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SIGMA-MODEL REPRESENTATION OF GAUGE THEORIES

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1. In this talk, I shall describe a novel formulation of gauge theories proposed in my recent papers[1,2]. This formulation explicitly demonstrates the common nature of the Yang-Mills theory with usual nonlinear σ-model and opens new avenues in attacking some fundamental problems of gauge fields.

Let me first outline the main aspects in which the sigma-model representation seems particularly promising.

It has been suggested recently[3,4] that the Yang-Mills theory is completely integrable and this property could be visualized by passing to suitable unconventional variables (closely related ideas were declared also in[5,6]). The main goal here is to represent the Yang-Mills equations as differential equations in some auxiliary space (for instance, as conditions of triviality of a certain connection) which would have the meaning of conditions of integrability of a certain spectral problem solvable by the inverse scattering method. Proceeding along this line, Aref'eva[3] and Polyakov[4] have succeeded in reformulating the Yang-Mills theory as a theory of the prinicipal chiral field on the space of closed paths (contours) and the sourceless Yang-Mills equations as first-order differential constraints for a vector form given in this space. The crucial point of their construction is an effective reduction of the space-time dimensionality by unity due to the reparametrization invariance of contour functionals; the D-dimensional Yang-Mills theory turns out to be equivalent, in a sense, to the D-1 dimensional theory of the principal chiral field. Thereby the 3-dimensional Yang-Mills proved to be equivalent to the 2-dimensional chiral theory which is known to be completely integrable (unfortunately, these arguments do not help in solving the 4-dimensional problem). The cost (perhaps, too high) is the essential nonlocality of this approach: the procedure of varying a contour is employed, etc. It is tempting to find an alternative scheme so that an ordinary differentiation be retained at each stage. It would be desirable also to understand the group-theoretical meaning of the contour variables, so called "string functionals" of gauge fields, which are basic ingredients of the approach of refs.[3,4] (see also refs.[7-10]). It will be argued that the sigma-model formulation of gauge theories is constructive in both respects.

One more important problem is the problem of adequate description of the symmetric, nonperturbative phase of the Yang-Mills theory. This phase is associated with the gauge-invariant vacuum and is expected to realize the colour confinement[11,12]. While the variables relevant to the standard, nonsymmetric phase of the Yang-Mills theory (perturbative phase) are well known and these are usual gauge fields, it is not so clear which variables are most suitable to represent the symmetric phase. Rendering gauge theories into the sigma-model notation provides an answer to this question. It becomes obvious that the nonsymmetric and symmetric Yang-Mills phases are related just in the same fashion as nonlinear and linear σ-models of conventional symmetries. This deep analogy points the way of how to treat the symmetric phase.

2. The formulation I am going to talk about is based upon the observation made earlier by Ogievetsky and myself. We have shown in[13,14] that any gauge theory can be looked upon as a generalized nonlinear σ-model for it results from the nonlinear realization of certain infinite-parameter group $K=G \rtimes \mathcal{S}$ ($\rtimes$ stands for a semi-direct product) with $G^0 \times \mathcal{S}$ as the vacuum stability subgroup (see also[15]). Here G^0 is the relevant global symmetry group, G is isomorphic to the connected component of the corresponding local group spanned by all gauge functions decomposable in the Taylor series around $x_\mu = 0$ and $\mathcal{S}$ is the ordinary Poincaré group. It has been understood in[13,14] that the Yang-Mills fields have the same meaning as, say, the pion fields in chiral dynamics; they can be viewed as coordinates parametrizing a certain homogeneous group space (namely, the coset space $K/G^0 \times \mathcal{L}$, $\mathcal{L}$ being the Lorentz group). In other words, they are simply the Nambu-Goldstone fields accompanying the spontaneous breakdown of symmetry with respect to the group K.

To make further consideration more clear it is worth recalling some details of the above-mentioned approach. The group G was represented as an abstract constant-parameter group generated by

the infinite set of symmetric tensor generators $Q^i, Q^i_\mu, \ldots Q^i_{\mu_1 \ldots \mu_n}, \ldots$ with Q^i being generators of the global subgroup G^0. The commutators between them and with the 4-translation generator P_μ are given by

$$
\begin{aligned}
&[Q^t_{\mu_1 \ldots \mu_k}, Q^l_{\mu_{k+1} \ldots \mu_n}] = i\, c^{tlm} Q^m_{\mu_1 \ldots \mu_n} \\
&[P_\rho, Q^t_\mu] = i\, \eta_{\rho\mu} Q^t \qquad\qquad (1) \\
&[P_\rho, Q^t_{\mu_1 \ldots \mu_n}] = i\,(\eta_{\rho\mu_1} Q^t_{\mu_2 \ldots \mu_n} + \ldots + \eta_{\rho\mu_n} Q^t_{\mu_1 \ldots \mu_{n-1}}) \; (n \geq 2).
\end{aligned}
$$

Here c^{tlm} are the totally antisymmetric structure constants of G^0, $\eta_{\rho\mu}$ = diag (1,-1,-1,-1).

Then, the nonlinear realization of $K = G \rtimes \mathcal{S}$ in the coset space $K/G^0 \times \mathcal{L}$ has been constructed following the standard prescriptions of refs.[16]. The infinite set of the tensor Goldstone fields $b^i_\mu(x), b^i_{\mu_1\mu_2}(x), \ldots b^i_{\mu_1 \ldots \mu_n}(x), \ldots$ with quantum numbers of generators $Q^i_\mu, Q^i_{\mu_1\mu_2}, \ldots Q^i_{\mu_1 \ldots \mu_n}$, has been introduced, x_ρ playing the role of Goldstonion associated with P_ρ. These fields were identified with coordinates of the coset space $K/G^0 \times \mathcal{L}$ and the group K was implemented as left multiplications of cosets. It has been found that in this particular realization the infinite-parameter subgroup G is represented by the standard gauge G^0-transformations, the vector Goldstone field $b^i_\mu(x)$ being transformed just as the corresponding Yang-Mills field. We have evaluated the covariant derivatives both of Goldstonions and extraneous, non-Goldstone fields $\psi_\alpha(x)$ (α is an index of the global subgroup G^0) and have shown that $b^i_\mu(x)$ is identical by its couplings with the standard Yang-Mills field on the group G^0. The remaining Goldstone fields $b^i_{\mu_1 \ldots \mu_n}(x) \; (n \geq 2)$ have been proved to be unessential, in the sense that they are covariantly expressible in terms of $b^i_\mu(x)$ and its derivatives by putting zero those parts of relevant covariant derivatives which are symmetric in tensor indices (the inverse Higgs phenomenon[17] works).

Now, I wish to show that this construction can be recast into an elegant and compact form by introducing an extra coordinate-Lorentz 4-vector y_μ and using the following particular representation for generators of the group K[1,2]

$$
P_\mu = i \frac{\partial}{\partial y^\mu}, \; L_{\mu\nu} = i(y_\mu \partial^y_\nu - y_\nu \partial^y_\mu), \; Q^i_\mu = y_\mu Q^i, \ldots Q^i_{\mu_1 \ldots \mu_n} = y_{\mu_1} \ldots y_{\mu_n} Q^i. \qquad (2)
$$

This choice is basically dictated by the condition standard for the nonlinear realizations that the generators entering into the group exponents commute with the related coset parameters.

The representation (2) is convenient in that it allows one to convert the infinite set of Goldstone fields $b^i_\mu(x), \dots b^i_{\mu_1\dots\mu_n}(x), \dots$ into the single object: the bilocal Goldstone field $b(x,y) \equiv b^i(x,y)Q^i = \sum_{n\geq 1}\frac{1}{n!} b^i_{\mu_1\dots\mu_n}(x)\, y^{\mu_1}\dots y^{\mu_n} Q^i$. It is completely specified by the requirement of its decomposability in the Taylor series over y_ρ and by the condition

$$b(x,0)=0 . \qquad (3)$$

An element of cosets $K/G^0\times L$ is given now by

$$\tilde G(x,b) = e^{ix^\rho P_\rho}\, e^{i\sum_{n\geq1}\frac{1}{n!} b^i_{\mu_1\dots\mu_n} Q^{i\mu_1\dots\mu_n}} = e^{-x^\rho\partial^y_\rho}\, e^{ib(x,y)}.$$

The transformation properties of component fields $b_\mu(x), \dots b_{\mu_1\dots\mu_n}(x), \dots$ with respect to the left action of the group K on $\tilde G$ /13,14/ imply for $b(x,y)$ the following transformation laws:

$$\mathcal{P}: \quad b'(x,y) = b(\Lambda^{-1}(x-a), \Lambda^{-1}y), \qquad (4)$$

$$G: \exp\{ib'(x,y)\} = \exp\{i\lambda(x+y)\}\exp\{ib(x,y)\}\exp\{-i\lambda(x)\} \qquad (5)$$

where $\lambda(y) = \lambda(0) + \sum_{n\geq1}\frac{1}{n!}\lambda_{\mu_1\dots\mu_n} y^{\mu_1}\dots y^{\mu_n}$ is the generating function for constant parameters of G. It is seen from (4) that coordinates x_μ and y_ρ have essentially different properties in the Poincaré group: the former undergoes both the Lorentz rotations and 4-translations while the latter only Lorentz rotations. One could say that x_μ behaves as the coordinate of centre of inertia of some extended object and y_ρ as the corresponding relative coordinate.

The covariant derivatives of Goldstonions $\nabla_\mu b_\rho(x), \dots \nabla_\mu b_{\rho_1\dots\rho_n}(x)$ combine into the bilocal Cartan form:

$$\omega_\mu(x,y) = -b_\mu(x) + \sum_{n\geq1}\frac{1}{n!}\nabla_\mu b_{\rho_1\dots\rho_n}(x)\, y^{\rho_1}\dots y^{\rho_n} \qquad (6)$$

which is introduced by the relation

$$i\omega_\mu(x,y) = \exp\{-ib(x,y)\}(\partial^x_\mu - \partial^y_\mu)\exp\{ib(x,y)\} \qquad (7)$$

and as a consequence satisfies the generalized Maurer-Cartan equation

$$(\partial^x_\mu-\partial^y_\mu)\omega_\rho(x,y) - (\partial^x_\rho - \partial^y_\rho)\omega_\mu(x,y) + i[\omega_\mu(x,y),\omega_\rho(x,y)] = 0 \qquad (8)$$

(which is equivalent to saying that the generalized Yang-Mills connection defined by a vector form $\omega_A(x,y)$ $(A=1,\dots,8)$ is trivial on the subspace $x=-y$ of 8-dimensional space x,y). It follows from the definition (7) that under transformations (5) $\omega_\mu(x,y)$ behaves like the Yang-Mills field (taken with minus):

$$\omega'_\mu(x,y) = \exp\{i\lambda(x)\}[\omega_\mu(x,y) - i\partial^x_\mu]\exp\{-i\lambda(x)\}. \qquad (9)$$

Using the bilocal notation, it becomes possible to render a simple geometrical meaning to the differential constraints by which unessential Goldstone fields $b_{\mu_1\dots\mu_n}(x)$ $(n\geq2)$ were eliminated in /13,14/ at the expense of $b_\mu(x)$. The infinite sequence of these conditions is now represented by the one manifestly covariant equation:

$$y^\mu[\omega_\mu(x,y) + b_\mu(x)] = 0 \qquad (10)$$

or, with making use of the definition (7):

$$y^\mu(\partial^x_\mu - \partial^y_\mu)\exp\{-ib(x,y)\} = i y^\lambda b_\lambda(x)\exp\{-ib(x,y)\}. \qquad (11)$$

The most simple way to solve this equation is as follows. One passes to new variables $t_\mu=(x+y)_\mu,\ y_\mu$ and makes rescaling $y\to\beta y$. As a result, (11) is rewritten in the form:

$$\frac{\partial}{\partial\beta}\exp\{-ib(t-\beta y,\beta y)\} = -i y^\lambda b_\lambda(t-\beta y)\exp\{-ib(t-\beta y, y)\} \qquad (11')$$

Taking into account the boundary condition (3), the solution of (11'), with setting $\beta=1$ at the end, is given by the formula (we have returned here to old variables):

$$\exp\{-i\bar b(x,y)\} = T\exp\{-i\int_0^1 d\gamma\, y^\rho b_\rho[x+(1-\gamma)y]\}, \qquad (12)$$

where the symbol T means the ordering in matrices Q^i within the interval $0\leq\gamma\leq1$. That expression is immediately recognized as the path integral of the Yang-Mills field along the straight line going from the point $x+y$ to x:

$$\exp\{-i\bar b(x,y)\} = T\exp\{i\int_{x+y}^{x} b_\mu(\xi)\,d\xi^\mu\}, \quad \xi_\mu = x_\mu + (1-\gamma)y_\mu .$$

Expanding both sides of (12) in powers of y_μ, one can be convinced that this formula exactly reproduces the expressions for tensor Goldstonions which have been obtained earlier by exploiting the inverse Higgs phenomenon at the component level /13,14/. Being expressed in terms of the minimal bilocal Goldstonion $\bar b(x,y)$, the Cartan form $\omega_\mu(x,y)$ reads as

$$\overline{\omega}_\mu(x,y) = -b_\mu(x) + \frac{1}{2}\mathcal{F}_{\mu\rho}(x)y^\rho + \sum_{n\geq 2}\frac{1}{(n+1)!}\nabla_{\rho_1}\cdots\nabla_{\rho_{n-1}}\mathcal{F}_{\mu\rho_n}(x)y^{\rho_1}\cdots y^{\rho_n}, \qquad (13)$$

where $\mathcal{F}_{\mu\rho} = \partial_\mu b_\rho - \partial_\rho b_\mu - i[b_\mu, b_\rho]$ is the standard Yang-Mills strength, $\nabla_\rho = \partial_\rho - i[b_\rho,$ is the Yang-Mills covariant derivative.

Thus, the string functional of gauge fields which is now under intensive study/3,4,7-10/ naturally arises in our approach as the most economical representation for cosets G/G^0 (its direct analog in chiral dynamics is $\exp\{i\mathcal{F}^i(x)\frac{1}{2}\tau^i\gamma_5\}$ which is an element of the coset space $SU^L(2)\times SU^R(2)/SU(2)$). While in /3,4,7-10/ these functionals are introduced "by hand", their appearance in our scheme is the result of the consistent application of methods of the general theory of nonlinear realizations/16/. We see that this theory prescribes quite definite rules of handling such functionals: covariants should be defined according to the formula (7), i.e., through ordinary differentiation of the end points of the path (which can be conceived as an infinitesimal rotation of the path as a whole around the point $x+y$). In the standard approach to the path integrals, covariants are defined in the essentially nonlocal fashion, through infinitesimal deformations of separate sections of the path.

The inverse Higgs phenomenon, in its standard minimal formulation/13,14/, picks out the straight path in a lot of paths between $x+y$ and x. However, without contradiction with the transformation laws (4), (5), it is possible to take as a representative of cosets G/G^0 also the string functional along any other path (this path, of course, should be such that the related $b(x,y)$ admits the power expansion about $y_\mu = 0$, i.e., the path should contract into the point x_μ when $y_\mu \to 0$). The choice of the curvilinear path corresponds to a certain modification of differential conditions of the inverse Higgs phenomenon. Namely, in this case the "straight-line" condition (10) is replaced by the more general one

$$y^\mu[\omega_\mu(x,y) + b_\mu(x)] = \Delta(x,y), \qquad (14)$$

where $\Delta(x,y)$ is a covariant functional of the strength $\mathcal{F}_{\rho\lambda}(x)$ and degrees of covariant derivatives of $\mathcal{F}_{\rho\lambda}(x)$. Knowing the structure of $\Delta(x,y)$ completely specifies the path configuration in the corresponding string functional. Indeed, the latter can always be

represented by the formula (12) in which $y^\rho b_\rho(x)$ is changed to $y^\rho b_\rho(x) - \frac{1}{\gamma}\Delta(x,\gamma y)$:

$$\exp\{i\tilde{b}(x,y)\} = T\exp\{i\int_0^1\{y^\mu b_\mu[x+(1-\gamma)y] - \frac{1}{\gamma}\Delta[x+(1-\gamma)y,\gamma y]\}d\gamma\} \qquad (12')$$

the straight path clearly corresponding to $\Delta(x,y)=0$. Note that any such generalized string functional is reducible, in the sense that it can be decomposed into the product of the minimal, straight-line factor and a nonminimal one:

$$\exp\{i\tilde{b}(x,y)\} = \exp\{i\bar{b}(x,y)\}\exp\{ih(x,y)\}, \qquad (15)$$

The meaning of the second factor is that it describes a deviation from the straight path*). It is expressed, in its y-expansion, through powers of covariant derivatives of $\mathcal{F}_{\rho\lambda}(x)$. The relation between functionals $h(x,y)$ and $\Delta(x,y)$ is as follows:

$$\Delta(x,y) = \frac{1}{i}\exp\{-ih(x,y)\}y^\mu(\nabla^x_\mu - \partial^y_\mu)\exp\{ih(x,y)\}.$$

So, there exist many inequivalent, but from the group-theoretical point of view equally acceptable ways to come down from the generally parametrized coset space $K/G^0\times L$ to its minimal connected invariant subspace characterized by the single field $b_\mu(x)$. More visually, we may form a "string" between the points $x+y$, x in various manners. Is it possible to indicate an extra dynamical principle choosing a definite string functional from all those admissible within the pure group-theoretical considerations? Some steps towards an answer to this question are outlined in the following Section. To avoid a possible misunderstanding, it is appropriate here to say that, in contrast to the approach of refs. /3,4,7-10/, there is no actual path-dependence in our formulation; as soon as a definite ansatz for excluding unessential degrees of freedom from $b(x,y)$ is chosen, the path in the resulting string functional is fixed once for all**)

*) Such a "polar" decomposition exists, of course, for any general coset representative $\exp\{ib(x,y)\}$ all the superfluous Goldstone fields $b_{\mu\rho_1\ldots\rho_n}(n\geq 2)$ being included into the corresponding functional $h(x,y)$.

**) In fact, it is not proved that any functional of the form (12') may be rewritten as the integral of b_μ along a certain path between $x+y$, x. It may happen that (12') describes more general situation and reduces to a path integral only under certain restrictions on $\Delta(x,y)$.

To conclude this part of my Talk I emphasize that the simple group meaning explained above can be attributed only to the "open string" functional of gauge fields. It is as yet unclear how to accomodate within the present scheme the closed paths (contours) which are of primary interest in papers/3,4/. The most direct way to embrace the case of contours is to admit paths which do not contract into point as $y_\mu \to 0$ (and so do not supply the condition (3)). It is likely that such an extension of the class of paths may naturally emerge upon allowing for the nontrivial topological structure of gauge group (i.e., including into play, along with G, also those components of the whole gauge group which are not connected with an identity element by continuous gauge transformations).

3. The basic relation (7) has the form typical for decompositions by which the Cartan forms are defined in nonlinear σ-models for principal chiral fields. Therefore, the Yang-Mills theory can be interpreted as a sector of the nonlinear σ-model for the 8-dimensional principal chiral field $\ell(x,y)$ on the group G^0. This sector is extracted by the condition (10) or, more generally, by (14) with $\ell(x,0)=0$, $\ell_\mu(x)=\partial^y_\mu \ell(x,y)|_{y=0}$ by definition.

In σ-models of such a type the equations of motion (with no sources) are written as the condition of vanishing of the divergence of the corresponding Cartan form (continuity equation). It is interesting to look whether it is possible to represent the standard source-free Yang-Mills equation

$$\nabla^\mu \mathfrak{F}_{\mu\rho}(x)=0 \qquad (16)$$

as an analogous closed differential condition on the bilocal Cartan form $\omega_\mu(x,y)$ (supplementary with respect to the "kinematical" conditions (8) and (10) or (14)). Keeping in mind the hypothesis of complete integrability of the Yang-Mills theory it is desirable that this condition be of the first order in derivatives.

In the Abelian case, the equation (16) (i.e., the free Maxwell equation) can easily be seen to be equivalent to the manifestly covariant condition of that the "straight-line" Cartan form $\overline{\omega}_\rho(x,y)$ (13) be divergenceless with respect to y-differentiation:

$$\partial^y_\mu \overline{\omega}^\mu(x,y)=0\,. \qquad (17)$$

Unfortunately, in the most interesting non-Abelian case such an equivalence (for the straight path) holds only up to the third

order in y_μ, in the sense that the coefficients of higher powers of y_ρ in $\partial^y_\rho \overline{\omega}^\rho(x,y)$ do not vanish in virtue of the Yang-Mills equation (16) alone. One may check that the equivalence cannot be restored without adding to the l.h.s. of (17) terms with higher derivatives of $\overline{\omega}_\mu(x,y)$.

Thus, so far as the straight-line Cartan form $\overline{\omega}_\mu(x,y)$ is considered one does not succeed in finding a simple representation for the Yang-Mills equations. A possible way out is as follows. As has been pointed out at the end of previous Sect., the straight path, though being the simplest one, is not favoured, from the group-theoretical point of view, over other paths between the points $x+y$, x. Accordingly, the most generic form of the string functional arising upon covariant exclusion of redundant Goldstone fields from $\ell(x,y)$ is given by the formula (12') with $\Delta(x,y)$ being nonzero in general. Therefore, the problem of recasting the Yang-Mills equations into the sigma-model notation can be thought about as search for the string functional in terms of which these equations have the simplest form. One can check that the only possible covariant differential constraint which has the first order in derivatives, incorporates the Yang-Mills equations (16) and is formulated solely in terms of $\omega_\mu(x,y)$ is just the condition of vanishing of the divergence of the latter with respect to ∂^y_μ. So, the question to be answered is: May we find a string functional $\exp\{i\tilde{\ell}^0(x,y)\}$ such that the associated Cartan form $\tilde{\omega}^0_\mu(x,y)$ satisfies the continuity equation

$$\partial^y_\mu \tilde{\omega}^{0\mu}(x,y)=0\,, \qquad (18)$$

when the Yang-Mills field $\ell_\mu(x)$ obeys the standard equation (16)? The answer turns out to be affirmative. The corresponding functional $h^0(x,y)$ (by which $\tilde{\ell}^0(x,y)$ is completely specified in virtue of (15)) is determined from the equation:

$$\partial^\mu_y\Big\{\exp\{-ih^0(x,y)\}\big[\partial^x_\mu-\partial^y_\mu+i\,\overline{\omega}_\mu(x,y)\big]\exp\{ih^0(x,y)\}\Big\}\Big|_{\nabla^\mu \mathfrak{F}_{\mu\rho}=0}=0 \qquad (19)$$

uniquely, up to possible terms vanishing on solutions of eq.(16). The equation (19) is obtained by substituting the decomposition (15) into the definition (7) and by imposing (18). Although the solution to eq.(19) in the closed form is still not found, the functional $h^0(x,y)$ can be evaluated to any desirable order in y_μ

by iterations. The difference between $\widetilde{\omega}_\mu^0$ and $\overline{\omega}_\mu$ begins from the fourth order in y_μ :

$$\widetilde{\omega}_\rho^0 - \overline{\omega}_\rho = \frac{i}{5!}\frac{1}{20}\partial_\rho^y (y^2 y^\lambda y^\sigma y^\alpha)\Big\{[\mathfrak{F}_{\beta\lambda}, \nabla_\sigma \mathfrak{F}_{\beta\alpha}] - \\ - \frac{3}{32}\eta_{\lambda\alpha}[\mathfrak{F}_{\beta\mu}, \nabla_\sigma \mathfrak{F}_{\beta\mu}]\Big\} + O(y^5).$$

It should be stressed that $\widetilde{\omega}_\rho^0(x,y)$ has the complicated nontrivial structure but this structure is such that all highest terms in the y -expansion of $\partial_\mu^y \widetilde{\omega}^{0\mu}$ become zero as soon as the lowest term (which is just $1/3!\,\nabla^\mu \mathfrak{F}_{\mu\rho} y^\rho$) vanishes. It is clear that the necessary condition for (18) to be fulfilled is that $b_\mu(x)$ obey the equation (16).

Now we may forget about all reasonings which led us to the string representation for $\exp\{i\mathcal{E}(x,y)\}$ and formulate the above result as the following Theorem.

Let the Yang-Mills field $b_\mu(x)$ on the group G^0 be given. Then there exists the bilocal functional $\exp\{i\mathcal{E}^0(x,y)\}$ of the form (12') with the following remarkable property. The vector form $\widetilde{\omega}_\mu^0(x,y)$ defined through $\mathcal{E}^0(x,y)$ according to the formula (7) is divergenceless with respect to y-differentiation iff $b_\mu(x)$ satisfies the source-free Yang-Mills equation.

If it is true that any functional of the type (12') can be represented as the contour integral of $b_\mu(x)$ along a certain path, then the path in $\exp\{i\mathcal{E}^0(x,y)\}$ should be essentially curvilinear (it becomes straight for an arbitrary y_ρ only in the Abelian case). This curve is likely formed by the Yang-Mills field itself (i.e., it is defined by a function explicitly depending on $b_\mu(x)$). So, one may expect the one-to-one correspondence between different classes of solutions of the Yang-Mills equation and permissible configurations of paths in $\exp\{i\mathcal{E}^0(x,y)\}$. Moreover, one may speculate on the possibility of the complete exclusion of $b_\mu(x)$ in favour of paths. To confirm these conjectures, it seems of primary importance to regain the equations (8),(14),(18) proceeding from a certain action principle. It is as yet unsolved task.

For the time being, I do not know to which extent the above considerations may be useful in proving the hypothetical complete integrability of the Yang-Mills theory. But the fact that the Yang-Mills equations can be represented as ordinary first-order differential constraints on a certain vector form in the extended

space (the condition of trivial connection (8) and the continuity equation (18)) seems unexpected and deserves further examination. It remains to find the corresponding spectral problem (if exists). Closely related is the question as to whether the equation (18) implies the existence of infinite series of currents conserved in the standard sense, with respect to X -differentiation.

4. The understanding of the fact that the Yang-Mills theory in the standard, perturbative phase is the nonlinear realization of the group $K = G\,G\,\mathfrak{S}$ with $G^0 \times \mathfrak{S}$ being the residual symmetry of vacuum led in ref.[14] to the problem of constructing linear, algebraic realizations of K which would naturally correspond to the completely K -invariant ground state. It has been pointed out in[14] that the relationship between theories associated with these two different kinds of the gauge group realization should resemble the well-known relation between nonlinear and linear σ -models of usual finite-parameter symmetries. It has also been conjectured that the linear σ -models of gauge groups might bear direct relation to dual theories of strong interactions*). One more important aspect is as follows. As suggested by the analogy with σ -models, just the linear realization of K should describe the symmetric, nonperturbative phase of the Yang-Mills theory. This phase is associated with fully gauge-invariant vacuum and is responsible, by the hypothesis of refs.[11,12], for the colour confinement. The knowledge of transformation laws of linear multiplets of K would allow one to construct the invariant Lagrangians relevant to the symmetric phase and to study the structure of this phase in the purely algebraic way, without any reference to the standard Yang-Mills theory. Such a consideration would be helpful, for instance, in clarifying the question as to whether the confinement is a direct consequence of gauge invariance of vacuum or extra dynamical assumptions are required for confinement to be valid.

The formulation I have described in previous Sections indicates a possible way in which the linear multiplets of the group K can be constructed. Indeed, once the Yang-Mills theory in the nonsymmetric phase admits embedding into the bilocal nonlinear σ -

*) This conjecture seems to be confirmed in the recent papers[18,19]

model on the group G^0 it is natural to assume that the symmetric phase of this theory can be interpreted within the corresponding bilocal linear σ-model. In other words, linear representations of K should operate on bilocal linear multiplets of G^0. I shall consider here the simplest multiplet of this kind. It will be shown how the gauge fields may appear within the linear realization of K. For simplicity, I take $G^0=SU(2)$.

The simplest multiplet can be constructed by completing the coset space $K/G^0\times L$ to a linear space just as, for instance, the vector multiplet of the group O(4) can be arrived at by completing a 3-dimensional sphere $\sim$ O(4)/O(3) to the 4-dimensional Euclidean space (σ-particle is added to three "pions").

Let us consider an <u>arbitrary</u> bilocal matrix $U(x,y)$ with the transformation properties (5):

$$U'(x,y)=U_0'(x,y)+\tfrac{1}{2}i\tau_k U_k'(x,y)=e^{i\lambda(x+y)}Ue^{-i\lambda(x)} \qquad (20)$$
$$(UU^+=U^+U\neq I)$$

It is not hard to see that all components in the decomposition of $U(x,y)$ in y_μ transform linearly and homogeneously (in contrast to components of bilocal Goldstonion $\ell(x,y)$):

$$\delta_G U_0(x)=0,\quad \delta_G U_k(x)=\varepsilon_{k\ell m}\lambda_\ell(x)U_m(x)$$
$$\delta_G U_{\mu k}(x)=U_0(x)\partial_\mu\lambda_k(x)-\varepsilon_{k\ell m}[\lambda_\ell(x)U^\mu_m(x)+\partial_\mu\lambda_\ell(x)U_m(x)], \qquad (21)$$
$$\delta_G U_{\mu\nu k}(x)=U_0(x)\partial_\mu\partial_\nu\lambda_k(x)+\ldots.$$

etc.

realizing thereby the <u>linear representation</u> of the group K.

If $\langle U_0\rangle_{vac}\neq 0$ the infinitesimal transformations of fields $U^k_\mu(x)$, $\ldots U^k_{\mu\nu}(x),\ldots$ start with inhomogeneous terms just typical for transformation laws of components of $\ell(x,y)$:

$$\delta U^k_\mu(x)=\langle U_0\rangle_{vac}\partial_\mu\lambda^k(x)+\ldots$$
$$\delta U^k_{\mu\nu}(x)=\langle U_0\rangle_{vac}\partial_\mu\partial_\nu\lambda^k(x)+\ldots$$
$$\vdots$$

Actually, in this case $U^k_\mu(x), U^k_{\mu\nu},\ldots$ can be equivalently related to Goldstonions $\ell^k_\mu, \ell^k_{\mu\nu},\ldots$ by means of the polar decomposition of matrix $U(x,y)$:

$$U(x,y)=\exp\{i\ell(x,y)\}\{\langle U_0\rangle_{vac}+\tilde{U}(x,y)+\tfrac{i}{2}\tau_k U_k(x)\}, \qquad (22)$$

where $\tilde{U}(x,y)$ is pure scalar with respect to the action of K. The matrix $U(x,y)$ exclusively belongs to the coset space $K/G^0\times L$ provided the following covariant conditions are fulfilled:

$$UU^+=U^+U=\langle U_0\rangle^2_{vac},\quad U(x,0)=\langle U_0\rangle_{vac}$$

or, in terms of polar components:

$$\tilde{U}(x,y)=0,\quad U^i(x)=0.$$

Thus, the bilocal linear σ-model constructed on the basis of representation (20) is expected to embody, after spontaneous breakdown of K-symmetry, the conventional massless Yang-Mills theory.

The main problem is the construction of the relevant Lagrangians with which the nonzero vacuum expectation values for different component fields would naturally emerge from the standard extremum conditions. It is not difficult to indicate the general form of the potential part of such an invariant Lagrangian:

$$\mathcal{L}^V\sim \mathrm{Tr}\, V(U(x,y)U^+(x,y)). \qquad (24)$$

As to invariants including derivatives of component fields, it is likely that in the case of exact K-symmetry it is not possible at all to construct bilinear invariants for components with internal indices (confinement?). At the same time, the invariant kinetic term for the G-scalar, "colourless" component $U_0(x)$ exists and has the standard form $\sim\partial_\mu U_0\partial_\mu U_0$. The simplest invariant with derivatives of "colour" fields is of the fourth order and is given by the lattice ansatz [11]:

$$\mathcal{L}^{kin}\sim \mathrm{Tr}\{U(x,y)U(x+z,-z)U^+(x+z,y)U^+(x+z+y,-z)\}, \qquad (25)$$

where the coordinate z_μ transforms under the Poincaré group just as y_μ. The main difficulty encountered when trying to expose the particle content of Lagrangians of the type (24),(25) is the presence of numerous mixings between component fields and their derivatives. So some diagonalization procedure is required.

Thus, the careful treatment of dynamics associated with Lagrangians of the type (24),(25) is the complicated business and it will be performed separately. Here we would like to focus on some things clear already at the pure group-theoretical level. In linear σ-models of gauge groups there are no Yang-Mills fields so far as the symmetry with respect to $K=G\otimes S$ is unbroken. The invariance is achieved without these fields, due to the specific form of transformation rules (20),(21) of linear K-multiplets.

Some vector components of the initial multiplet acquire the status of gauge fields only upon breaking of $\mathcal{G}$ -symmetry due to the appearance of nonzero vacuum expectation values of certain other components, in close parallel with the emergence of the Goldstone fields in the ordinary linear σ -models. It is instructive to see how the standard kinetic term of gauge fields arises in this picture. Suppose $\langle u_0\rangle_{vac} \neq 0$ by virtue of some dynamical reason (which we are not interested in for the moment). Then the Yang-Mills component of $U(x,y)$ is unambiguously defined by the polar decomposition (22). By substituting the latter into (25) and expanding (25) in powers of y_{μ}, z_{ρ} it can be shown that the first nonvanishing term of the Yang-Mills field $b_{\mu}(x)$ is given by the expression

$$\sim \langle u_0\rangle^2 [\tilde{u}(x,0)+\langle u_0\rangle]^2 \mathcal{F}^i_{\mu\nu}(x)\mathcal{F}^i_{\rho\lambda}(x)(y^{\mu}z^{\nu}-y^{\nu}z^{\mu})(y^{\rho}z^{\lambda}-y^{\lambda}z^{\rho})$$

which, upon appropriate integration over y, z, yields the conventional Yang-Mills Lagrangian.

Finally, let me point out once more that the linear σ -models of gauge groups are expected to manifestly realize the idea of gauge-invariant vacuum. They may be a useful tool for studying the structure of gauge theories in the region of phase transitions (which should manifest themselves in this language as the appearance or vanishing of vacuum expectation values of certain fields).

5. I have shown that the interpretation of gauge theories as theories of spontaneous breakdown/13,14/ naturally leads to their new description in terms of the 8-dimensional nonlinear sigma-model. Thereby, the intimate relevance of the latter to the gauge-field dynamics is established. Note the difference at this point from the consideration of refs./3,4/, the main idea of which is the reduction of the Yang-Mills theory to sigma-models in lower dimensions. On the other hand, the path integrals of gauge fields play the central role in both formulations. It still remains to understand at which more points these approaches are overlapped and what is the actual usefulness of each of them.

In conclusion, I list some open questions (apart from those already mentioned in the text). How to take into account, within the present scheme, the conformal invariance of the Yang-Mills theory? Is it possible to reformulate gravity and supergravity in the analogous fashion and what new consequences could follow from this? May the sigma-model formulation help in exposing

hypothetical hidden symmetries of gauge theories? We hope to answer these questions in due time.

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