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QUASI-CLASSICAL SOLITONS IN THE LINDNER-FEDYANIN MODEL AS COMPOUND BOSE-DROPS

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1. The one-dimensional Hubbard model[1] may be reduced, if one takes into account the electron-phonon interaction in the continuum approximation, to the following equation set for the Schrödinger wave functions

$$u = \phi_\uparrow + \phi_\downarrow \quad \text{and} \quad v = \phi_\uparrow - \phi_\downarrow ,$$

$$iu_t - \alpha_u u - u_{xx} - u(|u|^2 - \beta |v|^2) = 0, \tag{1}$$

$$iv_t + \alpha_v v + v_{xx} - v(|v|^2 - \beta |u|^2) = 0.$$

This system has been derived by Lindner and Fedyanin in ref.[2]. The same authors have investigated[3] some particular soliton-like solutions of the above system under the conditions: a) chirality, i.e., $|u|^2 + |v|^2$ = const and b) $\beta = 1$. In this case eqs (1) come to two independent Schrödinger equations with cubic nonlinearity (S3) for the functions u and v (note that for this the second condition b) is not necessary). The physical sense of the parameters α_i and β can be found in cited works.

In what follows we study soliton solutions to system (1) in quasi-classical limit.

2. The equations (1) are the Lagrange-Euler equations of the following variational problem*

$$\delta \int \mathcal{L} \, dx\, dt = 0, \tag{2}$$

where

$$\mathcal{L} = -\frac{i}{2}(u_t u^* - u_t^* u) - |u_x|^2 + \frac{1}{2}|u|^4 - \beta |u|^2 |v|^2$$
$$- \frac{i}{2}(v_t v^* - v_t^* v) + |v_x|^2 + \frac{1}{2}|v|^4 + \alpha_u |u|^2 - \alpha_v |v|^2 , \tag{3}$$

with the Hamiltonian density of the system being

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* N.Makhaldiani following Manakov[4] has obtained a (L-A) Lax pair for system (1) when $\beta = 1$.

$$\mathcal{H} = |u_x|^2 - \alpha_u|u|^2 - |v_x|^2 + \alpha_v|v|^2 - \frac{|u|^4 + |v|^4}{2} + \beta|u|^2|v|^2 . \tag{4}$$

The total system energy

$$E = \int \mathcal{H}\, dx$$

is the integral of motion, i.e., $dE/dt = 0$. In addition to this integral system (1) possesses at least three more ones:

-"particle number integrals"

$$N_u = \int |u|^2 dx, \quad N_v = \int |v|^2 dx, \quad dN_i/dt = 0$$

-"momentum integral"

$$P = i \int (u^* u_x + v v_x^* - u u_x^* - v^* v_x) dx = P_u + P_v$$

$$dP/dt = 0$$

and has plane wave solutions ("condensate")

$$u = a \exp\{i(\omega_u t - k_u x)\}, \quad \omega_u = k_u^2 - (a^2 - \beta b^2) - \alpha_u ,$$

$$v = b \exp\{i(k_u x - \omega_v t)\}, \quad \omega_v = k_v^2 - (\beta a^2 - b^2) - \alpha_v . \tag{5}$$

3. Consider first (1) at $\beta = 1$. In this case, as we shall see, the system possesses more rich spectrum of solitons and instead of (4) we have the Hamiltonian density

$$\mathcal{H} = |u_x|^2 - \alpha_u|u|^2 - |v_x|^2 + \alpha_v|v|^2 - \frac{1}{2}(|u|^2 - |v|^2)^2 . \tag{6}$$

To investigate soliton solutions let us come to the static representation

$$u = \tilde{u}(\xi) \exp\{i(\omega_u t - \frac{c}{2} x)\},$$

$$v = \tilde{v}(\xi) \exp\{i(\frac{c}{2} x - \omega_v t)\}, \quad \xi = x - ct .$$

We have now the stationary problem

$$-(\omega_u+\alpha_u)u-u_{xx}-u(|u|^2-|v|^2)=0,$$
$$(\omega_v+\alpha_v)v+v_{xx}+v(|v|^2-|u|^2)=0, \tag{7}$$

being the extremum condition for the following functional

$$F=E+\omega_v N_v-\omega_u N_u \tag{8}$$

(here and below we omit the tilde).

The system (7) in addition to the plane wave solutions has at $\beta=1$ at least three soliton type solutions which can be obtained with standard technique.

(i)
$$u=a\,\mathrm{sech}\,\kappa\xi, \quad \omega_u=\omega_1-\alpha_u\equiv\frac{c^2}{4}-\frac{a^2-b^2}{2}-\alpha_u$$
$$v=b\,\mathrm{sech}\,\kappa\xi, \quad \omega_v=\omega_1-\alpha_v, \quad \kappa^2=\frac{a^2-b^2}{2}>0 \tag{9}$$

(ii)
$$u=a\tanh\kappa\xi, \quad \omega_u=\omega_2-\alpha_u\equiv\frac{c^2}{4}+b^2-a^2-\alpha_u$$
$$v=b\tanh\kappa\xi, \quad \omega_v=\omega_2-\alpha_v, \quad \kappa^2=\frac{b^2-a^2}{2}>0 \tag{10}$$

(iii)
$$u=a\,\mathrm{sech}\,\kappa\xi, \quad \omega_u=\frac{c^2}{4}-\frac{a^2-b^2}{2}-\alpha_u$$
$$v=b\tanh\kappa\xi, \quad \omega_v=\frac{c^2}{4}+b^2-\alpha_v, \quad \kappa^2=\frac{a^2+b^2}{2}. \tag{11}$$

Calculate the energy (integral) characteristics of them* .

Note, that the frequencies ω_u and ω_v contain the constants α_u and α_v respectively. Terms proportional to these constants come also in the Hamiltonian density (6). We can omit in calculations such terms in the frequencies as well as in the Hamiltonian due to the transformations $u\to ue^{i\alpha_u t}$ and $v\to ve^{-i\alpha_v t}$ and restore them in the final expressions, e.g., using the formal substitution $c_i^2/4\to c_i^2/4-\alpha_i$.

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* Solutions (11) is a two-dimensional scalar analogue of that for SLAG-bag well-known in quantum field theory[5] .

4. Having used the solutions (5), formulae (6) and (8) one can easy find (at $k^2 = c^2/4$, $k_u = -k_v = k$)

$$E_k = -\frac{1}{2}\int (b^2 - a^2)(\frac{c^2}{2} + b^2 - a^2)dx$$

$$F_k = \frac{1}{2}\int (a^2 - b^2)^2 dx.$$

These expressions diverge and therefore are of the formal character. In the same way we obtain for the solution (9)

$$E_1 = \sqrt{2}(a^2 - b^2)^{1/2}(\frac{c^2}{2} - \frac{a^2 - b^2}{3})$$

$$N_u = 2\sqrt{2}\, a^2 (a^2 - b^2)^{-1/2}, \quad N_v = 2\sqrt{2}\, b^2 (a^2 - b^2)^{-1/2}.$$

Introducting $N_1 = N_u - N_v = 2\sqrt{2}(a^2 - b^2)^{1/2}$ one gets $E_1 = N(\frac{c^2}{4} - \frac{N_1^2}{48})$ and $\omega_1 = c^2/4 - N_1^2/16$. Where by

$$\frac{dE_1}{dN_1} = \omega_1(N_1). \tag{12}$$

The physical sense of this expression will be discussed below. The allowance for the constants α_u and α_v yields

$$dE_1 = \omega_u dN_u - \omega_v dN_v$$

with

$$\omega_i = \omega_1(N_1) - \alpha_i, \qquad i = u, v$$

instead of (12).

To obtain the integral characteristics for the solution (10) the adequate renormalization procedure has to be constructed since expressions for E and N_i are divergent.

Take the following procedure

$$\tilde{\Phi}[u,v] = \Phi[u,v] - \Phi[u_k, v_k], \tag{13}$$

where $\Phi[u,v]$ is the functional sought and $\Phi[u_k,v_k]$ is the same functional on the solution (5) (condensate). Using this rule we obtain

$$\tilde{E}_2 = \sqrt{2}(b^2 - a^2)^{1/2}(\frac{c^2}{2} + \frac{2}{3}(b^2 - a^2)), \quad N_i = -\tilde{N}_i[u,v]$$

$$N_u = 2\sqrt{2}\, a^2 (b^2 - a^2)^{-1/2}, \qquad N_v = 2\sqrt{2}\, b^2 (b^2 - a^2)^{-1/2}.$$

Introducing $N_2 = N_v - N_u = 2\sqrt{2}(b^2 - a^2)^{1/2}$ one has

$$\tilde{E}_2 = N_2\left(\frac{c^2}{4} + \frac{N_2^2}{24}\right), \qquad \omega_2 = \frac{c^2}{4} + \frac{N_2^2}{8}$$

and

$$\frac{dE_2}{dN_2} = \omega_2(N_2). \tag{14}$$

Finally

$$d\tilde{E}_2 = -\omega_u dN_u + \omega_v dN_v, \qquad \omega_i = \omega_2(N_2) - a_i.$$

Acting in the same way one can find for the solution (11)

$$\tilde{E}_3 = \sqrt{2}(a^2 + b^2)^{1/2}\left(\frac{c^2}{2} - \frac{a^2 - 2b^2}{3}\right) = (N_u + N_v)\left(\frac{c^2}{4} - (N_u - 2N_v)\frac{N_u + N_v}{48}\right)$$

$$N_u = \tilde{N}_u, \quad N_v = -\tilde{N}_v[u, v], \quad \omega_u \neq \omega_v \tag{15}$$

$$N_u = 2\sqrt{2}\, a^2 (a^2 + b^2)^{-1/2}, \qquad N_v = 2\sqrt{2}\, b^2 (a^2 + b^2)^{-1/2}$$

$$\omega_u = \frac{c^2}{4} - \frac{N_u - N_v}{16}(N_u + N_v), \qquad \omega_v = \frac{c^2}{4} + \frac{N_v}{8}(N_u + N_v).$$

Establish an analogous relation

$$d\tilde{E}_3 = \omega_u dN_u + \omega_v dN_v \tag{16}$$

is also valid in this case. For this we proceed from variables N_u, N_v to a, b then

$$dN_u = \frac{\partial N_u}{\partial a} da + \frac{\partial N_u}{\partial b} db,$$

$$dN_v = \frac{\partial N_v}{\partial a} da + \frac{\partial N_v}{\partial b} db.$$

Whence

$$d\tilde{E}_3 = \left(\omega_u \frac{\partial N_u}{\partial a} + \omega_v \frac{\partial N_v}{\partial a}\right)da + \left(\omega_u \frac{\partial N_u}{\partial b} + \omega_v \frac{\partial N_v}{\partial b}\right)db$$

on the other hand

$$d\tilde{E}_3 = \frac{\partial \tilde{E}_3}{\partial a}\, da + \frac{\partial \tilde{E}_3}{\partial b} db.$$

The relation (16) takes place if

$$\frac{\partial \tilde{E}_3}{\partial a} = \omega_u \frac{\partial N_u}{\partial a} + \omega_v \frac{\partial N_v}{\partial a}, \qquad \frac{\partial \tilde{E}_3}{\partial b} = \omega_u \frac{\partial N_u}{\partial b} + \omega_v \frac{\partial N_v}{\partial b}. \tag{17}$$

These equalities can be shown to be valid, in fact, using formulae (15).

Concluding this section note that in the case $\beta \neq 1$ only plane wave (5) and sech $\times$ tanh type (11) solutions survive. Thereat for the last one it should be $a = b$ and

$$\kappa^2 = (1+\beta)a^2/2, \quad \omega_u = \frac{c^2}{4} - \frac{1-\beta}{2} a^2, \quad \omega_v = \frac{c^2}{4} + a^2$$

and

$$d\tilde{E} = \omega_u dN_u + \omega_v dN_v .$$

5. The above results admit the following evident interpretation. The conventional WKB quantization formula assumes in our case the form

$$\int_0^T dt \int dx \left(\frac{\partial \mathcal{L}}{\partial u_t} u_t + \frac{\partial \mathcal{L}}{\partial u_t^*} u_t^* \right) = 2\pi n$$

or

$$\int_0^T dt \int dx\, i(u_t u^* - u_t^* u) = \omega \int_0^T dt \int dx\, |u|^2 = \omega \int_0^T dt N_u = \omega\, T N_u ,$$

whence $N_u = n$, $n \gg 1$. Here n is integer. The solution therefore presents to be the bound state of a large number of u-field boson quanta. As a result we can think of the soliton as a condensed Bose-drop (the potential of boson interactions decreases exponentially in it).

The equation

$$dE = \omega\, dN \qquad (18)$$

may be treated in this case as thermodynamical relation in which ω stands for the chemical potential, i.e., the energy per boson.

The equality of type (18) was known earlier for one-frequency solitons in various theories of condenced matter physics [6,7]*. Here we see that in the two-field model considered where the two-frequency solutions $\psi = \begin{pmatrix} u(\omega_u, \omega_v) \\ v(\omega_u, \omega_v) \end{pmatrix}$ are possible the equation (18) becomes

$$dE = \sum_i \omega_i dN_i \qquad (19)$$

known as thermodynamical relation for the mixtures.

It should be underlined that in the usual ideal gas statistics chemical potential is independent of particle numbers, $\partial\mu_i/\partial N_j = 0$, with equation (19) being linear in N_i. In our case the frequencies ω_i are the functions of particle numbers, $\omega_i(N_u, N_v)$, nevertheless, a "linear" form of relation (19) is conserved (that is, evidently, one more paradox of the soliton theory).

Thus, in quasi-classical approximation the solution obtained may be treated as boson drops composed of two components.

The plane wave solution (5) describes in this limit Bose-condensate since boson distribution function is $N(\epsilon) \propto \delta(\epsilon - \epsilon_k)$, where $\epsilon_k = \hbar\omega_k$. This condensate is the mixture of two "gases" with attraction inter-boson force (u-field) and repulsion one (v-field). There are three possible different situations depending on the condensate parameter relation, a and b.

1) $a > b$ (the u-gas density is greater than the v-gas one). Such a condensate turns out to be infinitesimally unstable with respect to decay into boson drops of sech $\times$ sech type. The solution of sech form may be treated as a liquid drop in a gas and tanh form as a gas bubble in a liquid (or a rarefaction drop, i.e., "antidrop"). A strongly excited condensate may transform into the solutions of sech $\times$ tanh type (i.e., " v-bubble" in " u-drop", thereat bubbles will be the nuclei of u-drop condensation).

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*Such a relation was evidently obtained for the first time by Kosevich and coauthors. In the case of relativistic systems the relation $dE = \omega dQ$ (Q is isocharge) appears naturally (see, e.g., ref. [8]).

2) $b < a$ (the u-gas density is less than the v-gas one). In this case the condensate is stable regarding to small perturbations. A strongly perturbed condensate transforms to the state of $\tanh \times \tanh$ type (i.e., u, v-gases with bubbles) or to the $\operatorname{sech} \times \tanh$ type (i.e., v-gas with bubbles in which u-drops are captured).

3) $a = b$ (the densities of both gases are equal). In this case there are possible only $\operatorname{sech} \times \tanh$ states (v-gas with bubbles-drops) appearing in a condencate with sufficiently strong excitation.

Now let us investigate stability of the solutions obtained.

In the first two cases when $\omega_u = -\omega_v$ one can employ the Q-theorem in the form $dN/d|\omega| > 0$ (for solitons at rest).

In the case $\omega_u \neq \omega_v$ the analoque of Q-theorem may be derived upon generalizing the technique described in review [8].

$$\left(\frac{\partial}{\partial\omega_u} + \frac{\partial}{\partial\omega_v}\right)(N_u + N_v) > 0. \qquad (20)$$

Note that getting this inequality one should use the renormalization procedure (13). Introducing (11) into (20) we find

$$\left(\frac{\partial}{\partial\omega_u} + \frac{\partial}{\partial\omega_v}\right)(N_u + \tilde{N}_v) \propto \frac{b^2}{\kappa} > 0,$$

i.e., the $\operatorname{sech} \times \tanh$ solutions are also stable.

To study the condensate stability let us take perturbation in the following form

$$\delta\begin{pmatrix} u \\ u^* \end{pmatrix} = a e^{i(kx-\Omega t)} \begin{pmatrix} \delta_1 e^{-i\omega_1 t} \\ \delta_2 e^{i\omega_1 t} \end{pmatrix}, \quad \delta\begin{pmatrix} v \\ v^* \end{pmatrix} = b e^{i(kx-\Omega t)} \begin{pmatrix} \delta_3 e^{i\omega_1 t} \\ \delta_4 e^{-i\omega_1 t} \end{pmatrix} \qquad (21)$$

with δ_i ($i = 1,2,3,4$) being constants and $\omega_1 = a^2 - b^2$. Substituting (21) into (1) and linearizing equations obtained over δ_i we get a homogeneous linear system of equations which possesses solutions when

$$\begin{vmatrix} -\Omega - k^2 + a^2 & a^2 & ab & ab \\ a^2 & \Omega - k^2 + a^2 & ab & ab \\ ab & ab & -\Omega + k^2 + b^2 & b^2 \\ ab & ab & b^2 & \Omega + k^2 + b^2 \end{vmatrix} = 0$$

or

$$\Omega^2 = k^2 [k^2 - 2(a^2 - b^2)]. \tag{22}$$

It follows from (22) that the condensate becomes unstable if $a^2 > b^2$ for infinitesimal perturbations of sufficiently long wavelengths, so that

$$\lambda = \frac{1}{k} > \frac{1}{\sqrt{2}} (a^2 - b^2)^{-1/2}.$$

Perturbations with $\lambda = (a^2 - b^2)^{-1/2}$ have the maximal growth rate

$$\gamma = a^2 - b^2 . \tag{23}$$

All this confirms the foregoing assumptions on the character of the above solutions.

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