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G.M.Sotkov*, R.P.Zaikov

ON THE STRUCTURE OF THE CONFORMAL COVARIANT N-POINT FUNCTIONS

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*Higher Pedagogical Institute, Shoumen, Bulgaria.

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I. INTRODUCTION

As is well known [1], the conformal invariance condition in QFT allows one to determine total two- and three-point functions up to a few constants. Any higher n-point function ($n \geq 4$), generally, depends on some set of arbitrary functions of $n(n-3)/2$ independent harmonical ratios. These functions can be determined from dynamical considerations only [1,2]. For the solution of this dynamical problem, it is convenient to use the partial wave expansion over three-point functions. Such an expansion allows one to write any n-point function in terms of $(n-1)$-point and three-point functions. Thus, one can obtain any n-point function from two- and three-point functions. However, $(n-3)$-intermediate integrations are necessary for this aim. Hitherto this programme is applied effectively only for scalar [4] and tensor [5] fields.

The aim of this paper is to find the general form (without dynamical hypothesis) of conformal covariant n-point functions for the basic fields (from a class Ia of [6]) with arbitrary spin and scale dimensions. The conformal covariant n-point functions for the fields which are transformed according to arbitrary irreducible representations (IR) of SL(2, C) are obtained by the method of homogeneous functions [7] for the realization IR of SL(2, C). This method is presented in our papers [8,9]. In Sec. III the explicit form of some four-point functions is given. The γ_5-invariance condition is discussed.

All our considerations are applied both for the Minkowski space-time and for the Euclidean space-time.

II. CONFORMAL COVARIANT N-POINT FUNCTION

Consider basic fields $\psi_a(x_a; z_a, \chi_a)$ $(a = 1, \ldots, n)$, where z is a complex two-component spinor $z = (z, \bar{z})$ [7] (in the Euclidean case $z = (z, \tilde{w})$ [10,11]), $\chi = \{d, j_1, j_2\}$ labels IR of SU(2, 2) according to which the field $\psi(x; z)$ is transformed. Here $d = d_\psi - j_1 - j_2$, d_ψ is the scale dimension of ψ, j_1 and j_2 label IR of the subgroup SL(2, C) (SU(2) $\otimes$ SU(2) in the Euclidean case).

For finite-dimensional representations $j_1, j_2 = 0, 1/2, 1, \ldots$ and for infinite-dimensional ones j_1, j_2 are generally complex numbers. The irreducibility for the field $\psi(x; z, \chi)$ with respect to the SL(2, C) (SU(2)⊗SU(2)) subgroup takes the form of a homogeneous condition:

$$\psi(x; \lambda z, \bar{\lambda}\bar{z}) = \lambda^{2j_1} \bar{\lambda}^{2j_2} \psi(x; z, \bar{z}) , \qquad (2.1)$$

where λ is a complex number.

Let us consider the n-point function

$$F^{(n)}_{\{\chi_a\}}(x_a; z_a) = \langle 0 | \prod_{b=1}^{n} \psi(x_b; z_b, \chi_b) | 0 \rangle, \quad (n = 2, 3, \dots) . \qquad (2.2)$$

The conformal covariance condition (in infinitesimal form) (with respect to SU(2, 2) or SU*(4)) has the form:

$$\sum_{k=1}^{n} X^k_{AB} F^{(n)}_{\{\chi_a\}}(x_a; z_a) = 0 . \qquad (2.3)$$

where X^k_{AB} ($k = 1, \dots, n$; $A, B = 0, 1, 2, 3, 5, 6$) represent the generators of SU(2, 2) which are realized in the space of the fields $\psi(x, z, \chi)$. The irreducibility condition (2.1) for an n-point function $F^{(n)}$ takes the form:

$$F^{(n)}_{\{\chi_a\}}(x_a; \lambda_a z_a) = \prod_{b=1}^{n} \lambda^{2j_1^b} \bar{\lambda}^{2j_2^b} F^{(n)}_{\{\chi_a\}}(x_a; z_a) . \qquad (2.4)$$

For the finite-dimensional representations of SL(2, C) eq. (2.4) must be added by the condition for polynomiality on z and $\bar{z}$.

If the functions $F^{(n)}$ depend on x_a, z_a and $\bar{z}_a$ by the independent relativistic invariants of these variables, then the relativistic invariance condition (i.e., $X_{AB} = P_\mu, M_{\mu\nu}$) is fulfilled. These invariants are constructed in Appendix A. Note that in this case invariance condition for the function $F^{(n)}_{\{\chi_a\}}(x_a; z_a)$ takes place instead of the covariance condition for $F^{(n)}_{\{\alpha_1\} \dots \{\alpha_n\}}$ /7/ .

It follows from (A.3) that the relativistic invariant n-point function is of the form:

$$F^{(n)}_{\{\chi_a\}}(x_a; z_a) = F^{(n)}_{\{\chi_a\}}(x^2_{jk}, z_a \epsilon z_b, \bar{z}_a \epsilon \bar{z}_b, z_a \underset{\sim}{x}_{jk} \bar{z}_b, \dots) . \qquad (2.5)$$

The irreducibility condition (2.4) will be required in final results

The change

$$x_0 \to x_4 = ix_0 , \quad (z, \bar{z}) \to (z, \bar{w}) ,$$
$$\sigma = (\sigma_0, \underline{\sigma}) \to \tau = (i\sigma_0, \underline{\sigma}) , \qquad (2.6)$$

realized the transition from the Minkowski space-time M_4 to the Euclidean space-time E_4. In M_4 the function $F^{(n)}$ is the Wightman's function ($x^2_{jk} = x^\mu_{jk} x^{jk}_\mu - i\epsilon x^0_{jk}$) and in E_4 - $F^{(n)}$ is the corresponding Schwinger function $S^{(n)}$.

The scale and special conformal invariance conditions (in an infinitesimal form) of the n-point function (2.5) lead to the equations:

$$\sum_{k=1}^{n} D_k F^{(n)}_{\{\chi_a\}}(x_a; z_a) = 0 , \qquad (2.7a)$$

$$\sum_{k=1}^{n} K^k_\mu F^{(n)}_{\{\chi_a\}}(x_a; z_a) = 0 , \qquad (2.7b)$$

where

$$D = i x^\mu \partial_\mu + \Delta ,$$

$$K_\mu = -i\{2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu - 2i x^\nu [g_{\mu\nu} \Delta + \Sigma_{\mu\nu}]\} ,$$

$$\Delta = i[d + \frac{1}{2}(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}})] , \qquad (2.8)$$

$$\Sigma_{\mu\nu} = \frac{1}{4}(z, \frac{\partial}{\partial \bar{z}})[\gamma_\mu, \gamma_\nu] \begin{pmatrix} \frac{\partial}{\partial z} \\ \bar{z} \end{pmatrix} ,$$

and γ_μ are the Dirac matrices in a representation where γ_5 is diagonal.

If we extend the conformal transformations with the γ_5 - transformation eqs. (2.7) take place instead of the condition

$$\sum_{k=1}^{n} \pi^{(k)} F^{(n)}_{\{\chi_a\}}(x_a; z_a) = 0 , \qquad (2.9)$$

where the generator of γ_5-transformation has the form:

$$\pi = i[\,\kappa + \frac{1}{2}(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})\,]. \qquad (2.10)$$

Here κ is the γ_5-dimension of the field $\psi(x;z)$.

In Appendix B the general solution of eqs. (2.7) is obtained:

$$F^{(n)}_{\{\chi_a\}}(x_a;z_a) = \prod_{j<k=1}^{n} (x^2_{jk})^{\delta_{jk}} \sum_{\{\ell,p,q,\ldots\}} \phi^{\{\ell,p,q,\ldots\}}(h_1,h_2,\ldots,h_{n(n-3)/2}) \times$$

$$\times \prod_{j\neq k=1}^{n} \left(\frac{z_j \underset{\sim}{x}_{jk} \bar{z}_k}{x^2_{jk}}\right)^{\ell_{jk}} \prod_{\substack{p,q=1\\ q\neq p,\,p+1}}^{n} \left(\frac{z_p \underset{\sim}{x}_{p,p+1} \bar{z}_p}{x^2_{p,p+1}} - \frac{z_p \underset{\sim}{x}_{p,q} \bar{z}_p}{x^2_{pq}}\right)^{p_{pq}} \times$$

$$\times \prod_{\{j,k,\ell\}=1}^{n} \left(\frac{z_j \underset{\sim}{x}_{jk} \tilde{x}_{k\ell}\, \epsilon\, z_\ell}{(x^2_{jk}x^2_{k\ell}x^2_{j\ell})^{1/2}}\right)^{q^k_{j\ell}} \prod_{\{j,k,\ell\}=1} \left(\frac{\bar{z}_j\, \epsilon\, \tilde{x}_{jk} \underset{\sim}{x}_{k\ell} \bar{z}_\ell}{(x^2_{jk}x^2_{k\ell}x^2_{j\ell})^{1/2}}\right)^{\bar{q}^k_{j\ell}} \times$$

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$$\times \prod_{\{k_1\ldots k_n\}=1}^{n} \left(\frac{z_{k_1} \underset{\sim}{x}_{k_1k_2} \tilde{x}_{k_2k_3} \cdots \underset{\sim}{x}_{k_{n-1}k_n} \bar{z}_{k_n}}{(x^2_{k_1k_2}x^2_{k_2k_3}\cdots x^2_{k_{n-1}k_n}x^2_{k_1k_n})^{1/2}}\right)^{r^{k_2\ldots k_{n-2}}_{k_1k_n}} \qquad (2.11)$$

Here

$$\delta_{jk} = \frac{1}{(n-1)(n-2)}\left[\sum_{\ell=1}^{n} d_\ell - (n-1)(d_j + d_k)\right], \qquad (2.12)$$

$\phi^{\{\ell,p,q\ldots\}}_{\{\chi\}}$ are arbitrary functions of the independent harmonical ratios. The spectrality, locality, and positivity conditions required some restriction on these functions. The last factor in (2.11) has the form $z_{k_1}\underset{\sim}{x}_{k_1k_2}\cdots\underset{\sim}{x}_{k_{n-1}k_n}\bar{z}_{k_n}$ at even number n and $z_{k_1}\underset{\sim}{x}_{k_1k_2}\cdots\tilde{x}_{k_{n-1}k_2}\,\epsilon\, z_{k_n}$ at odd n. The limits in the sums in (2.11) can be obtained from the irreducibility (2.4) and polynomiality conditions, which give

$$\sum_{\substack{k=1\\k\neq j}}^{n} \ell_{jk} + \sum_{\substack{m=1\\m\neq j,\,j+1}}^{n} p_{jm} + \sum_{k,\ell=1}^{n} q^k_{j\ell} + \ldots = 2j^j_1,$$

$$\sum_{\substack{j=1\\j\neq k}} \ell_{jk} + \sum_{\substack{m=1\\m\neq k,\,k+1}}^{n} p_{km} + \sum_{m,\ell=1}^{n} \bar{q}^m_{k\ell} + \ldots = 2j^k_2. \qquad (2.13)$$

$$(j,k = 1,\ldots,n).$$

Here $j+k \xrightarrow{j+k>n} j+k-n$. Hence, there exist 2n dependences between the powers in (2.11).

In the case when we require the γ_5-invariance in addition, it follows from (2.9,10) that:

$$\sum_{k=1}^{n} (j^k_1 - j^k_2) = \sum_{k=1}^{n} \kappa^k. \qquad (2.14)$$

Thus the invariant n-point function exists only for those representations of SL(2, C) for which eq. (2.14) is satisfied.

The corresponding n-point function in terms of spinor indices can be obtained from (2.11) by taking the $2j^k_1$-th and $2j^k_2$-th ($k = 1,\ldots,n$) derivatives with respect to z^k and $\bar{z}^k$, respectively, and performing a symmetrization.

One can verify that the function (2.11) is a totally conformal invariant.

III. COVARIANT FOUR-POINT FUNCTIONS

The known two- and three-point functions[9] were obtained from (2.11) when $n = 2, 3$. Consider in detail the four-point function. In this case only three of x_{jk} are independent and hence eq. (2.11) reduces to the term

$$\frac{z_j \underset{\sim}{x}_{jk} \tilde{x}_{k\ell} \underset{\sim}{x}_{\ell m} \bar{z}_m}{(x^2_{jk}x^2_{k\ell}x^2_{\ell m}x^2_{jm})^{1/2}}. \qquad (3.1)$$

Taking into account (A.5), we obtain that there are only twelve independent invariants of kind (3.1). Thus, it follows from (2.11) when $n = 4$ that:

$$F^{(4)}_{\{\chi_a\}}(x_a;z_a) = \prod_{j<k=1}^{4} (x^2_{jk})^{\frac{1}{6}[\sum_{\ell=1}^{4} d_\ell - 3d_j - 3d_k]} \sum_{\{\ell,q,q,\bar{q},r\}} \phi^{\{\ell,\ldots,r\}}(h_1,h_2) \times$$

$$\times \prod_{\substack{j,k=1\\j\neq k}}^{4} \left(\frac{z_j \underset{\sim}{x}_{jk} \bar{z}_k}{x^2_{jk}}\right)^{\ell_{jk}} \prod_{\substack{j,m=1\\m\neq j,\,j+1}} \left(\frac{z_j \underset{\sim}{x}_{j,j+1} \bar{z}_j}{x^2_{j,j+1}} - \frac{z_j \underset{\sim}{x}_{jm} \bar{z}_m}{x^2_{jm}}\right)^{p_{jm}} \times$$

$$\times \prod_{j,k,\ell=1}^{4} \left(\frac{z_j \underline{x}_{jk} \tilde{\underline{x}}_{k\ell} \epsilon z_\ell}{(x_{jk}^2 x_{k\ell}^2 x_{j\ell}^2)^{1/2}} \right)^{q_{j\ell}^{k}} \prod_{j,k,\ell=1}^{4} \left(\frac{\bar{z}_j \epsilon \bar{x}_{jk} \underline{x}_{k\ell} \bar{z}_\ell}{(x_{jk}^2 x_{k\ell}^2 x_{j\ell}^2)^{1/4}} \right)^{\bar{q}_{j\ell}^{k}} \times \quad (3.2)$$

$$\times \prod_{j,k,\ell,m=1}^{4} \left(\frac{z_j \underline{x}_{jk} \tilde{x}_{k\ell} \underline{x}_{\ell m} \bar{z}_m}{(x_{jk}^2 x_{k\ell}^2 x_{\ell m}^2 x_{jm}^2)^{1/4}} \right)^{r_{jm}^{k\ell}},$$

where $\phi_{\{\chi_a\}}^{\{\ell, p, q, \bar{q}, r\}}(h_1, h_2)$ are arbitrary function of the harmonical rations:

$$h_1 = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2}, \qquad h_2 = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2}. \qquad (3.3)$$

The irreducibility condition (2.13) (in the case $n = 4$) has the form:

$$\sum_{k \neq j = 1}^{4} \ell_{jk} + \sum_{\substack{m=1 \\ m \neq j,\, j+1}}^{4} p_{jm} + \sum_{\substack{k,\ell=1 \\ k \neq \ell \neq j}}^{4} q_{j\ell}^{k} + \sum_{\substack{m,k,\ell=1 \\ m \neq k \neq \ell \neq j}}^{4} r_{jm}^{k\ell} = 2j_1^{j},$$

$$(3.4)$$

$$\sum_{k \neq j = 1}^{4} \ell_{kj} + \sum_{\substack{m=1 \\ m \neq j,\, j+1}}^{4} p_{jm} + \sum_{\substack{k,\ell=1 \\ k \neq \ell \neq j}}^{4} \bar{q}_{j\ell}^{k} + \sum_{\substack{m,k,\ell=1 \\ m \neq k \neq \ell \neq j}}^{4} r_{mj}^{k\ell} = 2j_2^{j},$$

and the γ_5-invariance condition requires

$$\sum_{k=1}^{4} (j_1^k - j_2^k) = \sum_{k=1}^{4} \kappa^k. \qquad (3.5)$$

One may deduce from (3.2) the explicit form of the four-point functions for some fields:

a) Scalar fields $j_1^k = j_2^k = 0$ $(k = 1, \ldots, 4)$

$$F_{\{d_a\}}^{(4)}(x_a) = \prod_{j<k=1}^{4} (x_{jk}^2)^{\frac{1}{6}[\sum_{\ell=1}^{4} d_\ell - 3d_j - 3d_k]} \phi(h_1 h_2), \qquad (3.6)$$

where ϕ is an arbitrary function of the harmonical ratios. The γ_5-invariance condition (3.5) requires

$$\sum_{k=1}^{4} \kappa_k = 0. \qquad (3.7)$$

b) Tensor fields $2j_1^k = 2j_2^k = n_k$ $(k = 1, \ldots, 4)$

$$F_{\{\chi_a\}}^{(4)}(x_a; z_a) = \prod_{j<k=1}^{4} (x_{jk}^2)^{\frac{1}{6}[\sum_{\ell=1}^{4} d_\ell - 3d_j - 3d_k]} \times$$

$$(3.8)$$

$$\times \sum_{\{\ell, p\}} \phi^{\{\ell, p\}}(h_1, h_2) \prod_{j<k=1}^{4} \left(\frac{g(x_{jk})}{x_{jk}^2} \right)^{\ell_{jk}} \prod_{\substack{j,m=1 \\ m \neq j,\, j+1}}^{4} (\lambda_{j+1,m}^{j})^{p_{jm}}.$$

Here we denote

$$g(x_{jk}) = g^{\mu\nu}(x_{jk}) \xi_\mu^j \xi_\nu^k = (g^{\mu\nu} - \frac{x_{jk}^\mu x_{jk}^\nu}{x_{jk}^2}) z_j \sigma_\mu \bar{z}_j z_k \sigma_\mu \bar{z}_k$$

and

$$\lambda_{k\ell}^{j} = \left(\frac{x_{jk}^\mu}{x_{jk}^2} - \frac{x_{j\ell}^\mu}{x_{j\ell}^2} \right) \xi_\mu^j = \frac{1}{2} z_j \sigma^\mu \bar{z}_j \partial_\mu \ln\left(\frac{x_{jk}^2}{x_{j\ell}^2} \right).$$

One obtains from the irreducibility condition (3.4) the equalities

$$p_{j,j+2} = n_j - \sum_{\substack{k=1 \\ k \neq j}}^{4} q_{jk} - p_{j,j+3}, \quad j+k \xrightarrow{j+k>4} j+k-4.$$

The function (3.8) is γ_5-invariant when the condition (3.7) is satisfied.

c) Two scalar and two vector fields, i.e., $n_1 = n_4 = 0$, $n_2 = n_3 = 1$.
In this case it follows from (3.8) that

$$F^{(4)}(x_1, x_2; \xi_2, x_3; \xi_3, x_4) = \prod_{j<k=1}^{4} (x_{jk}^2)^{\frac{1}{6}[\sum_{\ell=1}^{4} d_\ell - 3d_j - 3d_k]} \times$$

$$\times\{\phi_1(h_1,h_2)\,\frac{g(x_{23})}{x_{23}^2}+\phi_2(h_1,h_2)\,\lambda_{13}^2\lambda_{12}^3+\phi_3(h_1,h_2)\,\lambda_{34}^2\lambda_{12}^3+ \qquad (3.9)$$

$$+\phi_4(h_1,h_2)\,\lambda_{13}^2\lambda_{24}^3+\phi_5(h_1,h_2)\,\lambda_{34}^2\lambda_{12}^3\}\,.$$

The $\phi_1,\ldots,\phi_5$ are arbitrary functions of harmonic ratios (3.3). The transition to the ordinary vector indices can be made by differentiating (3.9) with respect to ξ_2 and ξ_3, i.e.,

$$F^{(4)}_{\mu\nu}(x_a) = \frac{\partial^2}{\partial\xi_2^\mu\partial\xi_3^\nu}\,F^{(4)}(x_1,x_2;\xi_2,x_3;\xi_3,x_4)\,. \qquad (3.10)$$

Because of the fact that (3.8) and (3.9) depend only on ξ eqs. (3.8) and (3.9) can be written in tensor indices by the change of the variables $g(x_{jk})$ and $\lambda^j_{k\ell}$ by the tensors:

$$g_{\mu\nu}(x) = \frac{\partial^2 g(x_{jk})}{\partial\xi_j^\mu\partial\xi_k^\nu} = g_{\mu\nu} - \frac{x_\mu^{jk}\,x_\nu^{jk}}{x_{jk}^2}$$

and

$$(\lambda^j_{k\ell})_\mu = \frac{\partial}{\partial\xi_j^\mu}(\lambda_{k\ell}) = \frac{x_\mu^{jk}}{x_{jk}^2} - \frac{x_\mu^{j\ell}}{x_{j\ell}^2} = \frac{1}{2}\,\partial_\mu \ln\left(\frac{x_{jk}^2}{x_{j\ell}^2}\right)$$

with the subsequent symmetrization and subtraction of the traces with respect to the indices $\{\mu\}$ and $\{\nu\}$, separately.

d) Two scalar fields and two fields with spin -1/2, for example $\chi_1=\{d_1,½,0\}$, $\chi_2=\{d_2,0,½\}$ and $\chi_{3,4}=\{d_{3,4},0,0\}$.
One obtains from (3.2)

$$F^{(½)}_{(½)} = \prod_{j<k=1}^{4}(x_{jk}^2)^{\delta_{jk}}\{\phi_1\,\frac{z_1\tilde{x}_{12}\bar{z}_2}{x_{12}^2}+\phi_2\,\frac{z_1\tilde{x}_{14}x_{43}\tilde{x}_{32}\bar{z}_2}{(x_{14}^2x_{43}^2x_{32}^2x_{12}^2)^{½}}\}. \qquad (3.11)$$

If $\chi_1=\{d_1,½,0\}$ and $\chi_2=\{d_2,½,0\}$ one has:

$$F^{(½,½)} = \prod_{j<k=1}^{4}(x_{jk}^2)^{\delta_{jk}}\{\phi_1'\,\frac{z_1\tilde{x}_{13}\bar{x}_{32}\epsilon z_2}{(x_{12}^2x_{13}^2x_{23}^2)^{½}}+\phi_2'\,\frac{z_1\tilde{x}_{14}\bar{x}_{42}\epsilon z_2}{(x_{12}^2x_{14}^2x_{24}^2)^{½}}\}. \qquad (3.12)$$

The γ_5-invariance condition implies that eq. (3.11) exists if (3.7) is satisfied and for existence of (3.12) we must require

$$\sum_{k=1}^{4}\kappa_k = 1\,. \qquad (3.13)$$

e) The Dirac fields, i.e. $\chi_{1,2}=\{d_{1,2},½,0\}\oplus\{d_{1,2},0,½\}$

Because of the conditions (3.7) and (3.13) in this case the four-point function is a combination of the invariant functions (3.11) and (3.12) only.

$$F = \prod_{j<k=1}^{4}(x_{jk}^2)^{\delta_{jk}}\{\phi_1(h_1,h_2)\,\frac{\hat{x}_{12}}{x_{12}^2}+\phi_2(h_1,h_2)\,\frac{\hat{x}_{12}\hat{x}_{43}\hat{x}_{32}}{(x_{12}^2x_{14}^2x_{34}^2x_{23}^2)^{½}}\}, \qquad (3.14)$$

if

$$\sum_{j=1}^{4}\kappa_j = 0$$

and

$$F = \prod_{j<k=1}^{4}(x_{jk}^2)^{\delta_{jk}}\{\phi_1\,\frac{\hat{x}_{13}\hat{x}_{32}}{(x_{13}^2x_{23}^2x_{12}^2)^{½}}+\phi_2\,\frac{\hat{x}_{14}\hat{x}_{12}}{(x_{14}^2x_{24}^2x_{12}^2)^{½}}\} \qquad (3.15)$$

if $\sum_{j=1}^{4}\kappa_j = 1$. Here $\hat{x} = x^\mu\gamma_\mu$.

APPENDIX A

An arbitrary n-point function depends on x_j, z_j, and $\bar{z}_j$. Translational invariance requires that the dependence on x_j takes place by means of

$$x_{j,k} = x_j - x_k \quad (j,k=1,\ldots,n)\,. \qquad (A.1)$$

However, only $(n-1)$ of the variables (A.1) are independent. In general, we shall use the dependent variables. The question on independence will be discussed in the final result.

From the spinors z_j and $\bar{z}_j$ one can construct the following tensors:

$$z_j \epsilon z_k , \quad \bar z_j \epsilon \bar z_k , \quad z_j \underset{\sim}{\sigma}_\mu \bar z_k ,$$

$$z_j \underset{\sim}{\sigma}_\mu \tilde\sigma_\nu \epsilon z_k , \quad \bar z_j \epsilon \tilde\sigma_\mu \underset{\sim}{\sigma}_\nu \bar z_k ,$$

$$z_j \underset{\sim}{\sigma}_\mu \tilde\sigma_\nu \underset{\sim}{\sigma}_\lambda \bar z_k , \tag{A.2}$$

$$z_j \underset{\sim}{\sigma}_\mu \tilde\sigma_\nu \underset{\sim}{\sigma}_\lambda \tilde\sigma_\tau \epsilon z_k , \quad \bar z_j \epsilon \tilde\sigma_\mu \underset{\sim}{\sigma}_\nu \tilde\sigma_\lambda \underset{\sim}{\sigma}_\tau \bar z_k ,$$

. . . .

and so on, where $\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma = (\sigma_0 , \underset{\sim}{\sigma})$, $\tilde\sigma = (\sigma_0 , -\underset{\sim}{\sigma})$, $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, σ_j $(j = 1,2,3)$ are the Pauli matrices. The tensors (A.2) are not independent too.

From the four-vectors x_{jk} and tensors (A.2) one may form the following relativistic invariants.

$$x^2_{jk} , \tag{A.3}$$

$$z_a \epsilon z_b = -z_b \epsilon z_a , \quad \bar z_a \epsilon \bar z_b = -\bar z_b \epsilon \bar z_a , \tag{A.3a}$$

$$z_a x_{jk} \bar z_b , \tag{A.3b}$$

$$z_a \underset{\sim}{x}_{jk} \tilde x_{\ell m} \epsilon z_b , \quad \bar z_a \epsilon \tilde x_{jk} \underset{\sim}{x}_{\ell m} \bar z_b , \tag{A.3c}$$

$$z_a x_{jk} \tilde x_{\ell m} x_{pq} \bar z_g , \tag{A.3d}$$

$$z_a \underset{\sim}{x}_{jk} \tilde x_{\ell m} \underset{\sim}{x}_{pq} \tilde x_{rs} \epsilon z_b , \quad \bar z_a \epsilon \tilde x_{jk} \underset{\sim}{x}_{\ell m} \tilde x_{pq} \underset{\sim}{x}_{rs} \bar z_b , \tag{A.3e}$$

. . . .

where $\underset{\sim}{x} = x^\mu \underset{\sim}{\sigma}_\mu$ and $\tilde x = x^\mu \tilde\sigma_\mu$.

Using the equalities

$$\underset{\sim}{x} \tilde x = x^2 ,$$

$$\underset{\sim}{x} \tilde y \underset{\sim}{z} = (xy) \underset{\sim}{z} - (xz) \underset{\sim}{y} + (yz) \underset{\sim}{x} + i\epsilon^{\mu\nu\rho\tau} x_\mu y_\nu z_\rho \underset{\sim}{\sigma}_\tau \tag{A.4}$$

one can verify that, if some variables x_{jk} in (A.3) are dependent, then they may be expressed by means of the quantities

written above. All terms in (A.3) are products of no more than $(n-1)$ $x_{k\ell}$ as a consequence of the dependence of the n-point function on $n-1$ linear independent x_{jk} .

The relativistic invariant n-point function depends only on the invariants (A.3). For example, when $n = 2$, we have:

$$F^{(2)} = F(x^2_{12}, z_1 \epsilon z_2, \bar z_1 \epsilon \bar z_2, z_j \underset{\sim}{x}_{12} \bar z_k), \quad (j, k = 1,2).$$

APPENDIX B

To construct the conformal invariant n-point function, it is necessary to form, by means of relativistic invariants (A.3), conformal invariant variables, i.e., the quantities which satisfy eqs. (2.7). It is easy to verify that CI variables have the following form:

$$h^{mn}_{k\ell} = \frac{x^2_{\ell k} \, x^2_{mn}}{x^2_{km} \, x^2_{\ell n}} , \tag{B.1}$$

$$\frac{z_j \underset{\sim}{x}_{jk} \bar z_k}{x^2_{jk}} , \quad \lambda^j_{j+1,m} = \frac{z_j \underset{\sim}{x}_{j,j+1} \bar z_j}{x^2_{j,j+1}} - \frac{z_j \underset{\sim}{x}_{jm} \bar z_m}{x^2_{jm}} , \tag{B.2.1}$$

$$\frac{z_j \underset{\sim}{x}_{jk} \tilde x_{k\ell} \epsilon z_\ell}{(x^2_{jk} x^2_{k\ell} x^2_{j\ell})^{1/2}} , \quad \frac{\bar z_j \epsilon \tilde x_{jk} \underset{\sim}{x}_{k\ell} \bar z_\ell}{(x^2_{jk} x^2_{k\ell} x^2_{j\ell})^{1/2}} , \tag{B.2.2}$$

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$$\frac{z_{k_1} \underset{\sim}{x}_{k_1 k_2} \tilde x_{k_2 k_3} \cdots \underset{\sim}{x}_{k_{n-1} k_n} \bar z_{k_n}}{(x^2_{k_1 k_2} x^2_{k_2 k_3} \cdots x^2_{k_{n-1} k_n} x^2_{k_1 k_n})^{1/2}} \quad \text{if } n \text{ is even,} \tag{B.2.3}$$

$$\frac{z_{k_1} \underset{\sim}{x}_{k_1 k_2} \cdots \tilde x_{k_{n-1} k_n} \epsilon z_{k_n}}{(x^2_{k_1 k_2} \cdots x^2_{k_{n-1} k_n} x^2_{k_1 k_n})^{1/2}} \quad \text{if } n \text{ is odd} \tag{B.2.4}$$

The variables x_{jk} in (B.2.1 ÷ n) are linear independent. It follows from the equality

$$\lambda^j_{ab} + \lambda^j_{bc} = \lambda^j_{ac}$$

that only $n(n-2)-\lambda^{j}_{mn}$ are independent. There are only $n(n-3)/2$ independent harmonical ratios.

The general form of an n-point CI function is a polynomial in the variables (B.2) with the coefficient functions $f_n^{\{a\}}(x_{jk}^2)$. These functions satisfy the equations:

$$\Big\{ \sum_{j=1}^{n} x_{\mu}^{j} d_{j} + \sum_{j<k=1}^{n} (x_{\mu}^{j} + x_{\mu}^{k}) x_{jk}^{2} \frac{\partial}{\partial x_{jk}^{2}} \Big\} f_{n}^{\{a\}}(x_{ab}^{2}) = 0,$$

$$\Big\{ \sum_{j=1}^{n} d_{j} + 2 \sum_{j<k=1}^{n} x_{jk}^{2} \frac{\partial}{\partial x_{jk}^{2}} \Big\} f_{n}^{\{a\}}(x_{ab}^{2}) = 0, \tag{B.3}$$

i.e., $f_n^{\{a\}}$ satisfy the equations of invariance of scalar fields. It follows from (B.3) that $f_n^{\{a\}}$ has the form

$$f_{n}^{\{a\}}(x_{ab}^{2}) = \prod_{j<k=1}^{n} (x_{jk}^{2})^{\delta_{jk}} \phi_{n}^{\{a\}}(h_{1}, \dots, h_{n(n-3)/2}), \tag{B.4}$$

where δ_{jk} are given by (2.12). Here h_a are conformal invariant independent harmonical ratios (B.1).

REFERENCES

1. Midgal A.A. Phys.Lett., 1971, 37B, 98, p.386.
Ferrara S. et al. Phys.Lett., 1972, 38B, p.332.
2. Mack G., Todorov I.T. Phys.Rev., 1973, D8, p.1764.
3. Mack G. Acta Univ. Wr., 1971, 207, 2, p.1.
4. Dobrev V.K. et al. Phys.Rev., 1976, D13, p.887.
5. Пальчик М.Я., Фрадкин Е.С. Лекции ОИЯИ, 2-8847, Дубна, 1975.
6. Mack G., Salam A. Ann.Phys. (N.Y.), 1969, 53, p.174.
7. Todorov I.T., Zaikov R.P. J.Math.Phys., 1969, 10, p.2014.
8. Zaikov R.P., Bul.J.Phys., 1975, 2, p.89.
9. Sotkov G.M., Zaikov R.P. Rep.Math.Phys., 1977, 12, p.375.
10. Bargman V., Todorov I.T. J.Math.Phys., 1977, 18, p.1141.
11. Dobrev V.K., Petcova V.B. Rep.Math.Phys., 1978, 13, p.233.

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