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EQUATION OF MOTION FOR THE AXIAL GRAVITATIONAL SUPERFIELD

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1. Introduction

The action for $N=1$ -supergravity is just the volume of the superspace[1]

$$S_{SG} = \frac{1}{\varkappa^2}\int d^4x\, d^4\theta\ Ber\| E_M^A \| \equiv \frac{1}{\varkappa^2}\int d^4x\, d^4\theta\ E \quad , \tag{1.1}$$

where Ber is superdeterminant of the supervierbeins E_M^A, and $\varkappa$ is the gravitational coupling constant with dimension cm.

In the approach of Wess and Zumino[1,2] the supervierbeins are fundamental objects (potentials). However, they are not independent: They are restricted by a number of constraints on the torsion tensor[2]. This makes the variation of action (1.1) nontrivial[1].

In the minimal geometric approach[3,4] the supervierbeins as well as all other differential-geometry quantities are explicitly expressed in terms of a single prepotential: The axial gravitational superfield $\mathcal{H}^m(x,\theta,\bar\theta)$. No constraints have to be imposed, they follow from the simple geometric picture underlying the theory. Therefore, the equation of motion for the superfield $\mathcal{H}^m$ can easily be obtained just by varying $\mathcal{H}^m$ in action (1.1). In the present paper this procedure is carried out in detail.

In Section 2 the transformation properties of the variation of $\mathcal{H}^m$ (and of the equation of motion, correspondingly) are investigated. A new "vierbein" is introduced in order to have a variation transforming as a Lorentz vector.

In Section 3 the explicit form of the equation of motion is established. The use of the normal gauge earlier introduced[5], simplifies the derivation considerably. The left-hand side of the equation proves to be the component G_a of the torsion tensor, in agreement with Ref. .

In Section 4 the matter coupling is considered. Covariant variations of the matter superfields are introduced and their role is explained. The equation of motion in the presence of matter has the form

$$G_a = \varkappa^2 V_a \tag{1.2}$$

similar to the Einstein equation: G_a is the analogue of $R_{mn} - \frac{1}{2} g_{mn} R$, the supercurrent V_a is the analogue of the energy-momentum tensor T_{mn}.

In Section 5 the cosmological term in pure supergravity is investigated. The obtained equation of motion $(G_a = 0)$ implies that the torsion component $\bar{R}$ equals some constant (not necessarily zero, as claimed in Ref. /1/). This constant plays the role of the cosmological constant. The presence or absence of the cosmological term seems to be related to the behaviour of $\mathcal{H}^m$ at large x.

Appendices A and B contain proofs of some statements made in the text.

2. Transformation Properties of the Supergravity Equation of Motion

In order to derive the equation of motion one has to vary action (1.1). Suppose, this has been done, the derivatives acting on the variation $\tilde{\delta} \mathcal{H}^m$ have been removed by integration by parts and the result has been presented in the form

$$\tilde{\delta} S_{SG} = \frac{1}{\varkappa^2} \int d^4x \, d^4\theta \, E \, . \, \tilde{\delta} \mathcal{H}^m . \, \Gamma_m(x,\theta,\bar{\theta}) \, . \tag{2.1}$$

Now a question arises: How does $\tilde{\delta} \mathcal{H}^m$ (and correspondingly Γ_m) transform?

Recall that the gravitational superfield $\mathcal{H}^m$ itself transforms inhomogeneously /3/

$$\delta \mathcal{H}^m(x,\theta,\bar{\theta}) \equiv \mathcal{H}'^m(x,\theta,\bar{\theta}) - \mathcal{H}^m(x,\theta,\bar{\theta}) =$$

$$= -\frac{1}{2}\lambda^m(x_L,\theta) + \frac{1}{2}\bar{\lambda}^m(x_R,\bar{\theta}) - \tag{2.2}$$

$$-\left\{\frac{1}{2}\left[\lambda^n(x_L,\theta) + \bar{\lambda}^n(\bar{x}_R,\bar{\theta})\right]\partial_n + \lambda^\nu(x_L,\theta)\partial_\nu + \bar{\lambda}^{\dot{\nu}}(x_R,\bar{\theta})\bar{\partial}_{\dot{\nu}}\right\}\mathcal{H}^m.$$

Here λ^m, λ^μ and their conjugates $\bar{\lambda}^m, \bar{\lambda}^{\dot{\mu}}$ are chiral superfunctions-parameters of general coordinate transformations;

$$x_L^m = x^m + i\mathcal{H}^m(x,\theta,\bar{\theta}), \quad x_R^m = x^m - i\mathcal{H}^m(x,\theta,\bar{\theta}). \tag{2.3}$$

The first two terms in Eq.(2.2) cause the inhomogeneity, the rest are just coordinate translation terms. However, the variation $\tilde{\delta} \mathcal{H}^m$ transforms in a different way

$$\delta(\tilde{\delta} \mathcal{H}^m) = \tilde{\delta} \mathcal{H}^n \Big[\frac{1}{2} \partial_n^L \lambda^\kappa (1 - i\mathcal{H})_\kappa^{\ m} + \frac{1}{2} \partial_n^R \bar{\lambda}^\kappa (1 + i\mathcal{H})_\kappa^{\ m} -$$

$$- i\partial_n^L \lambda^\nu \partial_\nu \mathcal{H}^m + i\partial_n^R \bar{\lambda}^{\dot{\nu}} \bar{\partial}_{\dot{\nu}} \mathcal{H}^m \Big] + t.t. \quad . \tag{2.4}$$

Here

$$\partial_n^L = \frac{\partial}{\partial x_L^n}, \quad \partial_n^R = \frac{\partial}{\partial x_R^n}; \quad \mathcal{H}_\kappa^{\ m} = \partial_\kappa \mathcal{H}^m; \tag{2.5}$$

t.t. means translation terms of the same kind as in Eq.(2.2).

The transformation law (2.4) is already homogeneous. However, it does not coincide with the transformation law of a Lorentz vector as defined in Ref. /4/. To achieve this coincidence one has to introduce a new "vierbein" $v_a^{\ m}$ and its inverse $v_m^{\ a}$:

$$v_a^{\ m} = \frac{1}{4} F\bar{F} (\tilde{\sigma}_a)^{\dot{\alpha}\alpha} \left[\Delta_\alpha, \bar{\Delta}_{\dot{\alpha}}\right] \mathcal{H}^n (1 + \mathcal{H}^2)_n^{\ m}, \tag{2.6}$$

$$v_m^{\ a} v_a^{\ n} = \delta_m^n, \quad v_m^{\ a} v_b^{\ m} = \delta_b^a \, .$$

The operators Δ_α, $\bar{\Delta}_{\dot{\alpha}}$ were defined in Ref.[4]:

$$\Delta_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\frac{\partial}{\partial\theta^\alpha}\mathcal{H}^n(1-iH)^{-1\,m}_{\ n}\frac{\partial}{\partial x^m} = \partial_\alpha + i\Delta_\alpha\mathcal{H}^m\partial_m ,$$
$$\bar{\Delta}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\mathcal{H}^n(1+iH)^{-1\,m}_{\ n}\frac{\partial}{\partial x^m} = -\bar{\partial}_{\dot{\alpha}} - i\bar{\Delta}_{\dot{\alpha}}\mathcal{H}^m\partial_m ; \tag{2.7}$$

the quantities F, $\bar{F}$ were introduced there too:

$$F = 2^{\frac{2}{3}}\left(\det\|\hat{e}_a^{\ m}\|\right)^{-\frac{1}{3}}\left(\det\|\hat{\bar{e}}_a^{\ m}\|\right)^{\frac{1}{6}}, \quad \bar{F} = F^+, \tag{2.8}$$

where

$$\hat{e}_a^{\ m} = \tilde{\sigma}_a^{\ \dot{\alpha}\alpha}\Delta_\alpha\bar{\Delta}_{\dot{\alpha}}\mathcal{H}^m , \quad \hat{\bar{e}}_a^{\ m} = \tilde{\sigma}_a^{\ \dot{\alpha}\alpha}\bar{\Delta}_{\dot{\alpha}}\Delta_\alpha\mathcal{H}^m . \tag{2.9}$$

Now, as shown in Appendix A, the modified variation

$$\tilde{\delta}\mathcal{H}^a = v_m^{\ a}\,\tilde{\delta}\mathcal{H}^m \tag{2.10}$$

transforms according to the Lorentz law[4]

$$\delta(\tilde{\delta}\mathcal{H}^a) = \tilde{\delta}\mathcal{H}^b . 2\Omega_b^{\ a} + t.t. \tag{2.11}$$

with parameters

$$\Omega_{ab} = \frac{i}{4}\left[\Delta^\alpha(\sigma_{ab})_\alpha^{\ \beta}\lambda_\beta + \bar{\Delta}_{\dot{\alpha}}(\tilde{\sigma}_{ab})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\lambda}^{\dot{\beta}}\right] . \tag{2.12}$$

Then, the variation (2.1) can be rewritten in the form

$$\tilde{\delta}S_{SG} = \frac{1}{\varkappa^2}\int d^4x\, d^4\theta\, E\, \tilde{\delta}\mathcal{H}^a\,\Gamma_a , \quad \Gamma_a = v_a^{\ m}\Gamma_m . \tag{2.13}$$

Finally, from the invariance of action (1.1) and its variation (2.13) and from the transformation properties of $\tilde{\delta}\mathcal{H}^a$ (2.11) it follows that the quantity Γ_a transforms as a Lorentz vector too:

$$\delta\Gamma_a = -2\Omega_a^{\ b}\Gamma_b + t.t. \tag{2.14}$$

So, the equation of motion

$$\Gamma_a = 0 \tag{2.15}$$

obtained from Eq.(2.13) is Lorentz-covariant.

Having clarified this important question we can proceed further to find the explicit form of the equation of motion.

3. Variation of the Action, Normal Gauge and the Equation of Motion

In this Section we shall show that the left-hand side of Eq.(2.15) is in fact a component of the torsion tensor. There are two ways for doing this. The straightforward one is to write down the explicit expression for the density E, to vary the action and thus to find Γ_a in terms of the derivatives of $\mathcal{H}^m$. Then, after some algebraic rearrangement one is able to identify Γ_a with the expression for the torsion component G_a derived in Ref.[6].

The other proof is much simpler and elegant. It consists in constructive use of the normal gauge in supergravity[5,*]. In this gauge a number of the derivatives of $\mathcal{H}^m$ vanish at a given point z_0 in superspace. For instance, among the derivatives with dimension cm^K, $K \geq -1$, only the following ones do not vanish:

$$\partial_\nu\bar{\partial}_{\dot{\nu}}\mathcal{H}^m|_0 , \ \partial^\nu\partial_\nu\partial_m\mathcal{H}^m|_0 , \ \bar{\partial}_{\dot{\nu}}\bar{\partial}^{\dot{\nu}}\partial_m\mathcal{H}^m|_0 , \ \partial^\nu\partial_\nu\bar{\partial}_{\dot{\nu}}\bar{\partial}^{\dot{\nu}}\mathcal{H}^m|_0 . \tag{3.1}$$

Now, consider Γ_a at point z_0 in normal gauge. Γ_a is made of derivatives of $\mathcal{H}^m$, it is a vector, it has dimension cm^{-1} (mind that in Eq.(2.13) $\tilde{\delta}S_{SG}$ and E are dimensionless, $[\mathcal{H}^m] = cm$). Therefore, the only possible expression for Γ_a is

$$\Gamma_a|_0 = K\,\partial^\nu\partial_\nu\bar{\partial}_{\dot{\nu}}\bar{\partial}^{\dot{\nu}}\mathcal{H}_a|_0 , \tag{3.2}$$

where K is a numerical coefficient. On the other hand, in normal gauge the torsion component

$$G_a = -4i\,T_{\alpha a}^{\ \ \alpha} \tag{3.3}$$

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*) The helpfulness of geodesic and normal coordinate frames is well known in general relativity. For example, recall the variational derivation of the Einstein equation in the "Field Theory" textbook by Landau and Lifshitz.

looks like (see Ref. /6/)

$$G_a|_0 = -\frac{1}{3}\partial^{\nu}\partial_{\nu}\bar{\partial}_{\dot{\nu}}\bar{\partial}^{\dot{\nu}}\mathcal{H}_a|_0 . \tag{3.4}$$

So, one sees that in normal gauge Γ_a is proportional to G_a. This conclusion must be valid in any gauge because both Γ_a and G_a transform covariantly. Thus the proof is completed. Nevertheless, we are going to give some details of the straightforward proof too. This should give the reader an idea of how the quantities under consideration look like and how such calculations could be carried out in more complicated situations (e.g., in matter couplings).

In Ref. /4/ the following expression for the density E was found*)

$$E = 2^{\frac{4}{3}}\left(\det\|\hat{\tau}_a^{\ m}\|\right)^{-\frac{1}{6}}\left(\det\|\hat{\ell}_a^{\ m}\|\right)^{-\frac{1}{6}}\left(\det(1+\mathcal{H}^2)\right)^{\frac{1}{2}} \tag{3.5}$$

($\hat{\tau}$ and $\hat{\ell}$ were defined in Eqs.(2.9), (2.7), (2.5)). Putting Eq.(3.5) into Eq.(1.1) and varying the result one easily obtains

$$\tilde{\delta}S_{SG} = \frac{1}{\varkappa^2}\int d^4x\, d^4\theta\, E\Big[-\frac{1}{6}\tilde{\delta}\hat{\ell}_a^{\ m}\cdot\hat{\ell}_m^{\ a} - \frac{1}{6}\tilde{\delta}\hat{\tau}_a^{\ m}\cdot\hat{\tau}_m^{\ a} + \frac{1}{2}\tilde{\delta}(1+\mathcal{H}^2)_m^{\ n}(1+\mathcal{H}^2)^{-1}{}_n^{\ m}\Big] \tag{3.6}$$

where $\hat{\ell}_m^{\ a}$ is the inverse of $\hat{\ell}_a^{\ m}$, $\hat{\ell}_m^{\ a}\hat{\ell}_a^{\ n} = \delta_m^{\ n}$ and the same for $\hat{\tau}_m^{\ a}$.

From the definition (2.7) follows

$$\tilde{\delta}(\Delta_\alpha\mathcal{H}^m) = \tilde{\delta}(\partial_\alpha\mathcal{H}^m + i\Delta_\alpha\mathcal{H}^n\partial_n\mathcal{H}^m) = $$
$$= \tilde{\delta}(\Delta_\alpha\mathcal{H}^n)\, i\,\mathcal{H}_n^{\ m} + \Delta_\alpha\tilde{\delta}\mathcal{H}^m,$$

i.e.,

$$\tilde{\delta}(\Delta_\alpha\mathcal{H}^m) = \Delta_\alpha\tilde{\delta}\mathcal{H}^n(1-i\mathcal{H})^{-1}{}_n^{\ m} . \tag{3.7}$$

Analogously

$$\tilde{\delta}(\bar{\Delta}_{\dot{\alpha}}\mathcal{H}^m) = \bar{\Delta}_{\dot{\alpha}}\tilde{\delta}\mathcal{H}^n(1+i\mathcal{H})^{-1}{}_n^{\ m} .$$

*) Remarkably enough, $E = \det\|v_a^{\ m}\|$ (see Eqs.(2.8), (3.12)).

Further,

$$\tilde{\delta}(\bar{\Delta}_{\dot{\alpha}}\Delta_\alpha\mathcal{H}^m) = \bar{\Delta}_{\dot{\alpha}}\tilde{\delta}(\Delta_\alpha\mathcal{H}^m) - i\tilde{\delta}(\bar{\Delta}_{\dot{\alpha}}\mathcal{H}^n)\partial_n\Delta_\alpha\mathcal{H}^m =$$
$$= \Big[\bar{\Delta}_{\dot{\alpha}}\Delta_\alpha\tilde{\delta}\mathcal{H}^n - i\Delta_\alpha\tilde{\delta}\mathcal{H}^k(1-i\mathcal{H})^{-1}{}_k^{\ \ell}\,\bar{\Delta}_{\dot{\alpha}}\mathcal{H}_\ell^{\ n} - \tag{3.8}$$
$$- i\bar{\Delta}_{\dot{\alpha}}\tilde{\delta}\mathcal{H}^k(1+i\mathcal{H})^{-1}{}_k^{\ \ell}\,\Delta_\alpha\mathcal{H}_\ell^{\ n}\Big](1-i\mathcal{H})^{-1}{}_n^{\ m} .$$

Here the identity /4/

$$\partial_n\Delta_\alpha\mathcal{H}^m = \Delta_\alpha\mathcal{H}_n^{\ k}(1-i\mathcal{H})^{-1}{}_k^{\ n} \tag{3.9}$$

was used. The formula for $\tilde{\delta}(\Delta_\alpha\bar{\Delta}_{\dot{\alpha}}\mathcal{H}^m)$ is obtained from Eq.(3.8) by conjugation. Finally,

$$\tilde{\delta}(1+\mathcal{H}^2)_m^{\ n}(1+\mathcal{H}^2)^{-1}{}_n^{\ m} = 2\partial_m\tilde{\delta}\mathcal{H}^n\left(\frac{\mathcal{H}}{1+\mathcal{H}^2}\right)_n^{\ m} =$$
$$= \frac{i}{2}\{\Delta_\alpha,\bar{\Delta}_{\dot{\alpha}}\}\tilde{\delta}\mathcal{H}^n\mathcal{H}_n^{\ m}\frac{1}{F\bar{F}}v_m^{\ \alpha\dot{\alpha}} . \tag{3.10}$$

Here $v_m^{\ \alpha\dot{\alpha}} = v_m^{\ a}(\sigma_a)^{\alpha\dot{\alpha}}$, and the identity /4/

$$\{\Delta_\alpha,\bar{\Delta}_{\dot{\alpha}}\} = -i\,[\Delta_\alpha,\bar{\Delta}_{\dot{\alpha}}]\mathcal{H}^m\partial_m . \tag{3.11}$$

is used. Finally, putting Eq.(2.9), Eq.(3.8) and its conjugate, Eq.(3.10) into Eq.(3.6) and making use of the identities /4/

$$\hat{\ell}_a^{\ m} = -\frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}[\Delta_\alpha,\bar{\Delta}_{\dot{\alpha}}]\mathcal{H}^n(1+i\mathcal{H})_n^{\ m} = -\frac{2}{F\bar{F}}v_a^{\ n}(1-i\mathcal{H})^{-1}{}_n^{\ m},$$
$$\hat{\tau}_a^{\ m} = \frac{1}{2}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}[\Delta_\alpha,\bar{\Delta}_{\dot{\alpha}}]\mathcal{H}^n(1-i\mathcal{H})_n^{\ m} = \frac{2}{F\bar{F}}v_a^{\ n}(1+i\mathcal{H})^{-1}{}_n^{\ m}, \tag{3.12}$$

one finds

$$\tilde{\delta}S_{SG} = -\frac{1}{12\varkappa^2}\int d^4x\, d^4\theta\, E\Big[\Delta_\alpha\bar{\Delta}_{\dot{\alpha}}\tilde{\delta}\mathcal{H}^m(1-3i\mathcal{H})_m^{\ n} +$$
$$+ 2i\bar{\Delta}_{\dot{\alpha}}\tilde{\delta}\mathcal{H}^m(1+i\mathcal{H})^{-1}{}_m^{\ k}\Delta_\alpha\mathcal{H}_k^{\ n}\Big]\frac{v_n^{\ \alpha\dot{\alpha}}}{F\bar{F}} + h.c. \tag{3.13}$$

where h.c. means Hermitian conjugated terms.

In order to have this variation in the form (2.13) one has to integrate by parts and thus to take the derivatives off $\tilde{\delta}\mathcal{H}^m$. For this purpose the following formula together with its conjugate are to be used.

$$\int d^4x\, d^4\theta\, E\left[\Delta^\alpha \varphi . \psi_\alpha + (-1)^{p(\varphi)}\varphi\left(\Delta^\alpha \psi_\alpha + 2\Delta^\alpha \ln F . \psi_\alpha\right)\right] = 0 \qquad (3.14)$$

(the proof see in Appendix B; $p(\varphi)$ is 0 when φ is bosonic and 1 when it is fermionic). After performing this little algebra we finally obtain

$$\begin{aligned}\Gamma_a = \frac{1}{12} v_a{}^m \Big\{ & \bar\Delta_{\dot\alpha}\Delta_\alpha\Big[(1-3i\mathcal{H})_m^{\ n}\frac{v_n{}^{\alpha\dot\alpha}}{F\bar F}\Big] + 2i\bar\Delta_{\dot\alpha}\Big[(1+i\mathcal{H})^{-1\,k}_{m}\Delta_\alpha\mathcal{H}_k^{\ n}\frac{v_n{}^{\alpha\dot\alpha}}{F\bar F}\Big] + \\ & + \frac{2}{F\bar F}\bar\Delta_{\dot\alpha}\Delta_\alpha \ln F.(1-3i\mathcal{H})_m^{\ n} v_n{}^{\alpha\dot\alpha} - 4\Delta_\alpha \ln F.\bar\Delta_{\dot\alpha}\Big(\frac{v_m{}^{\alpha\dot\alpha}}{F\bar F}\Big) + \\ & + \frac{4}{F\bar F}\bar\Delta_{\dot\alpha}\ln\bar F\Big(\Delta_\alpha \ln F.\delta_m^n + i(1+i\mathcal{H})^{-1\,k}_{m}\Delta_\alpha\mathcal{H}_k^{\ n}\Big)v_n{}^{\alpha\dot\alpha}\Big\} + h.c. \end{aligned} \qquad (3.15)$$

Now, recall the expression for $G_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha} G_a$ derived in Ref. /6/:

$$\begin{aligned}G_{\alpha\dot\alpha} = 2\Big(& F\Delta^\beta\bar\omega_{\dot\alpha\alpha\beta} + 4\bar\omega_{\dot\alpha\alpha}{}^\beta\Delta_\beta F + \frac{\bar F}{F}\omega_{\alpha\dot\alpha\dot\beta}\bar\Delta^{\dot\beta}F + \\ & + \omega^\beta{}_{\beta\dot\alpha}\bar\omega^{\dot\beta}{}_{\dot\beta\alpha} + 2\bar F\bar\Delta_{\dot\alpha}\Delta_\alpha F + \bar F\Delta_\alpha\bar\Delta_{\dot\alpha}F + \Delta_\alpha\bar F.\bar\Delta_{\dot\alpha}F + \frac{\bar F}{F}\Delta_\alpha F.\bar\Delta_{\dot\alpha}\bar F\Big),\end{aligned} \qquad (3.16)$$

where $\omega_{\alpha\dot\beta\dot\gamma}$, $\bar\omega_{\dot\alpha\beta\gamma}$ are the spinor connections/4/. It costs certain algebraic efforts to show directly that Eq.(3.15) is the same as Eq.(3.16). Instead of doing this, we shall use normal-gauge arguments once again. In normal gauge the z_0 -value of Γ_a (3.15) is found as follows. One has/5/

$$\begin{aligned}& v_a{}^m|_0 = \delta_a^m, \quad F|_0 = \bar F|_0 = 1, \quad \Delta_\alpha F|_0 = 0, \ etc.; \\ & \Delta_\alpha \mathcal{H}_k^{\ n}|_0 = 0, \quad \Delta_\alpha v_n{}^a|_0 = -\left(v_n{}^b \Delta_\alpha v_b{}^m v_m{}^a\right)_0 = 0.\end{aligned} \qquad (3.17)$$

We are interested in the fourth-order spinor derivatives of $\mathcal{H}^m$ only, so Γ_a reduces to

$$\Gamma_a|_0 = \frac{1}{12}\delta_a^n\left[-\bar\partial_{\dot\alpha}\partial_\alpha\left(\frac{v_n{}^{\alpha\dot\alpha}}{F\bar F}\right) - 2\bar\partial_{\dot\alpha}\partial_\alpha F.\sigma_n{}^{\alpha\dot\alpha}\right]_0 + h.c. \qquad (3.18)$$

Finally, investigating the definitions of $v_n{}^{\alpha\dot\alpha}$ (2.6) and F, $\bar F$ (2.8) one easily finds (see Eq.(3.2))

$$\Gamma_a|_0 = \frac{1}{36}\partial^\gamma\partial_\gamma\bar\partial_{\dot\gamma}\bar\partial^{\dot\gamma}\mathcal{H}_a|_0 = -\frac{1}{12}G_a|_0 \quad . \qquad (3.19)$$

Hence, in an arbitrary gauge,

$$\Gamma_a = -\frac{1}{12}G_a \quad . \qquad (3.20)$$

The linearized form of G_a is obtained by putting $\mathcal{H}^m = \theta\sigma^m\bar\theta + \varkappa h^m$ and keeping the terms linear in $\varkappa$ only. It reads*) ($\mathcal{D}^0$ are the flat-superspace derivatives)

$$\begin{aligned}G_a^{lin} \sim \varkappa(\sigma_a)^{\alpha\dot\alpha}(\sigma_b)^{\beta\dot\beta}\{ & [\mathcal{D}^0_\alpha, \bar{\mathcal{D}}^0_{\dot\alpha}][\mathcal{D}^0_\beta, \bar{\mathcal{D}}^0_{\dot\beta}] + \\ & + 3[\mathcal{D}^0_\beta, \bar{\mathcal{D}}^0_{\dot\beta}][\mathcal{D}^0_\alpha, \bar{\mathcal{D}}^0_{\dot\alpha}]\}h^b \quad .\end{aligned} \qquad (3.21)$$

Concluding this section we stress that by varying action (1.1) with respect to the axial superfield $\mathcal{H}^m$ we are able to derive a single axial equation (2.15) and nothing more. In particular, the equations/1/

$$R = 0 \ , \quad \bar R = 0 \qquad (3.22)$$

do not follow from the action principle. Here R and $\bar R$ are some other components of the torsion tensor. We shall return to this point in Section 5.

4. Matter Couplings in Supergravity

The action for supergravity in the presence of matter has the form

*) Note that this result was first obtained back in 1976/7/.

$$S' = S_{SG} + \int d^4x\, d^4\theta .\, E .\, \mathcal{L}_M(\varphi, \mathcal{D}\varphi) . \tag{4.1}$$

Here S_{SG} is the pure-supergravity action (1.1) and $\mathcal{L}_M$ is a matter Lagrangian with derivatives replaced by covariant ones. In order to obtain equations of motion one has to vary both $\mathcal{H}^m$ and the matter superfield(s) φ. Here a peculiar situation occurs. Let φ be a scalar superfield. Its transformation law is

$$\begin{aligned} \delta\varphi = -\Big[\tfrac{1}{2}\big(\lambda^n(x+i\mathcal{H},\theta) + \bar\lambda^n(x-i\mathcal{H},\bar\theta)\big)\partial_n + \\ +\lambda^\nu(x+i\mathcal{H},\theta)\partial_\nu + \bar\lambda^{\dot\nu}(x-i\mathcal{H},\bar\theta)\bar\partial_{\dot\nu}\Big]\varphi . \end{aligned} \tag{4.2}$$

These are the familiar translation terms of Eq.(2.2). However, the right-hand side of Eq.(4.2) depends on φ and $\mathcal{H}^m$. Therefore, both $\tilde\delta\varphi$ and $\tilde\delta\mathcal{H}^m$ appear in the transformation law of $\tilde\delta\varphi$:

$$\begin{aligned} \delta(\tilde\delta\varphi) = -i\,\tilde\delta\mathcal{H}^m\Big[\tfrac{1}{2}(\partial^L_m\lambda^n - \partial^R_m\bar\lambda^n)\partial_n\varphi + \partial^L_m\lambda^\nu\partial_\nu\varphi - \\ -\partial^R_m\bar\lambda^{\dot\nu}\bar\partial_{\dot\nu}\varphi\Big] - \Big[\tfrac{1}{2}(\lambda^n+\bar\lambda^n)\partial_n + \lambda^\nu\partial_\nu + \bar\lambda^{\dot\nu}\bar\partial_{\dot\nu}\Big]\tilde\delta\varphi . \end{aligned} \tag{4.3}$$

So, $\tilde\delta\varphi$ does not transform like φ itself (Eq.(4.2). Hence, a group transformation will mix the equations for $\mathcal{H}^m$ and φ, they will not be covariant separately. To avoid this (apparent) noncovariance one has to introduce a new, covariant variation of φ. Its form for a scalar φ is

$$\begin{aligned} \tilde\delta_c\varphi = \tilde\delta\varphi + i\,\tilde\delta\mathcal{H}^n\, \mathcal{V}_n^{\ a}\Big[\tfrac{1}{2}(\ell_a^{\ m} - r_a^{\ m})\partial_m\varphi + \\ + \ell_a^{\ \mu}\partial_\mu\varphi - r_a^{\ \dot\mu}\bar\partial_{\dot\mu}\varphi\Big] \end{aligned} \tag{4.4}$$

where ℓ and r are the left and right supervierbeins defined in Ref./4/. One can verify that the new variation transforms properly, i.e.,

$$\delta(\tilde\delta_c\varphi) = -\Big[\tfrac{1}{2}(\lambda^n+\bar\lambda^n)\partial_n + \lambda^\nu\partial_\nu + \bar\lambda^{\dot\nu}\bar\partial_{\dot\nu}\Big]\tilde\delta_c\varphi . \tag{4.5}$$

The covariance of the equations derived from Eq.(4.1) is thus ensured. Let us write down the equation obtained by varying $\mathcal{H}^m$:

$$G_a = \varkappa^2 V_a . \tag{4.6}$$

The result of the variation of the matter term in Eq.(4.1) with respect to $\mathcal{H}^m$ is denoted by V_a and is called "supercurrent". Recall the algebraic property of $G_{\alpha\dot\alpha}$ /2,6/

$$\mathcal{D}^\alpha G_{\alpha\dot\alpha} = -\bar{\mathcal{D}}_{\dot\alpha}\bar R . \tag{4.7}$$

Here $\mathcal{D}^\alpha$, $\bar{\mathcal{D}}_{\dot\alpha}$ are covariant derivatives and $\bar R$ is some chiral scalar superfield $(\mathcal{D}_\alpha\bar R = 0)$. Then Eq.(4.6) implies

$$\mathcal{D}^\alpha V_{\alpha\dot\alpha} = -\frac{1}{\varkappa^2}\bar{\mathcal{D}}_{\dot\alpha}\bar R . \tag{4.8}$$

This is just the covariant form of the flat-superspace conservation law for the supercurrent proposed by Ferrara and Zumino/8/.

We would like to stress that equation (4.6) with conservation law (4.7) strongly resemble the familiar Einstein equation

$$R_{mn} - \tfrac{1}{2}g_{mn}R = \varkappa^2 T_{mn} , \tag{4.9}$$

where the energy-momentum tensor T_{mn} obeys the covariant conservation law

$$\nabla^m T_{mn} = 0 . \tag{4.10}$$

The close relationship between the energy-momentum tensor and the supercurrent (the latter is the supersymmetric generalization of the former) stimulated us to propose the form (4.6) of the supergravity equation of motion in 1976/7/.

5. Cosmological Term

Let us now try to elucidate an important question. In the case of pure supergravity we obtained the equation

$$G_{\alpha\dot\alpha} = 0 . \tag{5.1}$$

According to Eq.(4.7) this implies

$$\bar{\mathcal{D}}_{\dot\alpha}\bar R = 0 . \tag{5.2}$$

Further, $\bar{R}$ is a component of the torsion, which is chiral (see, e.g., Ref./5/)

$$\bar{\mathcal{D}}_{\dot{\alpha}} \bar{R} = 0 . \tag{5.3}$$

Eqs.(5.2) and (5.3) give

$$\bar{R} = const = \lambda . \tag{5.4}$$

Stress that λ is an arbitrary constant. It is not necessarily zero, as claimed in Ref./1/. What is its meaning?

To answer this question we have to turn to the component-field analysis. First, according to Eq.(5.4) all the components of the superfield $\bar{R}$ but the θ^0 -one vanish. The latter takes the constant value λ and this turns out to mean

$$S(x) = \lambda_1, \quad P(x) = \lambda_2 . \tag{5.5}$$

Here S and P are the scalar and pseudoscalar auxiliary fields introduced in the component-field formalism/9/; λ_1 and λ_2 are the real and imagianry parts of λ (Eq.(5.4)). Now, recall that the component-field action has the form/9/

$$S_{SG} = \int d^4x \; e \; \mathcal{L}_c + \int d^4x . e \left(S^2 + P^2 + A_m^2 \right), \tag{5.6}$$

where $\mathcal{L}_c$ is the Lagrangian for the physical fields e_m^a, ψ_m^α; S, P and A_m are the auxiliary fields; $e = \det \| e_m^a \|$. The variation of, say, S, produces the equation

$$S = 0 \tag{5.7}$$

instead of Eq.(5.5). What is the reason for this discrepancy?

The answer is related to the field content of the axial superfield $\mathcal{H}^m$. In the decomposition of $\mathcal{H}^m$ there is a vector S'^m and an axial P^m. As shown in Ref./3/, the fields S and P can be obtained by a noncanonical change of variables, e.g.,

$$S = e^{-1} \partial_m S'^m + \frac{i\varkappa}{8} \psi^m \sigma_{mn} \psi^n . \tag{5.8}$$

Now, putting Eq.(5.8) into Eq.(5.6) one finds the following term in the action

$$\int d^4x \, e S^2 \Rightarrow \int d^4x \, e \left(e^{-1} \partial_m S'^m + \frac{i\varkappa}{8} \psi^m \sigma_{mn} \psi^n \right)^2 \tag{5.9}$$

Varying this term with respect to S'^m (but not S !) one gets the equation

$$\partial_m \left(e^{-1} \partial_n S'^n + \frac{i\varkappa}{8} \psi^k \sigma_{kn} \psi^n \right) = 0 , \tag{5.10}$$

i.e., just Eq.(5.5), $S = \lambda_1$. Further, the variation of ψ_m^α and e_m^a with Eq.(5.5) taken into account gives the terms

$$\delta \psi_m^\alpha : \frac{i\varkappa\lambda_1}{2} e \left(\sigma^{mn} \psi_n \right)_\alpha ,$$

$$\delta e_m^a : e \left[e_a^m \left(-\lambda_1^2 + \frac{i\varkappa\lambda_1}{4} \psi^n \sigma_{nk} \psi^k \right) - \frac{i\varkappa\lambda_1}{2} \psi_a \sigma^{mn} \psi_n \right] . \tag{5.11}$$

These are just the mass term in Rarita-Schwinger equation and the <u>cosmological</u> term in Einstein equation.

Finally, note that the same terms (5.11) in the equations of motion are reproduced by the <u>cosmological</u> action term proposed in Ref./10/.

$$\begin{aligned} \int d^4x \; e \left[S^2 - 2\lambda S + \frac{i\varkappa\lambda}{4} \psi^m \sigma_{mn} \psi^n \right] &= \\ = \int d^4x \left(-2\lambda \partial_m S'^m \right) , \end{aligned} \tag{5.12}$$

i.e., it is a total derivative in our notation. This fact has a superfield expression, too. It turns out that the superspace cosmological action terms (proposed in Ref./11/)

$$\int d^4x \, d^4\theta \, \frac{E}{R} , \quad \int d^4x \, d^4\theta \, \frac{E}{\bar{R}} \tag{5.13}$$

do not give any contribution when varied with respect to $\mathcal{H}^m$.

We would like to point out the occurrence of the topological charge (5.12) related to the cosmological superfield invariant (5.13). It shows that the behaviour of the gravitational superfield $\mathcal{H}^m$ at large x has something to do with the cosmological term. In our opinion, this question is worthy of more detailed examination.

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Appendix A

To establish the transformation properties of $\mathcal{V}_a^m$ (2.6) we need some information from Ref. /4/. The derivatives $\Delta_\alpha, \bar\Delta_{\dot\alpha}$ (2.7) of a scalar superfield φ transform as follows

$$\delta(\Delta_\alpha\varphi) = -\Delta_\alpha\lambda^\beta\Delta_\beta\varphi + t.t. , \qquad (A.1)$$

$$\delta(\bar\Delta_{\dot\alpha}\varphi) = -\bar\Delta_{\dot\alpha}\bar\lambda_{\dot\beta}\bar\Delta^{\dot\beta}\varphi + t.t. .$$

The quantities $F, \bar F$ (2.8) have transformation laws

$$\delta F = -\frac{1}{2}(\Delta^\alpha\lambda_\alpha)F + t.t. , \quad \delta\bar F = -\frac{1}{2}(\bar\Delta_{\dot\alpha}\bar\lambda^{\dot\alpha})\bar F + t.t. . \qquad (A.2)$$

The parameters $\lambda, \bar\lambda$ are chiral

$$\bar\Delta_{\dot\alpha}\lambda = 0, \quad \Delta_\alpha\bar\lambda = 0 , \qquad (A.3)$$

hence (see Eqs.(3.11), (2.5)),

$$\bar\Delta_{\dot\alpha}\Delta_\alpha\lambda = \{\Delta_\alpha, \bar\Delta_{\dot\alpha}\}\lambda = -i[\Delta_\alpha, \bar\Delta_{\dot\alpha}]\mathcal{H}^m\partial_m\lambda = \qquad (A.4)$$

$$= -i[\Delta_\alpha, \bar\Delta_{\dot\alpha}]\mathcal{H}^m(1+i\mathcal{H})_m^n\partial_n^L\lambda .$$

Now, let us consider the different parts of $\mathcal{V}_a^m$ (2.6). First (see Eq.(2.2)),

$$\delta(\partial_n\mathcal{H}^m) = \partial_n(-\frac{i}{2}\lambda^m - \frac{1}{2}\lambda^k\partial_k\mathcal{H}^m - \lambda^\varkappa\partial_\varkappa\mathcal{H}^m) + h.c. =$$

$$= -\frac{i}{2}\partial_n\lambda^m - \frac{1}{2}\partial_n\lambda^k\partial_k\mathcal{H}^m - \partial_n\lambda^\varkappa\partial_\varkappa\mathcal{H}^m + t.t. + h.c. = \qquad (A.5)$$

$$= (1+i\mathcal{H})_n^z(-\frac{i}{2}\partial_z^L\lambda^m - \frac{1}{2}\partial_z^L\lambda^k\partial_k\mathcal{H}^m - \partial_z^L\lambda^\varkappa\partial_\varkappa\mathcal{H}^m) + t.t. + h.c.$$

Further (see Eqs.(A.1), (A.3), (A.4)),

$$\delta([\Delta_\alpha, \bar\Delta_{\dot\alpha}]\mathcal{H}^m) = -\Delta_\alpha\lambda^\beta\cdot\Delta_\beta\bar\Delta_{\dot\alpha}\mathcal{H}^m + \bar\Delta_{\dot\alpha}(\Delta_\alpha\lambda^\beta\Delta_\beta\mathcal{H}^m) +$$

$$+ \frac{i}{2}\bar\Delta_{\dot\alpha}\Delta_\alpha\lambda^m + t.t. + h.c. = -\Delta_\alpha\lambda^\beta[\Delta_\beta, \bar\Delta_{\dot\alpha}]\mathcal{H}^m -$$

$$-i[\Delta_\alpha, \bar\Delta_{\dot\alpha}]\mathcal{H}^n(1+i\mathcal{H})_n^k(\partial_k^L\lambda^\beta\Delta_\beta\mathcal{H}^m + \frac{i}{2}\partial_k^L\lambda^m) + t.t. + h.c. \qquad (A.6)$$

Combining Eqs.(A.5) and (A.6), and using the identity

$$\Delta_\alpha\mathcal{H}^m(1-i\mathcal{H})_m^n = \partial_\alpha\mathcal{H}^n$$

(see Eq.(2.7)) one gets

$$\delta([\Delta_\alpha, \bar\Delta_{\dot\alpha}]\mathcal{H}^n(1+\mathcal{H}^2)_n^m) = -\Delta_\alpha\lambda^\beta[\Delta_\beta, \bar\Delta_{\dot\alpha}]\mathcal{H}^n(1+\mathcal{H}^2)_n^m -$$

$$-i[\Delta_\alpha, \bar\Delta_{\dot\alpha}]\mathcal{H}^n(1+\mathcal{H}^2)_n^k[\partial_k^L\lambda^\beta\partial_\beta\mathcal{H}^m + \frac{i}{2}\partial_k^L\lambda^\ell(1-i\mathcal{H})_\ell^m] + t.t. + h.c. .$$

Finally, with the help of Eq.(A.2) we find

$$\delta\mathcal{V}_a^m = \frac{1}{4}(\tilde\sigma_a)^{\dot\alpha\alpha}\delta(F\bar F[\Delta_\alpha, \bar\Delta_{\dot\alpha}]\mathcal{H}^n(1+\mathcal{H}^2)_n^m) =$$

$$= -\frac{i}{2}\Delta^\alpha(\sigma_a{}^b)_\alpha{}^\beta\lambda_\beta\mathcal{V}_b^m +$$

$$+ \mathcal{V}_a^n[-i\partial_n^L\lambda^\beta\partial_\beta\mathcal{H}^m + \frac{1}{2}\partial_n^L\lambda^k(1-i\mathcal{H})_k^m] + t.t. + h.c.$$

The transformation law for $\mathcal{V}_m^a$, obtained from the definition $\mathcal{V}_m^a\mathcal{V}_a^n = \delta_m^n$, and Eqs.(2.4), (2.10) lead to Eq.(2.11).

Appendix B

To prove Eq.(3.14) consider first the integral

$$I = \int d^4x\, d^4\theta\, E[\Delta^\alpha(F\Psi_\alpha) + 2\Delta^\alpha F\cdot\Psi_\alpha] . \qquad (B.1)$$

Integrating the first term by parts and taking into account Eq.(2.7) one gets

$$I = \int d^4x\, d^4\theta\left(\Delta^\alpha \ln\frac{F^2}{E} - i\partial_m\Delta^\alpha\mathcal{H}^m\right)E\cdot F\cdot\Psi_\alpha . \qquad (B.2)$$

According to Eqs.(2.8), (3.5) and (3.12)

$$\frac{E}{F^2} = \det(1-i\mathcal{H}) .$$

Putting this into Eq.(B.2) and using Eq.(3.9) one finds

$$I = 0 .$$

This result easily leads to Eq.(3.14).

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