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# STATIONARY QUANTUM STATES IN YANG-MILLS THEORY AS THE UNITARY REPRESENTATIONS OF HOMOTOPY GROUP

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1. The dynamical description of a system is as a rule the construction of the system invariance group representations. In this paper an attempt is made to consider the stationary quantum states of Yang-Mills fields as the representation of the homotopy group.

2. We consider the Yang-Mills theory in gauge $A_0^a = 0$ with the Hamiltonian

$$H = \int d^3x \frac{1}{2} [(E_i^a)^2 + (B_i^a)^2], \tag{1}$$

$$E_i^a = \partial_0 A_i^a ; \quad B_i^a = \epsilon_{ijk} (\partial_j A_k^a + \frac{g}{2} \epsilon^{abc} A_j^B A_k^C ). \tag{2}$$

The finiteness of action and energy is one of the basic requirements of the theory [1]. According to this one considers usually fields $A_i^a$ as smooth functions (i.e., differentiable throughout $R(3)$ and vanishing at spatial infinity). The theory is invariant under the stationary gauge transformations

$$A_i'(x,t) = v(x)^{-1} (A_i(x,t) + \partial_i) v(x) ; \quad A_i = g \frac{\tau^a}{2i} A_i^a . \tag{3}$$

The matrices $v(x)$ are a smooth mapping from $R(3)$ to $SU(2)$ and, as is known, each matrix is characterized by an integer index $(n)$. The total gauge group $G$ is the production of a small gauge group $G_0(n = 0)$ and the homotopy group $\pi_3(SU(2)) = Z$. There is a quantity $\mathcal{N}(A)$ that realizes the homotopy-group representation [2]

$$\mathcal{N}(A) = \frac{g^2}{8\pi^2} \int d^3x \epsilon_{ijk} (\frac{1}{2} \partial_i A_j^a A_k^a + \frac{g}{3} A_i^a A_j^b A_k^c \epsilon^{abc} ), \tag{4}$$

$$\mathcal{N}(v^{(n)^{-1}} (A_i + \partial_i) v^{(n)}) = \mathcal{N}(A_i) + n . \tag{5}$$

3. Consider quantization of the theory. We realize the commutator $[\hat{E}(x), \hat{A}(y)] = i\delta^3(x-y)$ in the form $\hat{A} = A$; $\hat{E} = i\delta/\delta A$, as $A_i^a$ are smooth functions. Let $\Psi_\epsilon(A) = \langle A|\epsilon\rangle$ be an exact wave functional satisfying the Schrödinger equation:

$$H\Psi_\epsilon = \epsilon\Psi_\epsilon \tag{6}$$

and the auxiliary conditions:

$$\nabla_i^{ab}(A)E_i^b\Psi_\epsilon = 0; \qquad \nabla_i^{ab} = \delta^{ab}\partial_i + g\epsilon^{abc}A_i^c, \tag{7}$$

$$T\Psi_\epsilon = e^{i\theta}\ \Psi_\epsilon; \qquad T = \exp\{\frac{d}{d\,\mathfrak{N}(A)}\}. \tag{8}$$

Here eq. (7) is the invariance condition of $\Psi_\epsilon$ under the $G_0$ transformations; eq. (8) is the covariance condition under the homotopy group. The operator $T$ may be represented in the form:

$$T = \exp\{\frac{8\pi^2}{g^2}[i\int d^3x\, B^2]^{-1}\ \int d^3x\, B_i^a\hat{E}_i^a\} \tag{9}$$

(In fact, we have $Tf(\mathfrak{N}) = f(\mathfrak{N}+1)$).

Theorem 1: There is no complete system of physical solutions of eqs. (6) - (8).

Proof: The operators $H$ and $T$ do not commute: $[H,T] \neq 0$, $[H,[H,T]] \neq 0$, etc. Therefore, $H$ and $T$ have no complete system of common physical eigenstates. It is known that the existence of the physical solutions in this case is rather an exception to the rule than the rule. But there are exact nonphysical solutions of eqs. (6) - (8) a la "plane wave":

$$\Psi_0 = \exp\{\pm i(2\pi k + \theta)\mathfrak{N}(A)\}. \tag{10}$$

According to eq. (8) $k$ is integer, $-\pi \leq \theta \leq \pi$. $\Psi_0$ satisfies eq. (6) only under conditions $\epsilon = 0$ and

$$(2\pi k + \theta) = \pm i\frac{8\pi^2}{g^2}. \tag{11}$$

Therefore $\Psi_0$ is the nonnormalizable, nonphysical solution.

4. Thus we must carry out a noncontradictory "synthesis" of the "plane wave" and "oscillator state" (6).

Theorem 2: To construct the physical states as the unitary representation of the homotopy group, it is sufficient to introduce the new dynamical variable $N(t)$

$$\bar{\mathfrak{N}}(t) = \mathfrak{N}(A) + N(t) \tag{12}$$

so that the transformation (cyclic) properties of $\bar{\mathfrak{N}}(t)$ do not change (5), and the topological translation operator $(\frac{d}{d\bar{\mathfrak{N}}} \equiv \frac{d}{dN})$ commutes with the total Hamiltonian.

Proof: The dynamical variables in the gauge theory are defined by the equations of constraint [3,4] in arbitrary gauge.

$$\frac{\delta S}{\delta A_0} = 0 \implies \nabla_i^2(A) A_0 = \nabla_i(A)\partial_0 A_i , \tag{13}$$

where

$$S = \int dt L = -\int dt\, d^3x \frac{1}{4}(F^a_{\mu\nu})^2 , \tag{14}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu . \tag{15}$$

We introduce a zero-mode $\dot{c}(t)$ of the operator $\nabla_i^2$ by the explicit solution of eq. (13)

$$A^a_0 = \dot{c}(t)\Phi^a + (\frac{1}{\nabla_k^2}\nabla_i \partial_0 A_i)^a , \tag{16}$$

where $\Phi^a$ satisfies the equation

$$\nabla_k^2 \Phi = 0 . \tag{17}$$

Substituting the solution (16) into the Lagrangian (14) and to the definition of the Pontryagin index

$$\nu[A] \equiv \int dt \dot{N}(t) = \frac{g^2}{32\pi^2}\int dt\, d^3x F^a_{\mu\nu} F^a_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \tag{18}$$

we get*

$$L = \int d^3x \frac{1}{2}[(E^{(T)})^2 - B^2] + \frac{1}{2}(\dot{c})^2 [\int d^3x (\nabla\Phi)^2] - \dot{c}[\int d^3x E^{(T)} \nabla\Phi], \quad (19)$$

$$\dot{N}(t) = \dot{c}(t) \frac{g^2}{8\pi^2}[\int d^3x B \nabla\Phi] - \dot{\mathfrak{N}}(A); \quad \dot{\mathfrak{N}} = \frac{d\mathfrak{N}}{dt} = \partial_0 \mathfrak{N} , \quad (20)$$

where $\mathfrak{N}$ is defined by eq.(4)

$$E_i^{(T)a} = \partial_0 A_i^a - (\nabla_i \frac{1}{\nabla_k^2} \nabla_j \partial_0 A_j)^a \Rightarrow \nabla_i E_i^{(T)} = \nabla_i B_i = 0 . \quad (21)$$

For smooth transverse fields $B$, $E^{(T)}$, $\nabla\Phi$ (eqs.(17),(21)) all coefficients of the new variable $\dot{c}(t)$ in eqs.(19),(20) are equal to zero and we obtain a theory equivalent to that considered in parts 2 and 3. The new variable has physical meaning if the fields $B$, $E^{(T)}$, $\nabla\Phi$ are singular functions. We shall consider here stationary singularities. In other words, we represent the field $A$ in the form of a sum of the singular "Bose-condensate" and a smooth dynamical "quasiparticle" field.

$$A(x,t) = b(x) + \underset{\sim}{a}(x,t) . \quad (22)$$

According to (20) we may pass from the independent variables $\dot{c}$, $\underset{\sim}{a}$ to $\dot{N}$, $\underset{\sim}{a}$. The Lagrangian (19) with account of (20) ($\dot{c} = \dot{\bar{\mathfrak{N}}} [g^2 \int d^3x B \nabla\Phi]^{-1} 8\pi^2$)depends just on the combination of variables (12) in Theorem 2. The canonical momentum

$$P = \frac{\delta L}{\delta \dot{\bar{\mathfrak{N}}}} \equiv \frac{\delta L}{\delta \dot{N}} \quad (23)$$

is an integral of motion, as $L$ depends only on $\dot{N}$, $a$. Expressing the action (14) in terms of $P$ we get $S = \int dt \tilde{L}(b + \underset{\sim}{a})$, where

$$L(b+\underset{\sim}{a}) = \frac{1}{2}[\int d^3x (E^{(T)}(b+\underset{\sim}{a}))^2 - \langle E \rangle^2] - \frac{1}{2}[\int d^3x (B(b+\underset{\sim}{a}))^2 - \rho^2 \langle B \rangle^2], \quad (24)$$

* For more details see paper /4/.

$$\rho^2 = P^2 \left(\frac{8\pi^2}{g^2}\right)^2 \equiv (2\pi k + \theta)^2 \left(\frac{8\pi^2}{g^2}\right)^2 . \tag{25}$$

Here $\langle B \rangle$, $\langle E \rangle$ are quantities topologically invariant under smooth fields variations and dependent only on the "condensate":

$$\langle D \rangle^2 = (\int d^3x D \nabla\Phi)^2 / \int d^3x (\nabla\Phi)^2 ; \quad D = E^{(T)}(b) ; \ B(b) . \tag{26}$$

Let us show that the existence of a singular condensate does not contradict the basic assumptions of finite action and stability. Just from these assumptions we shall obtain an equation for the condensate and the quasiparticle spectrum. We expand the action (24) in powers of the smooth fields

$$S(b + \underset{\sim}{a}) = S(b) + S'(b)\underset{\sim}{a} + \frac{1}{2!} S''(b)\underset{\sim}{a}^2 + \ldots \tag{27}$$

The action finiteness for the stationary field is

$$S(b) = 0. \tag{28}$$

We require (28) to be fulfilled for both Euclidean and Minkowski spaces. Then we get two equations

$$\begin{aligned} &\int d^3x\, E^2(b) = \langle E \rangle^2 \Longrightarrow E(b) \sim \nabla(b)\Phi , \\ &\int d^3x\, B^2(b) = \rho^2 \langle B \rangle^2 . \end{aligned} \tag{29}$$

The stability condition is the vacuum-transition absence

$$S'(b)\underset{\sim}{a} = 0 . \tag{30}$$

The system will be stable if the fields $b_i^a$ satisfy the classical equation out of singularities and the fields $\underset{\sim}{a}$ have the zero boundary condition at the singularities. As the variational derivatives for $b_i^a$ are not defined, a unique way to satisfy (30) is to impose a condition of the type of (11) on the physical values of momentum $P$ :

$$\rho^2 = 1 \Longrightarrow (2\pi k + \theta)^2 = \left(\frac{8\pi^2}{g^2}\right)^2 . \tag{31}$$

That leads to the stationary duality equations:

$$E \sim B \sim \nabla\Phi. \tag{32}$$

Formula (31) completes the proof of Theorem 2.

5. Let us consider an exactly solvable example. Cylindrical-symmetric singular solutions of the self-dual stationary equation are found in ref. /5/

$$b^a_\mu = (\Phi^a, b^a_i)$$

$$\Phi^a = \frac{m}{g}\frac{x^a}{r}(\mathrm{ctg}\, z - \frac{1}{z}); \quad b^a_i = \frac{m}{g}\epsilon_{iak}\frac{x^k}{r}(\frac{1}{\sin z} - \frac{1}{z}),$$

where $z = mr$, $r = \sqrt{x^2_i}$, $m$ is a solution parameter with the dimensionality of mass. For illustration of the quasiparticle spectrum we consider an equation for a scalar coloured field in the same class of cylindrical-symmetric functions

$$([\nabla_\mu(b_\mu)]^2)^{cd}\,\underset{\sim}{a}^d = \kappa^2 \underset{\sim}{a}^c .$$

With the substitution

$$a^c = \sum_\ell \frac{x^c}{r^2}(\Psi_\ell(r)\,\hat{\xi}^{(+)}_\ell e^{iE_\ell t} + \Psi_\ell(r)\,\hat{\xi}^{(-)}_\ell e^{-iE_\ell t})$$

($\hat{\xi}_\ell$ are coefficients of the expansion over eigenfunctions). We obtain the equation for $\Psi_\ell(r)$

$$m^2[-\frac{\partial^2}{\partial z^2} + \frac{1}{2}(\frac{1}{\sin^2(z/2)} + \frac{1}{\cos^2(z/2)})]\Psi_\ell = (E^2 + \kappa^2)\Psi_\ell .$$

To solve this equation, it is sufficient to consider one "cell" of the periodic potential. The solution and spectrum look as follows

$$\Psi_\ell(r) = \sqrt{\frac{8}{\pi(\ell+3)(\ell+1)}}\,C^{(2)}_\ell(\cos z)\sin^2 z, \quad \int_0^\pi dz\,\Psi_{\ell_1}(r)\,\Psi_{\ell_2}(r) = \delta_{\ell_1,\ell_2}$$

$$(E^2_\ell + \kappa^2) = m^2(2+\ell)^2; \quad \ell = 0, 1, 2, \ldots ,$$

where $C_\ell^{(2)}$ is the Gegenbauer polynomial [6]. The eigenvalues of operator $[i\nabla_k(b)]^2$ are positive and nonzero and we have no infrared divergences in the new quasiparticle perturbation theory. Stable quasiparticle excitations of the system may only increase the energy, therefore the state of "condensate" is energetically favourable.

The considered solutions are in fact gluon bags [7] and we have provided here the mechanism of such a bag formation.

CONCLUSION

The approach assumed for constructing the colour field quantum states is physically appealing as it repeats the history of the theory of superfluidity (i.e., the theory in which for the first time the infrared catastrophe problem has been solved [8,9,10]).

I) The new variable $N(t)$ corresponds, in fact, to the London [8] cooperative variable describing the topological "rigid" excitation of the whole system.

II) According to the Bogolubov [10] microscopic theory we separate the c-number condencate and q-number quasiparticle fields. The very presence of the singular condensate provides the "rigid"-excitation condition.

III) The finiteness of action in field theory is the condition for the existence of the orthonormalizable basis diagonalizing the Hamiltonian. The conditions of "finiteness" and "stability" lead to the "Landau" [9] equation for the potential condensate ("superfluid component") and Bogolubov quasiparticle spectrum.

Points I, II, III constitute general features of the cooperative dynamics of systems with infinite number of degrees of freedom and with strong coupling in the infrared region. The criterion for the validity of such an approach to the infrared-catastrophe problem (i.e., the "necessary" analog of Theorem 2 ) is - as a rule - considered to be its self-consistency and the energetically favourable condensate.

We suggest to consider the non-Abelian fields as a physical medium like superfluid helium in a rotating bucket [11]. To calculate the parameters of the condensates singular configuration we must know the external boundary conditions and the helium atomic size. We do not know those for gluons, therefore, the "gluon-bag" parameters have to be determined experimentally.

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