объединенный
институт
ядерных
исследований
дубна

E2-80-123

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ON CONSTRUCTION OF INDEPENDENT LORENTZ-COVARIANTS FOR THE REACTIONS WITH THREE PARTICLES IN THE FINAL STATE

Submitted to **"Acta Physica Slovaca"**

1980

1. INTRODUCTION

The well-known prescription[1,2] in the M-function technique is to express the T-matrix as a linear combination of independent Lorentz covariants (ILC) contracted with the wave functions of external particles. Frequently, however, one knows the type of the interaction involved and using this input information can, therefore, analyze the structure of the T-matrix in more detail. It then may appear convenient to consider T as a product of two tensors which are individually expanded in terms of ILC and certain invariant functions ("form factors"). For the binary reaction the construction of ILC should be standard by now. The elimination of the dependent covariants becomes, however, technically difficult if the reactions with several final-state particles are considered. To our knowledge it has not been analysed as yet and, moreover, such amplitude expansions over heavily overcomplete sets of Lorentz covariants may be traced in a very recent literature.

In this note we address ourselves to the reactions with three particles in the final state. The traditional construction goes in the Cartesian basis. To eliminate the dependent covariants one indeed wishes to know first the number of ILC which span the needed set. In Sec. 2 we formulate a result obtained earlier[3] for the two-variable tensor spherical harmonics, which allow us to establish the number of ILC for the tensors considered. Then in Sec.3 we apply the well known[2] properties of the fully antisymmetric unit tensor $\epsilon_{\alpha\beta\gamma\delta}$ in the four-dimensional space to obtain several nontrivial coupling equations which allow us to eliminate all dependent Cartesian covariants for the case of reactions with spin-1/2 particles. The formulation though made for a particular reaction contains all ingredients necessary to analyze the general case of $1+2\to a+b+c$ processes. It should be noted that we do not consider the problem of the so-called kinematical singularities, which has to be studied independently in any case.

2. THE NUMBER OF INDEPENDENT COVARIANTS

The construction of multi-particle Cartesian covariants is described, e.g., by Hearn[4]. He considers the combinations of the so-called L-functions and C-bases. The L-functions are irreducible combinations of the Dirac gamma matrices (polarization vectors) for the case of fermion (boson) particles. The C-bases are combinations of the 4-momenta available in the process (tensor bases) and combinations of the 4-momenta with the antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$ (pseudotensor bases).

To give an example let us think of the radiative muon capture reaction on proton

$$\mu^-(p^{(\mu)})+p(p_i,j_i)\to\gamma(k)+\nu(p^{(\nu)})+n(p_f,j_f). \tag{1}$$

Assuming the effective interaction for the corresponding non-radiative process in the current×current form, we shall consider the individual pieces of the amplitude separately as linear combinations of the appropriate ILC. Then the complete reaction amplitude (disregarding the "radiation from the lepton line") can be written in the form

$$T(p^{(\mu)},p^{(\nu)},k,p_i,p_f;j_i,j_f)=\epsilon_\mu S_{\mu\lambda}(p_i,p_f,k;j_i,j_f)\cdot j_\lambda(p^{(\mu)},p^{(\nu)}), \tag{2}$$

where ϵ_μ is the photon polarization and j_λ is the leptonic weak charged current. To construct the hadronic tensor $S_{\mu\lambda}$ we have at our disposal the momenta $Q_\mu=(p_f+p_i)_\mu$, $q_\mu=(p_f-p_i)_\mu$, and k_μ, and similarly as in Ref.4 we write

$$\begin{aligned}&\text{the tensor basis: } \delta_{\mu\nu},\ Q_\mu,\ q_\mu,\ k_\mu,\ a_\mu b_\nu (a,b=Q,q,k),\ \ldots,\\ &\text{and L-functions: } 1,\ \gamma_\mu,\ \gamma_\mu\gamma_\nu,\ \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4,\ \gamma_5\gamma_\mu,\ \gamma_5\gamma_\mu\gamma_\nu,\end{aligned} \tag{3}$$

where γ_μ are the usual Dirac matrices. Using the components (3) and after all usual simplifications (apply the Dirac equation,...) we obtain 68 seemingly independent[5] covariants. Here one has to admit that there is apparently no possibility to establish the actual number of independent covariants if the construction is performed in the Cartesian bases as in Ref.[4].

It is the formalism of two-variable relativistic tensor harmonics in the spherical or helicity basis, which should be utilized here. Being fully equivalent to the Cartesian forms applied above it fortunately provides enormous technical simplifications. As an example of its use we establish here the number of ILC which span the complete set to be used for expansion of $S_{\mu\lambda}$.

It was shown in Ref.3 that starting with two 4-vectors, e.g., the vectors k_μ and q_μ introduced above, one can construct only $2\ell+1$ independent two-variable scalar spherical harmonics

$$\{(\ell_1\ell_2)\ell m\}=\sum_{m_1m_2}\begin{bmatrix}\ell_1 & \ell_2 & \ell\\ m_1 & m_2 & m\end{bmatrix} Y^{\ell_1}_{m_1}(\hat{k})\,Y^{\ell_2}_{m_2}(\hat{q}), \tag{4}$$

where $Y^{\ell_i}_{m_i}$ are the usual spherical harmonics and $[\because]$ stands for the Clebsch-Gordan coefficient, $\ell_1+\ell_2=\ell$ or $\ell+1$. All other harmonics $\{(\ell_1'\ell_2')\ell m\}$ can be expressed through the harmonics (4) as their linear combinations with scalar coefficients. It is then easy to see[3] that the number of independent tensor harmonics of an arbitrary order s is

$$N^{(s)}_j=\sum_r n^{(s)}_r\sum_{\ell=|j-r|}^{j+r}(2\ell+1)=\sum_r n^{(s)}_r(2r+1)(2j+1), \tag{5}$$

where $n^{(s)}_r$ is the statistical weight of the corresponding basis tensor $e_{\mu_1}, e_{\mu_2}, \ldots e_{\mu_s}$ connected with the vector Q, and j is the total angular momentum of the harmonics. In the case of 2nd order (s=2) spherical tensor harmonics we have[3] $n^{(2)}_0=2$, $n^{(2)}_1=3$, and $n^{(2)}_2=1$, therefore

$$N^{(2)}_j=16(2j+1) \tag{6}$$

$j=|j_f-j_i|, |j_f-j_i|+1, \ldots, j_f+j_i$. In particular, for the above example of reaction (1), where $j_f=j_i=1/2$, only $N^{(2)}_0+N^{(2)}_1=64$ independent Lorentz covariants exist. Four coupling equations needed shall be obtained in the next Section.

3. COUPLING EQUATIONS

Consider the form

$$U(a_\mu,b_\nu)=\gamma_5 a_\mu b_\nu \epsilon_{\alpha\beta\gamma\delta}\gamma_\alpha k_\beta p^{(f)}_\gamma p^{(i)}_\delta$$

and its contractions with the appropriate Dirac spinors:

$$\begin{aligned}&\bar{u}(p^{(f)})\gamma_5 a_\rho b_\eta \delta_{\rho\mu}\delta_{\eta\nu}\epsilon_{\alpha\beta\gamma\delta}\gamma_\alpha k_\beta p^{(f)}_\gamma p^{(i)}_\delta u(p^{(i)})=\\ &=\bar{u}(p^{(f)})U(a_\mu,b_\nu)u(p^{(i)}).\end{aligned} \tag{7}$$

The LHS of (7) may be expanded using subsequently twice the identity[2]

$$\epsilon_{\alpha\beta\gamma\delta}\,\delta_{\rho\nu} = \epsilon_{\nu\beta\gamma\delta}\,\delta_{\rho\alpha} + \epsilon_{\alpha\nu\gamma\delta}\,\delta_{\rho\beta} + \epsilon_{\alpha\beta\nu\delta}\,\delta_{\rho\gamma} + \epsilon_{\alpha\beta\gamma\nu}\,\delta_{\rho\delta} \tag{8}$$

which follows from antisymmetry of $\epsilon_{\alpha\beta\gamma\delta}$ in the 4-dimensional space. Substituting finally

$$\begin{aligned}\epsilon_{\alpha\beta\gamma\delta} = \tfrac{1}{6}(&\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta - \delta_{\alpha\beta}\gamma_\gamma\gamma_\delta - \delta_{\beta\gamma}\gamma_\alpha\gamma_\delta - \delta_{\gamma\delta}\gamma_\alpha\gamma_\beta \\ &+\delta_{\alpha\gamma}\gamma_\beta\gamma_\delta + \delta_{\beta\delta}\gamma_\alpha\gamma_\gamma - \delta_{\alpha\delta}\gamma_\beta\gamma_\gamma \\ &+\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta})\end{aligned} \tag{9}$$

one obtaines from (7) the needed form.

The three independent 4-vectors pertinent to the problem are called k_μ , $p_\mu^{(f)}$ and $p_\mu^{(i)}$. Any of them can be substituted both for a and for b in (7). In this way we obtain nine expressions which should be inspected individually in order to establish the possible couplings. All manipulations needed on this trail are completely straightforward, they become, however, extremely tedious due to the big number of operations. We doubt that this algebra can efficiently be done manually. To perform it we have used the algebraic manipulation system SCHOONSCHIP/6/ installed at the CDC 6500 computer in Dubna.

Out of nine relations obtained from eq. (7) there are only three which display couplings, namely those constructed with substitutions A1: $a = p^{(i)}, b = k$, A2: $a = p^{(f)}$, $b = k$, and A3: $a = p^{(i)}$, $b = p^{(f)}$. We list them in Appendix. (Three substitutions with a and b interchanged give again precisely the relations (A1)-(A3); three substitutions left $(a = b)$ lead to the trivial identity $0 \equiv 0$ only). As it should be expected according to (6), the identities (A1), (A2) and (A3) are dependent. They are connected via

$$-M\cdot(A1) + m\cdot(A2) + (mk\cdot p^{(f)} - Mk\cdot p^{(i)})(A3) = 0 \tag{10}$$

with

$$(\hat{p}^{(f)} - iM)u(p^{(f)}) = 0, \quad (\hat{p}^{(i)} - im)u(p^{(i)}) = 0.$$

As a result two independent coupling relations, e.g., eqs. (A1) and (A2) follow from the form (7). Other two independent couplings can be derived, if we consider the function

$$V(a_\mu, b_\nu) = a_\mu b_\nu \cdot \epsilon_{\alpha\beta\gamma\delta}\,\gamma_\alpha k_\beta p_\gamma^{(f)} p_\delta^{(i)}$$

instead of $U(a_\mu, b_\nu)$.

4. CONCLUSIONS

We have shown that the sets of Lorentz covariants constructed in Cartesian basis for the tensor decomposition of the amplitude of the reactions with three particles in the final state are overcomplete. Though we think that the existence of nontrivial identities derived may be of interest by itself, we wish also to stress two methodological points related with this work. Firstly, it is obvious that the popular Cartesian techniques become difficult to use in all cases beyond the simplest binary reactions and should be supplemented or even substituted by the covariant techniques which use the relativistic tensor harmonics. Secondly, the algebraic manipulation systems like REDUCE, SCHOONSCHIP, FORMAC, etc., which are by now available at a number of computation facilities certainly deserve to be much more widely exploited than it is customary nowadays since they may produce useful results which would be beyond reach if to stick to the "manual" derivations.

APPENDIX

$$\bar{u}(p^{(f)})A(p^{(i)}, m; p^{(f)}, M)u(p^{(i)}) = 0 \tag{A.1}$$

$$\bar{u}(p^{(f)})A(p^{(f)}, M; p^{(i)}, m)u(p^{(i)}) = 0 \tag{A.2}$$

$$\begin{aligned}A(a, m; b, M) = {}&\gamma_\lambda\gamma_\mu\hat{k}(m^2 k\cdot b + k\cdot a\ b\cdot a) \\ &+ \gamma_\lambda\gamma_\mu i(mk\cdot b\,k\cdot a - mk^2\,b\cdot a - M(k\cdot a)^2 - Mm^2k^2) \\ &+ (\gamma_\lambda k_\mu - \gamma_\mu k_\lambda)\hat{k}\,im(Mm + b\cdot a) + (\gamma_\lambda a_\mu - \gamma_\mu a_\lambda)\hat{k}i(Mk\cdot a - mk\cdot b) \\ &+ (\gamma_\mu k_\lambda - \gamma_\lambda k_\mu)(Mmk\cdot a + k\cdot a\,b\cdot a) + (\gamma_\mu b_\lambda - \gamma_\lambda b_\mu)(m^2k^2 + (k\cdot a)^2) \\ &+ (\gamma_\lambda a_\mu - \gamma_\mu a_\lambda)(Mmk^2 + k\cdot b\,b\cdot a) + \hat{k}(k_\lambda b_\mu - k_\mu b_\lambda)m^2 \\ &+ \hat{k}(k_\mu a_\lambda - k_\lambda a_\mu)Mm + \hat{k}(b_\mu a_\lambda - b_\lambda a_\mu)k\cdot a \\ &+ \hat{k}\delta_{\mu\lambda}(-m^2\,k\cdot b - k\cdot a\,b\cdot a) + (b_\mu k_\lambda - b_\lambda k_\mu)imk\cdot a \\ &+ (k_\mu a_\lambda - k_\lambda a_\mu)iMk\cdot a + (a_\mu b_\lambda - a_\lambda b_\mu)im\,k^2 \\ &+ \delta_{\mu\lambda}i(Mm^2k^2 + M(k\cdot a)^2 + mk^2\,b\cdot a - mk\cdot b\,k\cdot a)\end{aligned}$$

$$
\begin{aligned}
\bar{u}(p^{(f)})\{ & \gamma_\lambda \gamma_\mu \hat{k}(Mm - p^{(i)}\cdot p^{(f)}) + \gamma_\lambda \gamma_\mu i(mk\cdot p^{(f)} + Mk\cdot p^{(i)}) \\
& + (\gamma_\mu p_\lambda^{(f)} - \gamma_\lambda p_\mu^{(f)})\hat{k}\, im + (\gamma_\mu p_\lambda^{(i)} - \gamma_\lambda p_\mu^{(i)})\hat{k}\, iM \\
& + (\gamma_\mu k_\lambda - \gamma_\lambda k_\mu)(Mm - p^{(i)}\cdot p^{(f)}) + (\gamma_\lambda p_\mu^{(f)} - \gamma_\mu p_\lambda^{(f)})k\cdot p^{(i)} \\
& + (\gamma_\mu p_\lambda^{(i)} - \gamma_\lambda p_\mu^{(i)})k\cdot p^{(f)} + \hat{k}(p_\mu^{(i)} p_\lambda^{(f)} - p_\lambda^{(i)} p_\mu^{(f)}) + \hat{k}\delta_{\mu\lambda}(-M^2 m^2 + (p^{(i)} p^{(f)})^2) \\
& + (p_\mu^{(f)} k_\lambda - p_\lambda^{(f)} k_\mu)\, im + (p_\mu^{(i)} k_\lambda - p_\lambda^{(i)} k_\mu)\, iM \qquad \text{(A.3)} \\
& - \delta_{\mu\nu}\, i(mk\cdot p^{(f)} + Mk\cdot p^{(i)})\} \quad u(p^{(i)}) = 0
\end{aligned}
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Received by Publishing Department
on February 15 1980.