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TWO-DIMENSIONAL LORENTZ-INVARIANT FIELD THEORY MODELS WITH HIGHER INTEGRALS OF MOTION.

I. Complex Scalar Fields

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1. INTRODUCTION

A lot of recently discovered exactly solvable two-dimensional nonlinear models of the relativistic field theory integrable by the inverse scattering method (ISM) [1-13] makes important the problem of their "enumeration" and classification. This problem is a part of a more general problem of classification of all systems integrable by ISM (IS); its solution is, however, complicated because of the absence of direct regular methods for defining the possible integrability by ISM of an arbitrary given system of equations.

The unique mathematical properties of IS and wide applications make it unnecessary to prove importance of their research. Two approaches are possible for the IS classification. The first of them ("deductive"), most general, consists in the enumeration of all such systems and subsequent classification. The enumeration of all IS can, in principle, be realized within a method for constructing such systems, the most general of them is, probably, the method by Zakharov and Shabat [14] (see also refs. [5,6]); this method reduced the problem of enumeration of IS to the nontrivial problem of enumeration of reductions of systems of the general type still far from the general and complete solution. The second nontrivial problem is the selection of the systems with the Lagrange or Hamilton structure (without which the system is of no interest for applications) or the search for the nontrivial changes of variables that result in the IS Lagrange or Hamilton form, when the corresponding structures may be expected (see examples in refs. [5,12,13]). Another more particular method for constructing the Lorentz-invariant IS proposed by the author [13] is free of difficulties with reductions but not with the Lagrangian problem.

The second approach ("inductive") is the IS enumeration within the classification independent of methods for constructing IS and suitable for applications. This approach could be realized after the works by Wadati [15], Ibragimov and Shabat [16] since these works make it clear that the existence of the infinite set of local conserved densities (LCD) is not only the necessary but also sufficient condition of integrability by ISM of a wide class of nonlinear systems of equations. In papers [15,16] the Lax L-M pair has been reproduced and the relevant concequences have been established on the basis of the infinite set of LCD for the Corteveg-de Vries equation, the set being originated by the nontrivial symmetry group of Lie-Backlund. In paper [17] all integrable equations of the form

$$\phi_{\eta\xi} = g(\phi) \tag{1.1}$$

are found by analysing the LCD.

In ref. [18] it is shown that the existence of at least one higher-order LCD for the local translational-invariant equation entails the existence of an infinity of LCD. Thus, the second approach reduces to finding the systems of equations with at least one higher-order LCD and then by the method of refs. [15,16] the L-M pair is, in principle, restored. In this way, the problem is solved for enumeration of the IS having higher-order LCD within a given, e.g. (1.1), or a more general type of equations. Supposing each IS in a two-dimentional space to have higher LCD we find how to check the system integrability by ISM (in its present formulation): suffice it to list all the systems with higher-order LCD from the class that includes the given system. The method of enumeration of all such systems will be shown to be quite regular for a wide class of systems though technical difficulties can increase with the system complication.

This paper starts the series of works devoted to search and enumeration of the systems with higher LCD among the relativistic field-theory models in two-dimensional space-time. The simplest generalization to eq.(1.1) will be considered: the class of Lorentz-covariant Lagrange equations in two-dimensional space-time which possess higher-order LCD for one complex scalar field with the real (O(2)-symmetric) Lagrangian linear in field gradients squared.

Note that the higher-order LCD for eq.(1.1) were also considered earlier in papers [19,20].

2. BASIC RELATIONS

The most general form of the Lorentz- and gauge-invariant Lagrangian for one complex field ψ (or, the same, for an isovector field ψ^a with O(2) - symmetry) linearly dependent on field-gradients squared is:

$$\mathfrak{L} = V_1(|\psi|)|\psi_\mu|^2 + V_2(|\psi|)j_\mu^2 + V_3(|\psi|). \tag{2.1}$$

Notation is used: $\psi_\mu = \partial_\mu \psi$; $\nu,\mu = 0,1$; $|\psi_\mu|^2$ and $j_\mu^2 = (\psi_\mu\bar{\psi} - \bar{\psi}_\mu\psi)^2$ are the only gauge-invariant structures with derivatives of first order at most. Though the complex field ψ can be handled with directly, it is convenient to pass to its polar representation: by the change of variables $\psi = F(\phi)e^{i\beta}$ with a proper choice of F the Lagrangian (2.1) can be transformed as:

$$\mathfrak{L} = \frac{1}{2}\phi_\mu^2 + A\beta_\mu^2 - V, \tag{2.2}$$

where functions A and V depend on ϕ only. The relevant equations of motion read ($F' = dF/d\phi$)

$$\phi_{\mu\mu} - A'\beta_\mu^2 + V' = 0$$
$$(A\beta_\mu)_\mu = 0. \tag{2.3}$$

It can be readily verified that the second of eqs.(2.3) (a divergent-like) represents the law of current conservation of the field ψ.

Let us here formulate an important fact valid in the two-dimentional space-time as the following.

<u>Proposition</u>: A model with the Lagrangian (2.2) is equivalent to a model with the Lagrangian

$$\mathfrak{L} = \frac{1}{2}\phi_\mu^2 - c^2 A^{-1}\gamma_\mu^2 - V \tag{2.4}$$

(c^2 = const) up to the differential change of variables $\beta_\mu = A^{-1} c\epsilon_{\mu\nu}\gamma_\nu$ ($\epsilon_{\mu\nu}$ is a unit antisymmetric tensor, $\epsilon_{01} = 1$). Indeed, from (2.3b) it follows that $A\beta_\mu$ is a two-dimensional curl:

$$A\beta_\mu = c\epsilon_{\mu\nu}\gamma_\nu, \tag{2.5}$$

that results in the following equation for field γ

$$c(A^{-1}\gamma_\nu)_\nu = (\epsilon_{\lambda\mu}\beta_\mu)_\lambda = 0 \qquad (\epsilon_{\mu\lambda}\epsilon_{\mu\nu} = -\delta_{\lambda\nu}).$$

The Lagrangian corresponding to (2.3) by the change (2.5) takes the form (2.4). The models (2.2) and (2.4) will further be called dual and considered equivalent. Note that such a duality exists also in more general cases [13] when the fields β_μ belong to the algebra of SU(N) group: in particular, for SU(2) one can pass to the quaternionic field ψ. In this case that $\beta_\mu \in AU(1)$ may be regarded as a helpful duide in searching integrable generalizations of the sine-Gordon (S.-G.) equation [13].

For Lorentz-invariant equations the higher-order LCD are tensor currents of higher ranks $j_{\mu_1\mu_2\cdots\mu_n}$ dependent on the field functions and their derivatives. For the simplest IS of linear equations these currents are given explicitly in ref. [18]. As usual, a considerable simplification in the two-dimentional space-time is due to the transition to cone variables $2\eta = t + x$, $2\xi = t - x$. In these variables eqs.(2.3) becomes:

$$\phi_{\eta\xi} = A'\beta_\eta \beta_\xi - V' \qquad \beta_{\eta\xi} = -A'/2A(\phi_\eta \beta_\xi + \phi_\xi \beta_\eta) \tag{2.6}$$

and the conservation of an n-th-rank tensor $\partial_{\mu_1} j_{\mu_1\mu_2\cdots\mu_n} = 0$

$$I^{(n)}_\eta = J^{(n)}_\xi . \tag{2.7}$$

For scalar models of the type (1.1) there exist $I^{(n)}$ and $J^{(n)}$ that are polynomials in powers of the field derivatives with respect to ξ (or with respect to η due to symmetry $\eta \to \xi$; in what follows all statements will hold for $\eta \to \xi$, so there is a double set $I^{(n)}$, $J^{(n)}$). We shall call the rank of monomial the total order of the contained function derivatives with respect to ξ minus those with respect to η. Polynomials $I^{(n)}$ and $J^{(n)}$ consist of the monomials of equal rank and it will be called the polynomial rank. If the rank of $I^{(n)}$ is n, then that of $J^{(n)}$ is $(n-2)$ what is necessary to fulfil (2.7) (the product $\eta\xi$ is Lorentz invariant). For instance, for the S.-G. eq. $\phi_{\eta\xi} = \sin\phi$ the law of conservation of the first LCD, energy-momentum tensor, $T_{\mu\nu}$ has the form (2.7) with $I^{(2)} = \frac{1}{2}\phi^2_\xi$ and $J^{(2)} = \cos\phi$ (plus $\eta \to \xi$); for the first nonzero nontrivial LCD $I^{(4)} = \phi^2_{\xi\xi} - \frac{1}{4}\phi^4_\xi$, $J^{(4)} = \phi^2_\xi \cos\phi$. We shall call $I^{(n)}$, $J^{(n)}$ components of an n-th-rank current.

Note that in general the IS can have LCD of a form more general than polynomial or even only these LCD. Here we consider equations with polynomial LCD (PLCD).

Now let us formulate the method for searching and enumerating all systems of the type (2.6) with higher-order PLCD. For arbitrary n, $I^{(n)}$ and $J^{(n)}$ can explicitly be written as polynomials of rank n and $(n-2)$, resp, over all possible "tensor" structures composed of the ϕ- and β-fields derivatives with ϕ-dependent undefermined coefficients. The number of these structures is strongly limited by certain conditions formulated below. Substituting $I^{(n)}$ and $J^{(n)}$ into (2.7), using the equations of motion (2.6) and their differential consequences to eliminate mixed derivatives, and equating coefficients of the equal "tensor" structures give rise to a

system of nonlinear differential equations for coefficients and functions A and V. For $n \le 2$ the system is underdetermined and A and V remain arbitrary (the conservation of current and energy-momentum hold for any A and V); for $n > 2$ the system is overdetermined but consistent and with a finite number of solutions. The number of equations is of an order of 2^n. Solving the system we calculate both the PLCD and admissible Lagrangian.

In general, full analysis implies consideration of eq.(2.7) at arbitrary n, and the method of ref. [17] can be used for analysing a simpler equation with $\beta = 0$. However, the use of this method becomes rather complicated in the case under consideration and it is unnecessary: it is sufficient to employ the theorem of ref. [17] by which eq.(1.1) has a nontrivial group (i.e., nontrivial LCD) provided that *

$$g(\phi) = c_1 e^{a\phi} + c_2 e^{b\phi} , \tag{2.8}$$

where $a = -b$ or $a = -2b$. The system (2.6) goes into (1.1) at $\beta = 0$ $(V' = -g)$, , therefore PLCD of (2.6) should go into LCD (1.1) at $\beta = 0$ or vanish. Thus, the main problem reduced to finding the functions $A(\phi)$ from (2.6) compatible with $g(\phi)$ satisfying the above theorem. To solve this problem, it suffices to analyse just the first nontrivial PLCD that transform, as $\beta \to 0$, into the first nontrivial PLCD of eqs. with $g(\phi)$ obeying that theorem. Indeed, if for some $A(\phi)$ the system possesses a higher nontrivial PLCD of the same rank as for the corresponding limit equation for $\beta = 0$, then it should possess also the lower nontrivial PLCD of the limit equation. The inverse statement holds because of the proof [18] that the system with one higher nontrivial PLCD possesses an infinity of them.

The rank of the first nontrivial PCD (1.1) for $a = -b$ in (2.8) equals four; and for $a = -2b$, six. Thus, it is sufficient to examine the currents of fourth and sixth rank.

3. CURRENT OF 4TH RANK

The most general form of the 4th-rank PLCD for (2.6) is:

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* A similar result was obtained in ref. [20]; however, the authors made a wrong conclusion that at $a = -2b$ the set of higher-order integrals is finite.

$$I^{(4)} = \phi_{\xi\xi}^2 + c_1\phi_\xi^4 + a_1\beta_{\xi\xi}^2 + a_2\beta_\xi^4 + a_3\phi_{\xi\xi}\beta_\xi^2 + a_4\phi_\xi^2\beta_\xi^2,$$
$$J^{(4)} = b_1\phi_\xi^2 + b_2\beta_\xi^2 + b_3\beta_\xi^2\phi_\xi\phi_\eta + b_4\phi_\xi^2\beta_\xi\beta_\eta + b_5\beta_\xi^3\beta_\eta. \qquad (3.1)$$

Here $c_1 = \text{const}$; a_i, b_i are function of the field ϕ. Let us explain why the structures (3.1) are sufficient for our consideration:

1) The I - and J -components of a current of any rank are defined up to terms of a form of the total derivative with respect to ξ and η, resp. - for them the conservation law (2.7) holds trivially. For that reason, e.g., there is no term $\phi_{\xi\xi}\phi_\xi^2$.

2) Terms of the type $a\beta_{\xi\xi}\beta_\xi\phi_\xi = \frac{1}{2}[(a\beta_\xi^2\phi_\xi)_\xi - a'\beta_\xi^2\phi_\xi^2 - a\beta_\xi^2\phi_{\xi\xi}]$ reduce to the total derivative and structures (3.1).

3) Terms of an odd order in derivatives β are absent owing to the symmetry of system (2.6) relative to the change $\beta \to -\beta$.

4) In the limit $\beta \to 0$ the coefficients of remaining terms of the I-component are constant.

And finally, it can be shown that the current I-component does not contain η-derivatives of fields at all.

The "tensor" structure of J is uniquely reproduced from I by substituting I into eq.(2.7) - requiring the r.h.s. of (2.7) to contain the same structures as the l.h.s. Note that unlike the case $\beta = 0$, J contains the first η-derivatives of fields.

Substituting I and J (3.1), using eqs.(2.6) and their differential consequences, and equating coefficients of the same structures, we arrive at the overdetermined system of 15 nonlinear differential equations for 11 functions

$$\begin{cases} 1.\ \phi_{\xi\xi}\phi_\xi: & -g' = b_1 \\ 2.\ \phi_\xi^3: & -4c_1 = b_1' \end{cases}$$

$$\begin{cases} 3.\ \beta_{\xi\xi}\beta_\xi: & a_1 gA'/A = 2b_1 \\ 4.\ \beta_\xi^2\phi_\xi: & -a_3 g' - 2a_4 g = b_2' - b_4 g \end{cases} \qquad (3.2)$$

$$\begin{cases} 5.\ \phi_{\xi\xi}\beta_{\xi\xi}\beta_\eta: & 2A' - a_1A'/A = 0 \\ 6.\ \phi_{\xi\xi}\phi_\xi\beta_\xi\beta_\eta: & 2B_2 - a_3A'/A = 2b_4 \\ 7.\ \phi_{\xi\xi}\beta_\xi^2\phi_\eta: & -A'^2/A + a_3' - a_3A'/A = b_3 \\ 8.\ \beta_{\xi\xi}^2\phi_\eta: & a_1' - a_1A'/A = 0 \\ 9.\ \beta_{\xi\xi}\beta_\xi^2\beta_\eta: & -a_1A'^2/A + a_3A' = 3b_5 \\ 10.\ \beta_{\xi\xi}\phi_\xi^2\beta_\eta: & -2a_1B_1 = b_4 \\ 11.\ \beta_{\xi\xi}\beta_\xi\phi_\xi\phi_\eta: & -2a_1B_1 = 2b_3 \end{cases} \qquad (3.2)$$

$$\begin{cases} 12.\ \phi_\xi^3\beta_\xi\beta_\eta: & 4c_1A' - a_4A'/A = b_4' - b_4A'/2A \\ 13.\ \beta_\xi^4\phi_\eta: & a_2' - 2a_2A'/A - a_3A'^2/2A = -b_5A'/2A \\ 14.\ \beta_\xi^3\beta_\eta\phi_\xi: & -2a_2A'/A + a_3B_2 + 2a_4A' = b_3A' + b_5' - b_5A'/2A \\ 15.\ \beta_\xi^2\phi_\xi^2\phi_\eta: & a_4' - a_4A'/A = b_3' - b_4A'/2A. \end{cases}$$

Here $B_1 = A''/2A - 3A'^2/4A^2$; $B_2 = A'' - A'^2/2A$.

Equations (3.2) can be splitted into 4 groups (in braces). The first group gives the equation for g without A:

$$g'' + k^2 g = 0, \qquad (3.3)$$

where $-4c_1 = k^2$. From equations of the third group functions a_i and b_i can be eliminated thus giving the equation for A omitting the trivial case $A = \text{const}$ following from (3.2.5):

$$4A^2A'A''' - 4A^2A''^2 - 4AA'^2A'' + 3A'^4 = 0. \qquad (3.4)$$

It may be readily verified that this equation for A is obtained in studying the even-rank current of any order when we separate the coefficients of structures with higher ξ-derivatives of ϕ and β.

Equation (3.4) has the remarkable property of "homogeneity": if $A = f(\phi)$ is its solution, so is $A = m_1 f^{\pm 1}(m_2\phi + m_3)$ (m_i are constants). Twice integrating (3.4) we get

$$A'' = c_2 A'A^{1/2} + \frac{1}{2}A'^2A^{-1}$$
$$A' = c_2A^{3/2} + c_3A^{1/2}.$$

For $c_2=0$ or $c_3=0$ (the degenerate case) we have resp.

$$A=(c_4\phi+c_5)^{\pm 2}. \tag{3.5}$$

For $c_2\neq 0$, $c_3\neq 0$

$$A=c_6\,\mathrm{tg}^{\pm 2}(c_4\phi+c_5), \tag{3.6}$$

or

$$A=c_6\,\mathrm{th}^{\pm 2}(c_4\phi+c_5) \tag{3.7}$$

in dependence on the ratio between c_2 and c_3. The solutions (3.5)-(3.7) and the trivial one $A=\mathrm{const}$ exhaust all solutions to eq.(3.4). Note that "$\pm$" in the exponent in (3.5)-(3.7) corresponds to the existence of duality in the sense of the proposition proved in §2. It is also clear that solutions (3.6) and (3.7) differ unessentially. Now, with the given solution of A we determine algebraically from eqs.(3.2.3) coefficients a_1, a_3, b_3, b_4, b_5 ; and from 12-th and 14-th eqs.(3.2.4) (the others are identities), coefficients a_4 and a_2.

As to the eqs.(3.2.3-4) we can deduce from this group, with the given solution of A, one more first-order differential equation for g with A-dependent coefficients, which should be consistent with (3.3). Note that constants c_4, c_5 and c_6 in (3.5)-(3.7) can be taken arbitrary because it is always possible to redefine the fields ϕ, β and coordinates in the Lagrangian (2.2). Coefficients c_4, c_5, c_6 being fixed, all coefficients a_i, b_i, but a_2 and a_4 dependent on $c_1=-k^2/4$ are uniquely fixed and the function g is defined from the overdetermined system (3.2.1-4) with the only parameter k not all values of which can provide its compatibility. It suffices to analyse two essentially different cases.

1. Choose $A=\frac{1}{4}\mathrm{tg}^2\phi$, then calculating a_i, b_i from (3.2.5-15) and inserting them into (3.2.1-4) we get the system

$$g''+k^2g=0$$
$$3g'\mathrm{tg}\phi(1-\mathrm{tg}^2\phi)-g[3\,\mathrm{tg}^4\phi-(2+k^2)\,\mathrm{tg}^2\phi+3]=0. \tag{3.8}$$

Integrating the second equation

$$g=V'=\frac{m^2}{2}\sin 2\phi(\cos 2\phi)^{(k^2-4)/12},\quad m^2=\mathrm{const},$$

and then calculating the second derivative of g

$$g''=-k^2g+m^2\,\frac{(k^2-4)(k^2-16)}{72}(\cos 2\phi)^{(k^2-28)/12}\sin^3\phi$$

we conclude that eqs.(3.2) are compatible at $k=2$; $k=4$ only. As a result, for $A=\frac{1}{2}[\,\mathrm{tg}\,(-h)]^2(\phi)$ we obtain two possible values of V:

$$V_1=\frac{m^2}{2}[\sin(-h)]^2(\phi);\quad V_2=\frac{m^2}{8}[\sin(-h)]^2(2\phi).$$

The system with the Lagrangian

$$\mathfrak{L}=\frac{1}{2}(\phi_\mu^2+\mathrm{tg}^2\phi\,\beta_\mu^2-m^2\sin^2\phi) \tag{3.9}$$

(the "complex sine-Gordon") was invented and studied in refs. [2,4] ; in terms of variables $\psi=\sin\phi\, e^{i\beta}$ it takes the form [3]:

$$\mathfrak{L}=\frac{|\psi_\mu|^2}{1-|\psi|^2}-m^2|\psi|^2. \tag{3.10}$$

In ref. [21] the system is proved to be equivalent to the scalar-field nonlinear model in the Duffin-Kemmer formalism.

The system with the Lagrangian

$$\mathfrak{L}=\frac{1}{2}(\phi_\mu^2+\mathrm{tg}^2\phi\,\beta_\mu^2-m^2\sin^2 2\phi) \tag{3.11}$$

(that can be called the "complex sine-Gordon-II") has been found and investigated by the author [12]. In terms of variables $\psi=\sin\phi\, e^{2i\beta}$ it becomes:

$$\mathfrak{L}=\frac{|\psi_\mu|^2}{1-|\psi|^2}-m^2|\psi|^2(1-|\psi|^2). \tag{3.12}$$

The linear spectral problem for this model is expressed by 3 x 3 matrices [12].

2. Set $A=\frac{1}{2}\phi^2$, then through the analogous procedure we arrive at the system

$$g''+k^2g=0$$
$$3g'\phi-g(3-k^2\phi^2)=0, \tag{3.13}$$

that is compatible only at $k=0$, whence $g=m^2\phi$. Thus, we arrive at the integrable generalization of the linear equation with the Lagrangian

$$\mathcal{L}=\frac{1}{2}(\phi_\mu^2+\phi^2\beta_\mu^2-m^2\phi^2) \qquad (3.14)$$

that is trivial as the change of variables $\psi=\phi e^{i\beta}$ reduces the Lagrangian (3.14) to the Lagrangian for a free complex field (two noninteracting fields).

3. And finally, note that the system (3.8) has also the trivial solution: $g=0\ (m^2=0)$. The corresponding nonlinear model for two massless fields with the Lagrangian

$$\mathcal{L}=\frac{1}{2}(\phi_\mu^2+\mathrm{tg}^2\phi\,\beta_\mu^2) \qquad (3.15)$$

being nontrivial is a reduction of the model "complex sine-Gordon" $(m^2=0)$ and also it can be treated as the result of "complexification" of the linear massless equation $\phi_{\mu\mu}=0$. The model (3.15) resembles the well-known integrable nonlinear chiral model of $\vec{n}$-field [2] that takes values on the unit sphere with the Lagrangian (in the angle parametrization on the sphere)

$$\mathcal{L}=\frac{1}{2}(\phi_\mu^2+\sin^2\phi\,\beta_\mu^2). \qquad (3.16)$$

The model (3.15) may also have a bearing on an integrable chiral fields that takes values in some nonlinear manifold. The model (3.16) has not arisen in our consideration because of the nonpolynomiality of its simplest LCD [2].

4. 6TH RANK CURRENT; ODD-RANK CURRENTS

The consideration of the 6th rank current is analogous to that for the 4th rank current but it is more cumbersome. The system of 63 equations is again splitted into four groups. The third group gives eq.(3.4) for A; the first is reduced to the system of two differential equations for g:

$$g^{(IV)}-c_1g''+12c_2g=0$$
$$5g''-3c_3g'-2c_1g=0 \qquad (4.1)$$

(with $c_i=\mathrm{const}$) and the solution is $g=c_5e^{a\phi}+c_6e^{b\phi}$ at $a=-b$ or $a=-2b$ (for eq. with $a=-2b$ see refs.[8,17,20]).

A detailed examination of the second group of equations with A and ϕ shows that it is compatible with (4.1) again only at $g=\frac{m^2}{2}\sin2\phi\cos2\phi$ and $g=\frac{m^2}{2}\sin2\phi$ for $A=\frac{1}{2}\mathrm{tg}^2\phi$, and at $g=m^2\phi$ for $A=\frac{1}{2}\phi^2$. So, all the Lagrangians corresponding to IS are wholly calculated while studying the 4th runk current.

It is of interest also to examine the odd-rank currents which start with the usual "electromagnetic" current of 1st rank (2.3.2) and vanishes at $\beta=0$. From a detailed analysis it follows that for all ranks eq.(3.4) holds for A, and g is calculated in terms of A. For $A=\frac{1}{2}\mathrm{tg}^2\phi$ there is the only solution $g=\frac{m^2}{2}\sin2\phi=\frac{m^2}{2}(\sin^2\phi)'$ and for $A=\frac{1}{2}\phi^2$ $g=m^2\phi$. So, for the equation with Lagrangian (3.11) all odd currents starting from $n=3$ are omitted what is consistent with the consideration of the conservation laws within ISM [12].

Let us complete our consideration with the explicit form of the first nontrivial current for the equation "complex sine-Gordon" with the Lagrangian (3.10):

$$I^{(3)}=(1-\mathrm{tg}^2\phi)(\phi_\xi^2-\mathrm{tg}^2\phi\beta_\xi^2)\beta_\xi-2\mathrm{tg}\phi\phi_{\xi\xi}\beta_\xi$$
$$J^{(3)}=\sec^2\phi\,(\phi_\xi^2-\mathrm{tg}^2\phi\beta_\xi^2)\beta_\eta. \qquad (4.2)$$

5. CONCLUSION

We say that the equation $\phi_{\mu\mu}+V'(\phi)=0$ with the Lagrangian $\mathcal{L}=\frac{1}{2}\phi_\mu^2-V(\phi)$ and higher PLCD admits the <u>integrable Lagrange complexification</u> if there exists such a change of the field variable $\psi=\psi(\phi)$ that the transition to the complex field ψ and change in the corresponding Lagrangian $\psi\to|\psi|$, $\psi_\mu\to|\psi_\mu|$ again give the system with higher PLCD. (For instance, for $\mathcal{L}=\frac{1}{2}(\phi_\mu^2-\sin^2\phi)$ the change $\psi=\sin\phi$ gives $\mathcal{L}=\frac{\psi_\mu^2}{1-\psi^2}-\psi^2$ and for $\psi\to|\psi|$, $\psi_\mu\to|\psi_\mu|$ we obtain (3.10).

Then the main result is that the integrable complexifications exist only for the following three cases: for $V=\frac{m^2}{2}[\sin(-h)]^2(\phi)$ (two, (3.9-12), for $V=\frac{m^2}{2}\phi^2$ ((3.14), trivial), and for $V=0$

((3.15), nontrivial) within the duality transformation (2.5). The standard and "modified" [8,17,20] Liouville equations ($V=e^{\phi}$ and $V=c_1e^{\phi}+c_2e^{-2\phi}$) do not allow the integrable Lagrange complexifications.

The presented method can obviously be applied to study systems more general than (2.3). Much more systems with higher PLCD can be obtained via the same procedure but without requiring a system to be the Lagrange system, i.e., considering a system more general than (2.3):

$$\phi_{\mu\mu} + A_1(\phi)\beta_\mu^2 + A_2(\phi) = 0$$

$$(A_3(\phi)\beta_\mu)_\mu = 0$$

However, of greated interest are the Lagrange systems of nonlinear equations for a large number of both massive and massless fields. For instance, the reduction of the nonlinear CP(2) σ-model results in the system of nonlinear equations for three scalar fields [10,11].

And finally, the method may be applied not only to the Lorentz-invariant systems. For example, it can be shown that the general nonlinear Schrödinger equation $i\psi_t + \psi_{xx} + \psi V(|\psi|) = 0$ allows an infinity of PLCD at $V = c|\psi|^2$ only.

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