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ON HILBERT SPACES OF PATHS

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1. INTRODUCTION

The Feynman integrals are subject of constant interest of both physicists and mathematicians already for thirty years. Especially in the last decade a substantial progress has been achieved in mathematical theory of functional integration [1,2]. Since the path integrals cannot be treated in terms of the measure theory [3], various alternative definitions (reviewed in Refs. [1,2]) have been suggested. An important role among them plays that of Albeverio and Hoegh-Krohn [1,4] which employs a Parseval-type equality on the Hilbert space of paths. The more general definition of Dewitt-Morette [2,5,6] reduces to the previous one if the path space is not merely a locally compact Hausdorff but a Hilbert space [7]; we shall therefore abbreviate this definition further as DAH.

Truman [7] has shown that the DAH-definition can be extended to a wider class of functions via limiting procedure with polygonal approximation of paths. This method is unlike the standard sequential-limit expression for path integrals (cf., e.g., [8,9] or [10], sec.X.11) and more close to Feynman's heuristic considerations [11] because it deals with the exact classical action on polygonal paths instead of its Riemannian approximation [12]. It suggests also a way how to connect the DAH-type definitions with the other main line in functional integration based on analytic continuation of the Wiener integral [13].

For the simplest case of quantum-mechanical particle in one dimension Albeverio and Hoegh-Krohn use a path space which we shall denote as $AC_0[J^t;\mathbb{R}]$. It consists of all absolutely continuous real-valued functions γ on $J^t=[0,t]$ such that $\gamma(t) = 0$ and the derivative $\dot\gamma$ belongs to $L^2(J^t;\mathbb{R})$; the inner product is defined as follows

$$(\gamma_1,\gamma_2)=\int_0^t \dot\gamma_1(r)\dot\gamma_2(r)\,dr. \qquad (1)$$

However, Truman objects to this definition that it needs to factor out all the singularly continuous paths [7]. He proposes as an alternative to replace $\dot\gamma$ by the weak derivative and defines the path space via trigonometric series: $\gamma\in\mathcal{H}_T(J^t;\mathbb{R})$ iff there exist real a_0, $\{a_n\}$, $\{\beta_n\}$, $\sum_n(a_n^2+\beta_n^2) < \infty$, , such that

$$\gamma(r) = a_0(r - t) + \sum_{n=1}^{\infty} \frac{a_n t}{2\pi n} \sin(\frac{2\pi n r}{t}) + \sum_{n=1}^{\infty} \frac{\beta_n t}{2\pi n} (1 - \cos(\frac{2\pi n r}{t})) \tag{2}$$

for all $r \in J^t$. The inner product $(.,.)_T$ is given by

$$(\gamma, \gamma')_T = t a_0 a_0' + \frac{t}{2} \sum_{n=1}^{\infty} (a_n a_n' + \beta_n \beta_n'). \tag{3}$$

This definition is intended to give an extension of $AC_0[J^t; R]$. If it would so, serious problems would arise. Firstly, the Feynman-Ito formulae derived in Refs. /1,7/ would represent different assertions connected nontrivially by the Fubini theorem. Secondly, one would be obliged to give meaning to the classical action on singularly continuous paths.

We shall prove, however, that this is not the case. Both the Hilbert spaces coincide, and this assertion generalizes easily to n degrees of freedom. Truman's proposal may be thus regarded as a useful equivalent expression of the conventional Hilbert space of paths.

2. SOME NOTATIONS

$C_0[J^t; R] = \{\gamma: J^t \to R: \gamma$ continuous in J^t, $\gamma(t) = 0\}$;
$AC_0[J^t; R] = \{\gamma \in C_0[J^t; R]: \gamma$ absolutely continuous, $\dot{\gamma} \in L^2(J^t; R)\}$;
$\dot{\gamma}$, derivative of γ: $\dot{\gamma}(r) = \lim_{h \to 0} h^{-1}(\gamma(r+h) - \gamma(r))$;
$\dot{\gamma}_w$, weak derivative of γ: $\int_0^t \dot{\gamma}_w(r)\phi(r)\,dr = -\int_0^t \gamma(r)\dot{\phi}(r)\,dr$ for all $\phi \in C_0^{\infty}[J^t]$, where the last symbol denotes the set of all infinitely differentiable functions with a compact support contained in J^t;
$s_N(r) = a_0 + \sum_{n=1}^{N} a_n \cos(\frac{2\pi n r}{t}) + \sum_{n=1}^{N} \beta_n \sin(\frac{2\pi n r}{t})$;
$\dot{\gamma}_m$, L^2-norm limit of $\{s_N\}$: $\dot{\gamma}_m = s - \lim_{N \to \infty} s_N$;
$\dot{\gamma}_p$, pointwise limit of $\{s_N\}$: $\dot{\gamma}_p(r) = \lim_{N \to \infty} s_N(r)$ whenever the limit exists;
$\mathcal{H}_T = \mathcal{H}_T(J^t; R)$ see (2), (3);
$AC_0[J^t; R^n] = AC_0[J^t; R] \oplus \ldots \oplus AC_0[J^t; R]$, Hilbert space of R^n-valued functions which belong componentwise to $AC_0[J^t; R]$, analogously $\mathcal{H}_T(J^t; R^n)$ equals to direct sum of n identical

copies of $\mathcal{H}_T(J^t;R)$: the coefficients α_0, α_n, β_n are elements of R^n in this case and their products are replaced by R^n-inner products

3. AUXILIARY STATEMENTS

Proposition 1: $\mathcal{H}_T$ is real separable Hilbert space, $\mathcal{H}_T \subset C_0[J^t;R]$.

Proof: Let us take an arbitrary γ from $\mathcal{H}_T$; it is clearly real-valued and $\gamma(t) = 0$. Further the inequality

$$\sum_{n=1}^{\infty}\frac{\alpha_n t}{2\pi n}\sin\left(\frac{2\pi n\tau}{t}\right)\leq\sum_{n=1}^{\infty}\frac{|\alpha_n| t}{2\pi n}\leq\left(\sum_{n=1}^{\infty}\alpha_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\frac{t^2}{4\pi^2 n^2}\right)^{1/2}$$

holds together with an analogous estimate for the other series in (2), thus both the series are majorized by convergent τ-independent series and converge therefore uniformly in J^t. Continuity of the partial sums implies now continuity of γ.

$\mathcal{H}_T$ is a real vector space with respect to addition and scalar multiplication defined in the standard way. If $\gamma = 0$, then $\gamma(t) = 0$ gives $\alpha_0 = 0$ and

$$\gamma(\tau) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n\tau}{t}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n\tau}{t}\right) = 0$$

holds for all $\tau \in J^t$, where $-A_0 = \sum_{n=1}^{\infty} A_n$, $A_n = -\frac{\beta_n t}{2\pi n}$, $B_n = \frac{\alpha_n t}{2\pi n}$. The interval J^t is closed so that continuity of γ implies its integrability. The Fourier coefficients of γ are thus uniquently given; it means that the mapping $\{\alpha_0, \{\alpha_n\}, \{\beta_n\}\}\mapsto\gamma$ is bijective. Then the mapping $(.,.)_T : \mathcal{H}_T\times\mathcal{H}_T\to R$ is an inner product and the exictence of an isomorphism between $\mathcal{H}_T$ and $\ell^2(R)$, the Hilbert space of real square-summable sequences, is easily established. ■

Proposition 2 (Dirichlet-Jordan theorem) : Let $f \in L(J^t;R)$ have a bounded variation in an interval $[u,v]\subset J^t$, then

(a) for any $\tau\in[u,v]$ the Fourier series of f converges to $\frac{1}{2}(f(\tau+0) + f(\tau-0))$,

(b) moreover, if f is continuous, then the convergence is uniform in each closed subinterval $\subset(u,v)$.

Proof is standard and may be found in textbooks (cf. e.g., Ref. /14/, theorem 185).
This statement has the following easy consequence:

Proposition 3: Let $f \in L(J^t; R)$ have a Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n\tau}{t}) + \sum_{n=1}^{\infty} \beta_n \sin(\frac{2\pi n\tau}{t}) \tag{4}$$

(which is not assumed to converge in any sense), then the function

$$F: F(\tau) = \int_0^\tau f(\xi)d\xi - a_0\tau$$

fulfils $F(0) = F(t) = 0 = 0$ and its Fourier series

$$\sum_{n=1}^{\infty} \frac{a_n t}{2\pi n} \sin(\frac{2\pi n\tau}{t}) + \sum_{n=1}^{\infty} \frac{\beta_n t}{2\pi n}(1 - \cos(\frac{2\pi n\tau}{t})) \tag{5}$$

converges to $F(\tau)$ uniformly in J^t.

Proof: Since $f \in L(J^t; R)$, F is absolutely continuous, thus it has a bounded variation in any bounded interval. Clearly $F(0) = 0$, further $a_0 = \frac{1}{t}\int_0^t f(\xi)d\xi$ implies $F(t) = 0$. According to Proposition 2 continuity of F implies that its Fourier series converges to $F(\tau)$ for any $\tau \in J^t$, moreover uniformly in J^t, because $F(0) = F(t)$ and Proposition 2 may be by the same right applied to the periodic extension of f, say to the interval $[-t, 2t]$. The Fourier coefficients of F are obtained through integration by parts

$$A_n \equiv \frac{2}{t}\int_0^t F(\tau)\cos(\frac{2\pi n\tau}{t})d\tau = -\frac{2}{t}\frac{t}{2\pi n}\int_0^t f(\tau)\sin(\frac{2\pi n\tau}{t})d\tau = -\frac{\beta_n t}{2\pi n}$$

and analogously $B_n = \frac{a_n t}{2\pi n}$. Finally $F(0) = 0$ implies $A_0 = -\sum_{n=1}^{\infty} A_n$. ∎

Proposition 4: $AC_0[J^t; R] \subset \mathcal{H}_T$.

Proof: If $\gamma \in AC_0[J^t; R)$, then $\dot\gamma(\tau)$ exists a.e. in J^t and $\dot\gamma \in L^2(J^t; R) \subset L(J^t; R)$. Consequently, $\dot\gamma$ is L^2-norm limit of some sequence $\{s_N\}$ with $\sum_n (a_n^2 + \beta_n^2) < \infty$ and its Fourier series is of the form (4). Proposition 3 applied to $f = \dot{\gamma}$ states that $F(r) = \gamma(r) - \gamma(0) - a_0 r$ equals (5). Finally $F(t) = \gamma(t) = 0$ gives $\gamma(0) = -a_0 t$ so that $\gamma \in \mathcal{H}_T$. ■

Proposition 5: The function $\dot{\gamma}_p$ is defined a.e. in J^t and $\dot{\gamma}_p(r) = \dot{\gamma}_m(r)$ for almost all $r \in J^t$, if $\sum_n (a_n^2 + \beta_n^2) < \infty$.

Proof: The first assertion was proved by Carleson[15]; it is worth mentioning that it represents the solution to the highly nontrivial problem formulated by Luzin in 1915 (cf.Refs.16,17 for more details). It holds therefore $\dot{\gamma}_p(r) = \lim_{N\to\infty} s_N(r)$ for all $r \in J^t - M_p$, $m(M_p) = 0$, being the Lebesgue measure on R. On the other hand, $\dot{\gamma}_m = \text{s-}\lim_{N\to\infty} s_N$ implies that there exists a subsequence $\{s_{N_k}\}$ such that $\dot{\gamma}_m(r) = \lim_{k\to\infty} s_{N_k}(r)$ for all $r \in J^t - M_m$, $m(M_m) = 0$ (cf., e.g., Ref.[17], sec.VII.2). Consequently, $\dot{\gamma}_p(r) = \dot{\gamma}_m(r)$ for all $r \in J^t - (M_p \cup M_m)$. ■

4. THE MAIN RESULT

We have verified up to now that the elements of $\mathcal{H}_T$ are continuous (and therefore interpretable as paths) and that each $\gamma \in AC_0[J^t;R]$ is contained in $\mathcal{H}_T$ so that $\mathcal{H}_T$ might represent an extension of the conventional path space. Now we shall prove the statement indicated in the introduction.

Theorem: (a) The Hilbert spaces $\mathcal{H}_T(J^t;R]$ and $AC_0[J^t;R]$ are identical.

(b) The functions $\dot{\gamma}$, ${}_t\dot{\gamma}_w$, $\dot{\gamma}_m$, $\dot{\gamma}_p$ exist and equal mutually a.e. in J^t for any $\gamma \in \mathcal{H}_T$.

Proof: Let γ be an arbitrary element of $\mathcal{H}_T$ corresponding to $\{a_0, \{a_n\}, \{\beta_n\}\}$, $\sum (a_n^2 + \beta_n^2) < \infty$, then there exists $\dot{\gamma}_m \in L^2(J^t;R) \subset L(J^t;R)$. Proposition 3 applied to the function

$$F : F(r) = \int_0^r \dot{\gamma}_m(\xi)\,d\xi - a_0 r$$

asserts that $F(0) = F(t) = 0$ and $F(r)$ is expressed by (5). Together with (2) it implies

$$\gamma(r) = a_0(r-t) + F(r) = -a_0 t + \int_0^r \dot{\gamma}_m(\xi)\,d\xi$$

so that γ is absolutely continuous because $\dot{\gamma}_m \in L(J^t; R)$. Then γ is differentiable a.e. in J^t and

$$\dot{\gamma}(r) = \dot{\gamma}_m(r) \qquad \text{for almost all } r \in J^t. \qquad (*)$$

Further $\dot{\gamma}_m \in L^2(J^t; R)$, $\gamma(t) = 0$ gives $\mathcal{H}_T \subset AC_0[J^t; R]$ and both the sets are qual due to Proposition 4. Truman /12/ proved that $\dot{\gamma}_w$ exists and equals $\dot{\gamma}_m$ a.e. in J^t. Combining this result with (*) and Proposition 5, we obtain the equality

$$\dot{\gamma}(r) = \dot{\gamma}_w(r) = \dot{\gamma}_m(r) = \dot{\gamma}_p(r) \qquad \text{a.e. in } J^t \qquad (6)$$

for each $\gamma \in \mathcal{H}_T$ which proves (b). It remains to verify that $\mathcal{H}_T$ and $AC_0[J^t; R]$ are equal as Hilbert spaces. The relations (1), (3), (6) together with the Parseval equality imply

$$||\gamma||^2 = \int_0^t \dot{\gamma}^2(r)\,dr = ||\gamma_m||^2 = t\alpha_0^2 + \frac{t}{2}\sum_{n=1}^{\infty}(\alpha_n^2 + \beta_n^2) = ||\gamma||_T^2;$$

proof is completed by the polarization identity. ■

In view of the definitions of the path spaces for $n > 1$ the proved theorem implies immediately:

Corollary: The Hilbert spaces $\mathcal{H}_T(J^t; R^n)$ and $AC_0[J^t; R^n]$ are identical.

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