^2 + c^2} - \Psi'^2 - c^2\right)$$

Direct integration gives

$$\Psi' = \frac{\cos\alpha\,(1 - \sin^2\alpha\, e^{4\cos\alpha(x+c_1)})}{1 + 2e^{2\cos\alpha(x+c_1)} + \sin^2\alpha\, e^{4\cos\alpha(x+c_1)}} .$$

here we put $c = \sin\alpha$ since the localized solutions exist only for $|c| \leq 1$. And finally, in an arbitrary reference frame, by using (3.13)-(3.14) and having respect to that the solution of (3.2) is determined up to an arbitrary phase factor ε , we arrive at the most general form of the one-soliton solution dependent on 4 parameters $V, \alpha, x_1^{(0)}, \varepsilon$:

$$\Psi_2 = \Psi' + i\sin\alpha = e^{i\varepsilon}\frac{1 + 2\sin\alpha\, e^{z+\bar{z}+i(\frac{\pi}{2}-\alpha)} + \sin^2\alpha\, e^{2[z+\bar{z}+i(\frac{\pi}{2}-\alpha)]}}{1 + 2e^{z+\bar{z}} + \sin^2\alpha\, e^{2(z+\bar{z})}} =$$

$$= e^{i\varepsilon}\frac{[1 + \sin\alpha\, e^{z+\bar{z}+i(\frac{\pi}{2}-\alpha)}]^2}{1 + 2e^{z+\bar{z}} + \sin^2\alpha\, e^{2(z+\bar{z})}} . \tag{4.3}$$

At $\alpha = \varepsilon = 0$ the solution Ψ_2 (4.3) degenerates into the solution $\Psi_1 |_{\alpha=0}$ (3.15) with the opposite topological charge $(x \to -x)$.

Note also an interesting fact (an analogous fact holds for the equation "complex S.G.-I" [3]):

$$|\Psi_1|^2 + |\Psi_2|^2 = 1$$

what is, probably, valid (like in case [3]) for multisoliton solutions.

5. Hirota formalism

Multisoliton solutions can be easily found by the Hirota method [13]; we do not write them here because of their cumbersome form and cite only the system of equations in the bilinear form. Details of calculations are quite analogous to those of [3]. It is remarkable that both Ψ_1 (3.15) and Ψ_2 (4.3) satisfy the same system. Substitution $\Psi = g/f$, where f is a real function, allows us to write eq. (3.2) as a system of two equations $(m^2 = 1)$

$$\mathcal{D}_\mu^2 f \cdot f = 8(|g|^2 - |g|f)$$

$$g(\mathcal{D}_\mu^2 + 4) g \cdot f = 4g^2|g| - \tfrac{1}{2}(f - 2|g|)\mathcal{D}_\mu^2 g \cdot g \,, \qquad (5.1)$$

so that both Ψ_1 (3.15) and Ψ_2 (4.3) (g and f are numerator and denominator of Ψ_1 or Ψ_2, respectively) obey the system (5.1). Here $\mathcal{D}_\mu^2$ are Hirota Lorentz-invariant operators [3]:

$$\mathcal{D}_\mu^2 g \cdot f = g\partial_\mu^2 f - 2\partial_\mu g \partial_\mu f + f\partial_\mu^2 g$$

with the following property most important in applications

$$\mathcal{D}_\mu^2 e^{z_i} \cdot e^{z_j} = (k_\mu^i - k_\mu^j)^2 e^{z_i + z_j}; \quad z_i = k_\mu^i x_\mu \,; \quad k_\mu^{i\,2} = -1 \,.$$

The second of eqs. (5.1) is trilinear in functions g and f ; however, introducing an auxiliary function h by the relation

$$\mathcal{D}_\mu^2 g \cdot g = 2gh \quad ^{1)}$$

we arrive at the system of equations in the standard bilinear form

$$\begin{cases} \mathcal{D}_\mu^2 f \cdot f = 8|g|(|g| - f) \\ \mathcal{D}_\mu^2 g \cdot g = 2gh \\ (\mathcal{D}_\mu^2 + 4) g \cdot f = 4g|g| - h(f - 2|g|). \end{cases} \qquad (5.2)$$

Functions g, f, h are looked for as series of the form

$$G = G_0 + \sum_{n=1}^{4N} \varepsilon^{2n} G_{2n} \,, \quad \text{here} \quad G_{2n} \sim \exp\left(\sum_{i=1}^{2n} z^{(i)}\right),$$
$$f_0 = 1 \,, \quad h_0 = 0 \,; \quad g_0 = 0$$

for the solutions with zero asymptotics, $g_0 = 1$ for those with nonzero asymptotics; an arbitrary parameter ε can be put to equal unity at the end of calculations. It is not difficult to show that in this model the interaction of soliton is of the character standard for simple models with the trivial dynamics: in collisions solitons are elastically scattered acquiring the complex-phase shift.

6. "U-V-Formalism"

In this section, the "Lax representation" for (2.7) will be deduced in a different way, within a rather general scheme. Let $2\eta = t + x$, $2\xi = t - x$ be "cone" variables in the two-dimensional space-time, $C, \Gamma, \mathcal{X}$ are complex (NxN)-matrix functions of η, ξ. Consider the system of equations ("U-V-system") [14-15]:

$$\begin{aligned} iC_\eta &= [A_0, \Gamma] \\ i\Gamma_\xi &= [\Gamma, C] \,, \end{aligned} \qquad (6.1)$$

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1) The author is thankful to Professor Hirota who has drawn his attention to this "device".

where A_0 is a diagonal matrix independent of η, ξ. This system is manifestly Lorentz-invariant; it is the condition of compatibility (identically in the spectral parameter λ) of the system of linear equations

$$\begin{cases} i\chi_\xi = (C + \lambda A_0)\chi \\ i\chi_\eta = \lambda^{-1}\,\Gamma\chi \ . \end{cases} \tag{6.2}$$

A happy choice of C, Γ, A_0 consistent with eqs.(6.1) (the system reduction) diminishes the number of unkown functions and leads to physically meaningful equations.

Let $\Gamma = \Psi\Gamma_0\Psi^+$ and

$$\Gamma_0 = A_0 = diag(0, 0, \ldots 0, 1) . \tag{6.3}$$

Obviously, $(\Gamma_0)_{ab} = \delta_{ab}\delta_{bN}$ $(a, b = 1, 2, \ldots N)$; then $\Gamma_{ab} = \Psi_{aN}\bar\Psi_{bN}$ where $\bar\Psi$ is a function complex conjugated to Ψ. Γ is a bivector: $\Gamma_{ab} = a_a\bar a_b$, $a_a = \Psi_{aN}$. Substituting Γ into (6.1a) we see that C should be of the form

$$C = \left(\begin{array}{ccc|c} & & & c_1 \\ & 0 & & \vdots \\ & & & c_{N-1} \\ \hline \bar c_1 & \cdots\cdots & \bar c_{N-1} & 0 \end{array}\right)$$

where c_j obey the equation

$$ic_{j\eta} = \bar a_N a_j \ . \tag{6.4}$$

Here $i, j = 1, \ldots N-1$. The second of eqs (6.1) gives the equations for a_i and a_N:

$$i a_{j\xi} = -c_j a_N$$
$$i a_{N\xi} = -\sum_{j=1}^{N-1}\bar c_j a_j \ .$$

Denoting $b = a_N$ we obtain finally

$$\begin{cases} ic_{j\eta} = -\bar b a_j \\ i a_{j\xi} = -b c_j \\ i b_\xi = -\sum_{i=1}^{N-1}\bar c_i a_i \ , \quad j = 1, \ldots N-1 \end{cases} \tag{6.5}$$

The system (6.5) has "the conservation law". Multiplying (6.5.2) by $\bar a_j$, summing over j, adding to (6.5.3) multiplied by $\bar b$ and then adding the complex conjugated result, we arrive at the expression

$$\frac{\partial}{\partial\xi}\left(\sum_{j=1}^{N-1}|a_j|^2 + |b|^2\right) = 0 \ . \tag{6.6}$$

Further reduction is to take real c_j, a_j and imaginary $b = i\tau$. Then (6.5) becomes

$$\begin{aligned} c_{j\eta} &= \tau a_j \\ a_{j\xi} &= -\tau c_j \\ \tau_\xi &= \sum c_i a_i \ . \end{aligned} \tag{6.7}$$

With the help of eq. (6.6) variable τ can be eliminated (equating, without loss of generality, the integration constant to unity):

$$\tau = \sqrt{1 - \sum a_i^2} \ .$$

Inserting this τ into (6.7) gives

$$\begin{cases} c_{j\eta} = a_j\sqrt{1 - \sum a_i^2} \\ a_{j\xi} = -c_j\sqrt{1 - \sum a_i^2} \end{cases} \tag{6.8}$$

and eliminating c_j we obtain

$$\partial_\eta \frac{a_{j\xi}}{\sqrt{1 - \sum a_i^2}} + a_j\sqrt{1 - \sum a_i^2} = 0 \ , \quad j = 1, \ldots N-1 \ . \tag{6.9}$$

In the simplest case N=2, setting $a_1 = \sin\varphi$ we arrive at the S.G. equation for φ:

$$\varphi_{\eta\xi} + \sin\varphi\cos\varphi = 0 \ .$$

For $N > 2$ eq. (6.9) gives an infinite series of systems generalizing the S.G. equation. In such a form these systems (and more general systems (6.5)), except for the case $N=2$ have no Lagrangian; determination of the Lagrangian requires nontrivial changes of variables. The transformations given in this section are found by V.E.Zakharov/16/.

Now we demonstrate what is the way to bring the systems (6.9) and (6.5), in two simplest cases, into the Lagrange form. At $N=3$ in (6.9) the complex variable $a = a_1 + i a_2$ can be introduced, and in parametrization $a = \sin\varphi\, e^{i\omega}$ the system (6.9) becomes

$$\begin{cases} \varphi_{\eta\xi} - tg\varphi\, \omega_\eta \omega_\xi + \sin\varphi \cos\varphi = 0 \\ \omega_{\eta\xi} + ctg\varphi\, (\varphi_\xi \omega_\eta + \sec^2\varphi\, \omega_\xi \varphi_\eta) = 0. \end{cases} \tag{6.10}$$

Introducing the new variable β

$$\begin{cases} \omega_\xi = \frac{1}{2} \beta_\xi \cos\varphi \sec^2 \varphi/2 \\ \omega_\eta = \frac{1}{2} \beta_\eta \sec^2 \varphi/2 \end{cases} \tag{6.11}$$

we obtain from (6.18)

$$\begin{cases} \varphi_{\eta\xi} - \frac{1}{2} tg\frac{\varphi}{2} \sec^2\frac{\varphi}{2} \beta_\eta \beta_\xi + \sin\varphi\cos\varphi = 0 \\ \sin\varphi\, \beta_{\eta\xi} + \varphi_\eta \beta_\xi + \varphi_\xi \beta_\eta = 0. \end{cases} \tag{6.12}$$

The condition of compatibility of the overdetermined system (6.11) coincides with the second equation (6.12) and this ensures the construction to be consistent. And finally, in variables x and t we obtain the system (2.7):

$$\begin{cases} \varphi_{\mu\mu} - \frac{1}{2} tg\frac{\varphi}{2} \sec^2\frac{\varphi}{2} \beta_\mu^2 + \sin\varphi\cos\varphi = 0 \\ (tg^2\frac{\varphi}{2}\, \beta_\mu)_\mu = 0. \end{cases}$$

The second system that can be brought into the Lagrange form follows from (6.5) at $N=2$. With due regard to (6.6) one can introduce new variables by the relations $a_1 = \sin\varphi\, e^{i\omega_1}$,

and $b = \cos\varphi\, e^{i\omega_2}$ then in the system obtained for φ and $\omega = \omega_1 - \omega_2$ perform the change of variables (6.11); as a result, for the variable $\psi = \sin\varphi\, e^{i\beta}$ we get the system with the Lagrangian (1.1) ("complex S. G.-I").

Finally, we show that the system (6.10) is a result of the nontrivial reduction of the system of three S.G. fields interacting with nontrivial dynamics studied in ref. [15]. If in system (1.9) of ref. [15b] (obtained also within the U-V-system (6.1)) we set $E_{12} = \gamma_{12} = 0$ that corresponds to our choice of matrices Γ_0 and A_0 in the form (6.3), then this system takes the form

$$\begin{cases} \theta_{\eta\xi} + \varphi_\eta \psi_\xi \cos\theta - \sin\theta\cos\theta = 0 \\ \varphi_{\eta\xi} - \varphi_\eta \theta_\xi\, tg\theta - \theta_\eta \psi_\xi \sec\theta = 0 \\ \psi_{\eta\xi} - \theta_\eta \psi_\xi\, tg\theta - \varphi_\eta \theta_\xi \sec\theta = 0 \end{cases} \tag{6.13}$$

Multiplying the second equation (6.13) by $\sin\theta$ and substracting the third one, we get

$$(\varphi_\eta - \psi_\eta \sin\theta)_\xi = 0$$

whence it can be put

$$\psi_\eta = \varphi_\eta \csc\theta \tag{6.14}$$

Substituting (6.14) into (6.13) and changing the notation $\varphi \to \omega$, $\theta = \varphi + \frac{\pi}{2}$ we arrive at the system (6.10).

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