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E2 - 80 - 26

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Submitted to the International Seminar "Group-Theoretic Methods in Physics", Zvenigorod, November 1979.

1980

E2 - 80 - 26

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Простейший геометрический подход к супергравитации

В докладе сделан краткий обзор результатов по простейшему геометрическому подходу к супергравитации, основанному на теории аксиального суперполя.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1980

E2 - 80 - 26

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The Simplest Geometric Approach to Supergravity

At present there exist several approaches to supergravity. All of them describe the same physical problem and therefore are equivalent in essence. However, they differ in their complexity and adequacy.

The first successful one has been the so-called "component" approach (see Ref. /1/ and references therein). There supergravity group has been realized as a group of complicated transformations of a set of ordinary fields. The transformation laws and the invariant action have skillfully been guessed without entering deeply into the corresponding geometry. Recently, despite of great technical difficulties, certain positive results have been obtained /2/ even for $\mathcal{N}=2$ - extended supergravity. However, the rapidly increasing complexity of such a scheme hardly allows one to proceed further, for $\mathcal{N}>2$.

An alternative framework seems preferable. It is based on the concept of (curved) superspace and its principal advantage consists in using powerful geometric tools. One such an approach has been developed by Wess and Zumino /3/ (see also Ref. /4/ *)). They generalize directly all the concepts of differential geometry (such as vierbeins, connections, torsion, curvature, etc.) to the superspace $\{(x^m, \theta^\mu, \bar\theta^{\dot\mu})\}$. The gauge group includes the supergroup of general coordinate transformations (GCT) in superspace and the basic superfields (potentials) in the theory are the supervierbeins $E_A^M(x,\theta,\bar\theta)$ and superconnections. Unfortunately, both the group and potentials are much larger than the physical situation requires. Therefore, the gauge has to be strongly fixed and a set of (guessed) algebraic constraints on the components of torsion have to be imposed to reduce the superfluous freedom in the theory**). So, the straightforward generalization of differen-

*) Recently some progress has been achieved in the $\mathcal{N}=2$ - supergravity case also /5/.

**) In their recent papers Siegel and Gates /6/ and Stelle and West /7/ investigate the classification of constraints and search for possible recipes for guessing them. Our opinion is that the main role of these constraints is to ensure the expressibility of the theory in terms of the axial superfield H^m (to be discussed in this talk).

tial geometry to superspace does not provide a minimal description of supergravity.

A minimal geometric superspace approach has been proposed in 1976 /8/ and now completed for the case of N=1 - supergravity /9/ (see also the papers of Siegel and Gates /10/). Here we are going to give an outline of this formulation of supergravity. Its main features are:

a) The approach is based on a clear and simple geometric picture;

b) The gauge supergroup is as small as possible;

c) Only a single axial superfield $H^m(x,\theta,\bar\theta)$ is introduced as an independent dynamical variable. It is the minimal superfield capable of describing supergravity multiplet. Its occurrence can be understood in the following way. In Yang-Mills theory the potentials are the connection coefficients $A_m(x)$. In general relativity the connection can be expressed in terms of a primary potential: the vierbein (or metric). In N=1 - supergravity there appears one more step, the prepotential H^m, and the supervierbeins as well as connections are given in terms of its derivatives;

d) The components of torsion and curvature are directly calculated in terms of the derivatives of H^m. No constraints have to be imposed, they follow automatically from the initial geometric construction;

e) The action principle is straightforward. The equation of motion is obtained just by simple variation of the action with respect to H^m. Its left-hand side is an invariant tensor (the superanalogue of $R_{mn}-\frac{1}{2}g_{mn}R$ in general relativity) and the right-hand side is the supercurrent (the superanalogue of energy-momentum tensor).

Now let us explain in brief these points.

1) Supergroup. To start with, we shall replace the ordinary real 4+4-dimensional superspace $\{(x^m, \theta^\mu, \bar\theta^{\dot\mu})\}$ by an 8+4-dimensional superspace $\{(x_L^m, \theta_L^\mu ; x_R^m, \bar\theta_R^{\dot\mu})\}$ where $x_R^m = (x_L^m)^*$ is a complex space-time fourvector coordinate and $\bar\theta_R^{\dot\mu} = (\theta_L^\mu)^+$. Then our supergroup is defined /11/ just as the GCT group in the left $\{(x_L^m, \theta_L^\mu)\}$ as well as in the conjugated right $\{(x_R^m, \bar\theta_R^{\dot\mu})\}$ superspaces:

LSS RSS

$$x_L'^m = x_L^m + \lambda^m(x_L,\theta_L) \qquad x_R'^m = x^m + \bar\lambda^m(x_R,\theta_R) \tag{1}$$
$$\theta_L'^\mu = \theta_L^\mu + \lambda^\mu(x_L,\theta_L) \qquad \bar\theta_R'^{\dot\mu} = \bar\theta_R^{\dot\mu} + \bar\lambda^{\dot\mu}(x_R,\theta_R),$$

where

$$\bar\lambda^m(x_R,\theta_R) = (\lambda^m(x_L,\theta_L))^+, \quad \bar\lambda^{\dot\mu}(x_R,\theta_R) = (\lambda^\mu(x_L,\theta_L))^+.$$

Owing to the fact that the parameters λ depend on θ_L (or $\bar\theta_R$) only, this group is already much smaller than the GCT group in the 4+4 -superspace. Moreover, it can be further restricted by the condition that the supervolume in LSS (as well as in RSS) should be preserved

$$\mathrm{Ber}\left\| \frac{\partial(x_L', \theta_L')}{\partial(x_L,\theta)} \right\| = 1 \tag{2}$$

(here Ber means Berezinian or superdeterminant of transformations) or, infinitesimally,

$$\frac{\partial}{\partial x_L^m}\lambda^m - \frac{\partial}{\partial\theta_L^\mu}\lambda^\mu = 0 . \tag{2'}$$

The general group (1) corresponds to Weyl supergravity and the subgroup (2) to Einstein supergravity.

2. Gravitational Axial Superfield

Our 8+4 -superspace $\{(x_L^m, \theta_L^\mu; x_R^m, \bar\theta_R^{\dot\mu})\}$ has four superfluous bosonic coordinates. When introducing the usual physical 4+4-superspace $\{(x^m, \theta^\mu, \bar\theta^{\dot\mu})\}$ one can put:

$$x^m = \frac{1}{2}(x_L^m + x_R^m), \quad \theta^\mu = \theta_L^\mu, \quad \bar\theta^{\dot\mu} = \bar\theta_R^{\dot\mu} \tag{3}$$

and regard the remaining fourvector (the imaginary part of x_L^m) as a function of the general form of x , θ , $\bar\theta$:

$$x_L^m - x_R^m = 2i\,H^m[\tfrac{1}{2}(x_L + x_R), \theta_L, \bar\theta_R] =$$

$$= 2iH^m(x,\theta,\bar\theta) \tag{4}$$

rather than as an independent coordinate. The geometric meaning of this trick is quite clear. Eq. (4) can be interpreted as equation of a 4+4-dimensional (curved) hypersurface in the 8+4-superspace and it is the way how the physical superspace is embedded in the complex one. The supergroup (1), (2) is now realized nonlinearly with respect to H^m :

$$\begin{aligned} x'^m &= x^m + \tfrac{1}{2}\lambda^m(x+iH,\theta) + \tfrac{1}{2}\bar\lambda^m(x-iH,\bar\theta), \\ \theta'^\mu &= \theta^\mu + \lambda^\mu(x+iH,\theta), \\ \bar\theta'^{\dot\mu} &= \bar\theta^{\dot\mu} + \bar\lambda^{\dot\mu}(x-iH,\bar\theta), \end{aligned} \tag{5}$$

$$H'^m(x',\theta',\bar\theta') = H^m(x,\theta,\bar\theta) - \tfrac{i}{2}\lambda^m(x+iH,\theta) + \tfrac{i}{2}\bar\lambda^m(x-iH,\bar\theta) .$$

The nonpolynomiality of these transformations can be avoided if a proper gauge is chosen (an analogue of the WZ-gauge in supersymmetric Yang-Mills theory /12,9/). Then the remaining transformations are just the ones discussed in the "component" approach: GCT, local Lorentz and local supersymmetry transformations (all of them in 4-dimensional space-time).

A comparison of different approaches is given in Table 1 to illustrate the degrees of freedom involved.

Table 1

General SS approach	Minimal SS approach	Component approach
Field Variables		
Supervierbeins $E_M^A(x,\theta,\bar\theta)$ or 1024 fields	Gravitational axial superfield $H^m(x,\theta,\bar\theta)$ or 64 fields	Graviton $e_m^a(x)$, gravitino $\psi_{m,\alpha}(x)$ and auxiliary fields $A_m(x)$, $s(x)$, $p(x)$ or 38 fields
Gauge Parameters:		
GCT in real SS $\{(x,\theta,\bar\theta)\}$ or 128 gauge parameters Lorentz supervierbein group or 96 gauge parameters. Altogether 224 gauge parameters.	GCT in LSS and RSS with supervolume-preservation condition or 40 gauge parameters.	GCT, local Lorentz and local supersymmetry transformations in space-time or 14 gauge parameters.
1024 - 224=800(≫24)	64-40=24=12+12	38-14=24=12+12

3. Lorentz Supervierbein Group

As can be seen in Table 1, we have no independent Lorentz supervierbein group. However, it is a useful concept, especially when developing differential geometry formalism. Now we are going to introduce superfields with Lorentz indices transforming according to the law

$$\delta\Phi_A = i\Omega^{cd}(\Lambda_{cd})_A{}^B\Phi_B . \tag{6}$$

Here $A=(a,\alpha,\dot\alpha)$ and Λ_{cd} are the ordinary Lorentz-group generators. The main peculiarity of Eq. (6) is that the parameters $\Omega^{cd}(x,\theta,\bar\theta)$ are not arbitrary superfunctions (as in the general SS approach) but they are given by the parameters of our basic group (1), (2):

$$\Omega^{cd} = \frac{i}{4}\Delta^\alpha(\sigma^{cd})_\alpha{}^\beta\lambda_\beta + \frac{i}{4}\bar\Delta_{\dot\alpha}(\bar\sigma^{cd})^{\dot\alpha}{}_{\dot\beta}\bar\lambda^{\dot\beta}, \tag{7}$$

$$\Delta_\alpha = \partial/\partial\theta^\alpha + i\frac{\partial H^m}{\partial\theta^\alpha}(\delta_m^n - i\partial_m H^n)^{-1}\partial/\partial x^n , \quad \bar\Delta_{\dot\alpha} = (\Delta_\alpha)^+ . \tag{8}$$

In other words, our local (in superspace) Lorentz transformations are locked to the world-superspace ones. Thus, we can benefit by the technical advantages of considering Lorentz-like objects without introducing an additional gauge freedom.

It should be mentioned that the law (6)-(7) was not just invented; it was extracted from the transformation properties of the derivative $\frac{\partial \Phi}{\partial \theta^{\alpha}}$ in a certain basis in superspace (see Ref. /9/).

4. Differential Geometry. Having defined our world-superspace group (1), (2) and the local Lorentz one (6), (7), we can proceed further in developing the differential geometry formalism. First, we introduce the concept of covariant derivatives.

4.1. Spinor covariant derivatives:

$$\mathcal{D}_\alpha \Phi_B = E_\alpha^{\ M} \frac{\partial}{\partial Z^M} \Phi_B + \omega_{\alpha B}^{\ \ \ C} \Phi_C \tag{9}$$

Here $Z=(x,\theta,\bar\theta)$, $B=(b,\beta,\dot\beta)$, $M=(m,\mu,\dot\mu)$; $E_\alpha^{\ M}$ and $\omega_{\alpha B}^{\ \ \ C}$ are supervierbeins and connections with standard transformation laws. All of them can be expressed in terms of H^m, e.g.,

$$E_\alpha^{\ \mu} = F\delta_\alpha^{\mu}, \quad E_\alpha^{\ \dot\mu} = 0, \quad E_\alpha^{\ m} = iF\Delta_\alpha H^m,$$
$$\omega_{\alpha\beta\gamma} = (\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta})\Delta^\delta F, \tag{10}$$

where *)

$$F = 2^{\frac{2}{3}}\left[\det\|\Delta\sigma_a\bar\Delta H^m\|\right]^{-1/3}\left[\det\|\bar\Delta\tilde\sigma_a\Delta H^m\|\right]^{1/6}. \tag{11}$$

4.2. Vector covariant derivatives. The most natural (although not unique) way to define them is by direct generalization of the corresponding flat-supersymmetry relation

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*) In ref. /9/ the normalizing factor $2^{2/3}$ in Eq. (11) was absent. It is needed to have familiar flat-superspace limit in the theory.

$$\mathcal{D}_a = \frac{i}{4}\tilde\sigma_a^{\ \dot\alpha\alpha}\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} \tag{12}$$

This choice causes certain formal differences with the results of Ref. /13/ which we shall discuss below.

4.3. Torsion and curvature. They are defined by the (anti) commutator ($T_{AB}^{\ \ C}$ and $R_{AB,\mathcal{D}}^{\ \ \ \ E}$ are the tensors of torsion and curvature, respectively)

$$[\mathcal{D}_A, \mathcal{D}_B\} \Phi_{\mathcal{D}} = T_{AB}^{\ \ C}\mathcal{D}_C \Phi_{\mathcal{D}} + R_{AB,\mathcal{D}}^{\ \ \ \ E}\Phi_E \tag{13}$$

and thus can be simply calculated in terms of H^m. Here are some examples.

The components of torsion tensor with all indices being spinorial vanish. The components $T_{\alpha\dot\beta}^{\ \ c}$ remain the same as in flat supersymmetry

$$T_{\alpha\dot\beta}^{\ \ c} = -2i\sigma_{\alpha\dot\beta}^{c}. \tag{14}$$

The components $T_{\dot\alpha b}^{\ \ \gamma}$ have a simple form

$$T_{\dot\alpha b}^{\ \ \gamma} = \frac{i}{4}(\sigma_b)_{\dot\alpha}^{\ \gamma} R^*, \quad R^* = \Delta^\alpha\Delta_\alpha(F^2). \tag{15}$$

The rest components are either zero or are expressed in terms of the basic superfields $G_{\alpha\dot\alpha}(x,\theta,\bar\theta)$ and $W_{\alpha\beta\gamma}(x,\theta,\bar\theta)$, $\bar W_{\dot\alpha\dot\beta\dot\gamma}(x,\theta,\bar\theta)$ (the latter are totally symmetric in their indices and chiral). They are connected by means of Bianchi identities /13/ which occur in our case just as Jacobi identities for double (anti) commutators (see Eqs. (13), (12)). Here we give some of these components, e.g.,

$$T_{\alpha b}^{\ \ \gamma} = -\frac{i}{8}(\sigma_b)_{\alpha\dot\beta}\, G^{\gamma\dot\beta}, \tag{16}$$

$$G_{\alpha\dot\alpha} = F\{\Delta^\beta\bar\omega_{\dot\alpha\alpha\beta} + \bar\omega_{\dot\alpha\alpha}^{\ \ \ \beta}\Delta_\beta(F^4\bar F^{-1}) + \omega^{\beta}_{\ \dot\alpha\dot\beta}\,\bar\omega^{\dot\beta}_{\ \alpha\beta} - \tag{17}$$

$$-\Delta_\alpha\left[\bar F\bar\Delta_{\dot\alpha}\ln(\bar F^2F^{-1})\right]+2\bar F\Delta_\alpha\ln F\bar\Delta_{\dot\alpha}\ln\bar F\} + \text{Herm. Conj.}$$

The results obtained turn out to agree with the constraints of Wess and Zumino except for a nonprincipal difference. The point is that our definition (12) of vector covariant derivative fixes the values of the vector connection $\omega_a{}^{fg}$ and (as follows from Eq. (13)) of the curvature component $R_{\alpha\dot\alpha}{}^{fg}=0$. However, one can redefine $\omega_a{}^{fg}$ by adding a certain tensor (thus not changing the transformation properties of $\omega_a{}^{fg}$):

$$\omega'_a{}^{fg}=\omega_a{}^{fg}+A\,\varepsilon_{ab}{}^{fg}(\tilde\sigma^b)^{\dot\alpha\alpha}G_{\alpha\dot\alpha}\ , \tag{18}$$

where A is a constant. So, one can change the value of $\omega_a{}^{fg}$ in order to reproduce exactly the WZ-results (if necessary).

5. Action Principle. As we have seen above, in our approach all the quantities such as supervierbeins, etc.,have been expressed manifestly in terms of a single superfield H^m. This circumstance appears to be especially important when discussing an action for supergravity which has the form /14/

$$S=\frac{1}{\varkappa^2}\int d^4x\,d^4\theta\,\mathrm{Ber}\|E_M^A\|+\int d^4x\,d^4\theta\,\mathrm{Ber}\|E_M^A\|\,\mathcal{L}(\Phi,\mathcal{D}_B\Phi)\,. \tag{19}$$

Here $\varkappa$ is the gravitational coupling constant and $\mathcal{L}$ is some matter Lagrangian with its derivatives replaced by covariant ones. Now, the difficulties arise in the general superspace approach /14/ when one has to vary this action to obtain equations of motion. These difficulties are due to the fact that the supervierbeins E_M^A are not independent variables. In the general SS approach /14/ they are subject to the constraints on torsion. So, the variational procedure can not be straightforward; the constraints must be taken into account /14/. This is a rather nontrivial task, especially when matter is present.

In our case E_M^A are the functions of H^m and the direct variation of H^m produces the desired equations.

Note that the explicit calculation is significantly simplified in a special gauge. We call it "normal" due to its analogy with the so-called "normal coordinates" in general relativity. The coordinate frame can be chosen so that at a certain point $z_0=(x_0,\theta_0,\bar\theta_0)$ in superspace

$$H^m|_0=0\,,\ \partial_N H^m|_0=0\,,\ \ \partial_N\partial_K H^m|_0=0 \tag{20}$$

except for

$$\partial_\nu\bar\partial_{\dot\lambda}H^m|_0=-\sigma^m_{\nu\dot\lambda} \tag{21}$$

also the connections vanish at this point

$$\omega_{AB}{}^C=0\,. \tag{22}$$

This gauge is very convenient in a number of applications, not only here.

The equation of motion for supergravity is thus obtained in the form

$$G_{\alpha\dot\alpha}\ =\ \varkappa^2\,V_{\alpha\dot\alpha} \tag{23}$$

torsion component (17); analogue of $R_{mn}-\frac{1}{2}g_{mn}R$ in general relativity	supercurrent; analogue of energy - momentum tensor T_{mn}

This simple form (23) was suggested in 1976 when the realization of the idea that supergravity was the theory of an axial superfield generated by the supercurrent started. The correctness and fruitfulness of this idea is now completely confirmed.

We would like to stress that Eq. (23) is the only equation of motion. The second one, $R=0$, mentioned in the case of pure supergravity by Wess and Zumino /14/ cannot be reproduced. In this case we have only

$$R=\mathrm{Const} \tag{24}$$

as a corollary of Eq. (23) with $V_{\alpha\dot\alpha}=0$. Note, that a nonvanishing constant in Eq. (24) corresponds to a theory with the cosmological term. This peculiarity is connected with the fact that in our case the auxiliary field $S(x)$ is in fact the divergence $\partial_m S^m$ of a vector field $S^m(x)$ in the decomposition of $H^m(x,\theta,\bar\theta)$ /9/. The presence or absence of the cosmological term is thus related to the behaviour of H^m at large x.

In conclusion we shall point out that one can introduce chiral superspaces and axial superfield H^m in the extended-supergravity case too /11/. However, there H^m is not more the minimal superfield describing the corresponding supergravity multiplet. It should be, perhaps, constrained in a certain way. Anyhow, even in extended supergravity H^m is much more simple object than the supervierbeins and connections.

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Received by Publishing Department
on January 11 1980.