сообщения
объединенного
института
ядерных
исследований
дубна

E2-80-110

E.-M.Ilgenfritz, M.Mueller-Preussker

THE PERMEABILITY OF THE INTERACTING YANG-MILLS INSTANTON GAS

1980

1. Introduction

In their recent work Callan, Dashen, and Gross have proposed to consider the Euclidean Yang-Mills vacuum as a magnetizable medium in the sense of fourdimensional magnetostatics, responding to external (classical) "magnetic" fields /1/. The sources of the latter are thought to be heavy static quarks. In this picture the main role is played by instanton solutions /2/. They ought to dominate the vacuum-to-vacuum amplitude within the dilute gas approximation. In other words, the Euclidean vacuum time histories are described in terms of a grand canonical ensemble for a four-dimensional gas of dipole-like instantons and antiinstantons. It has been suggested /1/ that such a gas can undergo a first-order phase transition driven by an external Minkowskian electric field, in this sense providing a microscopic basis for the MIT bag model /3/. From another point of view, instantons turned out to renormalize the coupling constant of the Yang-Mills theory beyond the behaviour dictated by the perturbative β-function /4/. With instanton contributions included the β-function interpolates between the perturbative one at small coupling and the strong coupling β-function determined within Euclidean lattice QCD /5/. The central notion appearing in both questions turns out to be the permeability μ of the instanton gas.

In a recent paper /6/ we reinvestigated the existence of the phase transition on the basis of a corrected, soft suppression of instantons by an external electric field. This was necessary due to the fact that the thermodynamical, "magnetostrictive" mechanism fails to provide strong exponential suppression of large-size instantons. A first order phase transition was nevertheless shown to exist, even in absence of interactions among instantons. In Ref. /6/ all calculations were based on the one-instanton amplitude according to the Pauli-Villars regularization scheme, and relatively large coupling constants had to be taken into account in order to obtain a significant influence of the instanton gas onto the equation of state of the vacuum. In this paper we will extend the analysis to other regularization schemes.

In the preceding paper we studied the influence of instanton interactions using different effective field methods. The ansatz proposed in Ref. /1/ (based on Onsager's treatment of strongly dielectric media /7/) was found to modify the phase transition only slightly compared to the noninteracting case. An a priori as well possible mean field ansatz (inspired by Weiss' theory of ferromagnetism), however, changed this picture and opened the possibility that the first order transition becomes lost. Instead of this spontaneous polarization would show up. In view of this ambiguity it seems worthwhile to treat the instanton interactions in a more systematic, i.e., microscopic way, a problem which is solved to some extent in this paper.

We will investigate the grand partition function of the instanton-antiinstanton gas with account of the dipole-dipole interaction. We succeeded to calculate the logarithm of the partition function up to second order in the external field obtaining in this way the permeability of the interacting instanton gas. We arrived at the approximate formula

$$\mu = 1 + \frac{4\pi^2 \chi_0}{1 - (\pi^2 \chi_0)^2}\,, \qquad (1)$$

where χ_0 denotes the susceptibility of the noninteracting gas. This formula exhibits an irregular behaviour of the interacting gas to be expected as $\pi^2\chi_0$ approaches 1. It has to be compared with the corresponding expression

$$\mu = 4\pi^2 \chi_0 + \sqrt{1 + (4\pi^2\chi_0)^2} \qquad (2)$$

used by Callan, Dashen, and Gross /1,4/ and based on the linear response to Onsager's local field exerted onto an instanton in a cavity by the continuous medium surrounding it.

Since we do not know the logarithm of the partition function at arbitrary strong external fields, we shall use formula (1) in order to introduce an effective field analogous to that used in connection with expression (2) and consider again the first order phase transition in terms of the equation of state. Compared with the noninteracting case the behaviour is qualitatively reproduced within small deviations.

Furthermore, expression (1) for the permeability will be applied to renormalize the coupling constant by instanton effects. So far as the interpolation to the strong coupling β-function is concerned, the results obtained in Ref. /4/ remain qualitatively unaltered.

In section 2 the ansatz for the partition function is written down, and some formulae are collected. In section 3 the standard method of functional averaging is used to deal with the dipole-dipole interaction. From the full expression for the permeability the approximation (1) is derived. The estimation of the validity of the Gaussian approximation for the functional average is relegated to the appendix. We apply formula (1) to the equation of state and the coupling constant renormalization in section 4 and conclude in section 5.

2. Partition function of the interacting instanton gas

We are going to consider the vacuum-to-vacuum transition amplitude for the pure SU(N) Yang-Mills theory. The Euclidean functional integral will be calculated within the quasiclassical approximation by expanding the action around field configurations given by superpositions of N_+ instantons and N_- antiinstantons (in singular gauge)

$$R_\mu^a(x) = \sum_{i=1}^{N_+ + N_-} D_{\mu\nu}^{\varepsilon_i a} \frac{2\,(x - x_i)_\nu}{(x - x_i)^2\left[(x - x_i)^2 + \rho_i^2\right]} = \sum_{i=1}^{N_+ + N_-} A_{i\mu}^{\varepsilon_i a}(x - x_i) \qquad (3)$$

considered as approximate stationary points. Here

$$D_{\mu\nu}^{\varepsilon_i a} = \frac{\rho_i^2}{g} R_i^{a\alpha} \eta_{\alpha\mu\nu}^{\varepsilon_i} \qquad (4)$$

denotes the dipole moment of the i-th instanton ($\varepsilon = +1$) or antiinstanton ($\varepsilon = -1$), respectively. x_i, ρ_i and $R_i^{a\alpha}$ are the usual collective coordinates, the latter matrix determining the global gauge orientation. The η symbols have been introduced in Ref. /8/: $\eta^+_{\alpha\mu\nu} \equiv \bar{\eta}^\alpha_{\mu\nu}$, $\eta^-_{\alpha\mu\nu} \equiv \eta^\alpha_{\mu\nu}$.

We write the vacuum-to-vacuum amplitude in the form of a grand canonical partition function corresponding to the instanton-antiinstanton gas in a four-volume V,

$$Z[\tilde{H},\zeta_\pm,V] = \sum_{N_+,N_-} \frac{1}{N_+!\,N_-!} \prod_{j=1}^{N_++N_-} \int_V d^4x_j \int^{\varrho_c} \frac{d\varrho_j}{\varrho_j}\, n_0(\varrho_j) \int [dR_j] \times$$
$$\times\, e^{\zeta_{\varepsilon_j}(x_j,\varrho_j,R_j)}\, e^{2\pi^2 D^{\varepsilon_j a}_{\mu\nu} \tilde{H}^a_{\mu\nu}}\, e^{V_{int}}, \qquad (5)$$

where the one-loop single-instanton amplitude is given by /8/

$$n_0(\varrho) = C_{SU(N)} \frac{1}{\varrho^4} x(\varrho)^{2N} e^{-x(\varrho)}, \qquad x(\varrho) = \frac{8\pi^2}{g^2(\varrho)}. \qquad (6)$$

In the partition function has been introduced the classical interaction with the external field $H^a_{\mu\nu}$ =const.,

$$-\delta S = 2\pi^2 D^{\varepsilon a}_{\mu\nu} \tilde{H}^a_{\mu\nu}, \qquad \tilde{H}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} H^a_{\lambda\sigma} \qquad (7)$$

and, for later use, a chemical potential $\zeta_\varepsilon = \zeta_\varepsilon(x,\varrho,R)$.

As to the one-instanton amplitude, the effective coupling constant will be taken according to the one-loop approximation for the β-function, i.e.,

$$-\frac{d\varrho}{\varrho} = \frac{3}{11N}dx, \qquad x(\varrho) = \frac{11N}{3}\ln\frac{1}{\varrho\Lambda}, \qquad (8)$$

where Λ depends on the regularization scheme, as well as the factor $C_{SU(N)}$ in equ.(6) does. (The dependence of $C_{SU(N)}$ both on N and the regularization scheme adopted will be specified in section 4.)

The integration over instanton sizes ϱ will be cut off at some $\varrho_c = \varrho(x_c)$ as usual /1/. Appropriate physical conditions to determine x_c in either case are discussed later, too.

From the partition function will be obtained, e.g., the average instanton density

$$\langle n_\varepsilon(x,\varrho,R)\rangle = \left.\frac{\delta \ln Z[\tilde{H},\zeta_\pm,V]}{\delta\zeta_\varepsilon(x,\varrho,R)}\right|_{\zeta_\pm=0} \qquad (9)$$

or the "magnetization" of the instanton gas

$$4\pi^2\tilde{M}^a_{\mu\nu} = \frac{\partial}{\partial\tilde{H}^a_{\mu\nu}}\left\{\frac{1}{V}\ln Z[\tilde{H},\zeta_\pm,V]\Big|_{\zeta_\pm=0}\right\}. \qquad (10)$$

For small external field H one expects

$$M^a_{\mu\nu} = \chi H^a_{\mu\nu}, \qquad (11)$$

where χ is called "susceptibility" of the instanton gas. One introduces a microscopic field B

$$B^a_{\mu\nu} = H^a_{\mu\nu} + 4\pi^2 M^a_{\mu\nu} \underset{H\to 0}{\sim} \mu H^a_{\mu\nu} \qquad (12)$$

including both the external field and the average field of the (anti)instanton dipoles. The "permeability" of the instanton "medium" is then given by

$$\mu = 1 + 4\pi^2\chi. \qquad (13)$$

Our main task in this paper is to calculate the permeability μ with account of the instanton-antiinstanton interaction, schematically indicated in equ. (5) as V_{int}. In general, interaction terms not factorizable within the partition function arise on classical and quantum level. Classical interactions stem from overlap contributions between different (anti)instantons to the action; interactions on the quantum level appear from multiscattering expansions of multiinstanton determinants and due to the fact that the field configurations to expand about are not true minima of the action. Moreover, there are corrections coming from the Jacobian with respect to all collective coordinates. This has been extensively studied by Levine and Yaffe /9/. They have classified all interaction terms with respect to their dependence on the distance between (anti)instantons, expressable in terms of $\varrho_i\varrho_j/(x_i-x_j)^2$, and have shown that there are no contributions falling less rapidly than $\varrho_i^2\varrho_j^2/(x_i-x_j)^4$, typical e.g., for the classical dipole-dipole interaction. Therefore we will concentrate upon the dipole-dipole interaction,

$$V_{int} \approx -S_{int} \approx -\frac{1}{2}\sum_{i,j}\int d^4y\, \overset{+}{A}{}^a_{i\mu}(y-x_i)\,\Box_y\, \bar{A}^a_{j\mu}(y-x_j). \qquad (14)$$

3. Functional averaging method for the partition function of the interacting instanton gas

We rewrite the interaction factor $\exp V_{int}$ in equ. (5) in the form of a functional integral

$$\exp\Big(-\frac{1}{2}\sum_{i,j}\int d^4y\, \overset{+}{A}{}^a_{i\mu}(y-x_i)\,\Box_y\, \bar{A}^a_{j\mu}(y-x_j)\Big) = \exp\Big(-\frac{1}{4}A^T_M\cdot\hat{\Box}\cdot A_M\Big)$$
$$= (\det\hat{\Box}^{-1})^{\frac{1}{2}}\int\delta h_M \exp\Big(-\frac{1}{2}h^T_M\cdot\hat{\Box}^{-1}\cdot h_M - \frac{i}{\sqrt{2}}A^T_M\cdot h_M\Big), \qquad (15)$$

where we used the matrix notation

$$A_M = \begin{pmatrix} \sum_i \overset{+}{A}{}^a_\mu(y-x_i) \\ \sum_j \overset{-}{A}{}^a_{j\mu}(y-x_j) \end{pmatrix}, \quad h_M = \begin{pmatrix} \overset{+}{h}{}^a_\mu(y) \\ \overset{-}{h}{}^a_\mu(y) \end{pmatrix} \tag{16}$$

$$\hat{\Box} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Box_y\, \delta(y-y')\, \delta_{aa'}\, \delta_{\mu\mu'} \quad .$$

Space integration and summation over Lorentz and colour indices are understood. Then we are able to sum up the noninteracting partition function with fluctuating external fields $h^{\pm}$ under the sign of functional averaging and obtain instead of equ. (5)

$$Z = (\det \hat{\Box}^{-1})^{\frac{1}{2}} \int \delta h_M \exp\Big\{ -\tfrac{1}{2} h_M^T \cdot \hat{\Box}^{-1} \cdot h_M + \tag{17}$$

$$+ \int d^4x \int_0^{\rho_c} \frac{d\rho}{\rho}\, n_0(\rho) \int [dR] \sum_{\varepsilon=\pm} \exp\Big[S_\varepsilon + 2\pi^2 \overset{\varepsilon}{D}{}^a_{\mu\nu} \tilde{H}^a_{\mu\nu} - \tfrac{i}{\sqrt{2}} \int d^4y\, \overset{\varepsilon}{A}{}^a_\mu(y-x)\, \overset{\varepsilon}{h}{}^a_\mu(y) \Big] \Big\} .$$

We integrate by expanding the exponent up to second order in $h^{\pm}$

$$Z = Z_0 (\det \hat{\Box}^{-1})^{\frac{1}{2}} \int \delta h_M \exp\Big\{ -\tfrac{1}{2} h_M^T \cdot (\hat{\Box}^{-1} + \hat{\chi}) \cdot h_M - J_M^T \cdot h_M \Big\} (1 + O(h^3)) , \tag{18}$$

where

$$Z_0 = \exp\Big\{ \int d^4x \int \frac{d\rho}{\rho}\, n_0(\rho) \int [dR] \sum_{\varepsilon=\pm} \exp\Big[S_\varepsilon + 2\pi^2 \overset{\varepsilon}{D}{}^a_{\mu\nu} \tilde{H}^a_{\mu\nu} \Big] \tag{19}$$

represents the partition function for the noninteracting gas, and where $\tilde{\chi}$ and J_M denote

$$\tilde{\chi} \equiv \begin{pmatrix} \overset{+}{\chi}{}^{aa'}_{\mu\mu'}(y-y') & 0 \\ 0 & \overset{-}{\chi}{}^{aa'}_{\mu\mu'}(y-y') \end{pmatrix} , \qquad J_M = \begin{pmatrix} \overset{+}{J}{}^a_\mu(y) \\ \overset{-}{J}{}^a_\mu(y) \end{pmatrix} \tag{20}$$

with

$$\overset{\varepsilon}{\chi}{}^{aa'}_{\mu\mu'}(y-y') = \tfrac{1}{2} \int \frac{d\rho}{\rho}\, n_0(\rho) \int [dR] \exp\Big[S_\varepsilon + 2\pi^2 \overset{\varepsilon}{D} \tilde{H} \Big] \int d^4x\, \overset{\varepsilon}{A}{}^a_\mu(y-x)\, \overset{\varepsilon}{A}{}^{a'}_{\mu'}(y'-x) \tag{21}$$

$$\overset{\varepsilon}{J}{}^a_\mu(y) = \frac{i}{\sqrt{2}} \int \frac{d\rho}{\rho}\, n_0(\rho) \int [dR] \exp\Big[S_\varepsilon + 2\pi^2 \overset{\varepsilon}{D} \tilde{H} \Big] \int d^4x\, \overset{\varepsilon}{A}{}^a_\mu(y-x) . \tag{22}$$

Some of these expressions have only a formal meaning, but there remain only welldefined quantities, when they are arranged as indicated. Notice that for vanishing external field H=0 we have $J^a_\mu(y)=0$, since

$$\int [dR]\, R^{\alpha a} = 0 \quad . \tag{23}$$

In this case the correction term $O(h^3)$ in formula (18) is in fact of order $O(h^4)$. In the appendix we estimate its magnitude to check the validity of the Gaussian approximation for $Z[\tilde{H}=0]$. (This check is sufficient also for H near zero which is needed to define the permeability.)

The Gaussian approximation of the functional integral (18) is immediately known. We expand the determinant of $(1+\hat{\Box}\hat{\chi})^{-1}$ and the operator itself and obtain

$$Z = Z_0 \exp\Big(\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \mathrm{Tr}(\hat{\Box}\cdot\hat{\chi})^n \Big) \exp\Big(\sum_{n=0}^{\infty} \frac{(-1)^n}{2} J_M^T \cdot (\hat{\Box}\cdot\hat{\chi})^n \cdot \hat{\Box} \cdot J_M \Big) . \tag{24}$$

Spelling out the matrix structure of $\hat{\Box}$ and $\hat{\chi}$ and introducing Fourier transforms we arrive at

$$Z = Z_0 \exp\Big\{ \frac{V}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int \frac{d^4k}{(2\pi)^4}\, k^{4n} \Big[\overset{-}{\chi}{}^{a_1a_1'}_{\mu_1\mu_1'}(k)\, \overset{+}{\chi}{}^{a_1'a_2}_{\mu_1'\mu_2} \cdots \overset{-}{\chi}{}^{a_na_n'}_{\mu_n\mu_n'}\, \overset{+}{\chi}{}^{a_n'a_1}_{\mu_n'\mu_1}(k) \Big] \Big\} \times$$

$$\times \exp\Big\{ \tfrac{1}{2} \sum_{n=0}^{\infty} \int \frac{d^4k}{(2\pi)^4}\, k^{4n+4} \Big[\overset{+}{J}{}^a_\mu(-k)\, \overset{-}{\chi}{}^{aa_1'}_{\mu\mu_1'}(k)\, \overset{+}{\chi}{}^{a_1'a_2}_{\mu_1'\mu_2} \cdots \overset{+}{\chi}{}^{a_n'a_{n+1}}_{\mu_n'\mu_{n+1}}\, \overset{-}{\chi}{}^{a_{n+1}a'}_{\mu_{n+1}\mu'}\, \overset{+}{J}{}^{a'}_{\mu'}(k) \Big] +$$

$$+ \Big(\overset{+}{J}, \overset{-}{\chi} \longleftrightarrow \overset{-}{J}, \overset{+}{\chi} \Big) \Big\} . \tag{25}$$

The first exponential (due to the determinant) can be visualized as a sum of ring graphs containing instantons and antiinstantons in alternating sequence, **whereas the second one is re**presented by alternating chains with "currents" J at the ends. Chains with different J^+, J^- at their ends vanish identically and are therefore omitted in equ.(25).The Fourier transforms of $\overset{\varepsilon}{\chi}$, $\overset{\varepsilon}{J}$ have the form

$$k^2 \overset{\varepsilon}{\chi}{}^{aa'}_{\mu\mu'}(k) = -\tfrac{1}{2} \int \frac{d\rho}{\rho}\, n_0(\rho) \int [dR] \exp\big(S_\varepsilon + 2\pi^2 \overset{\varepsilon}{D} \tilde{H} \big)\, k^2 \overset{\varepsilon}{A}{}^a_\mu(k)\, \overset{\varepsilon}{A}{}^{a'}_{\mu'}(k) \tag{21'}$$

$$k\, \overset{\varepsilon}{J}{}^a_\mu(k) = \frac{i}{\sqrt{2}} \int \frac{d\rho}{\rho}\, n_0(\rho) \int [dR] \exp\big(S_\varepsilon + 2\pi^2 \overset{\varepsilon}{D} \tilde{H} \big)\, k\, \overset{\varepsilon}{A}{}^a_\mu(k) \cdot (2\pi)^4 \delta(k) \tag{22'}$$

where the Fourier transformed (anti)instanton solution is

$$k\,\overset{\epsilon}{H}{}^a_\mu(k) = 4\pi^2 i\,\overset{\epsilon}{D}{}^a_{\mu\nu}\,\hat{k}_\nu\,F(k\rho) \qquad (26)$$

with $\hat{k}_\nu = k_\nu/k$ and $F(x) = \frac{4}{x^2}\left(1-\frac{x^2}{2}K_2(x)\right)$. (27)

We notice that $F(x)$ is monotonically falling from $F(0)=1$ to $F(x\to\infty)=0$.

To calculate the permeability μ we expnnd lnZ up to second order in H. Then, due to the form of $k^2\overset{\epsilon}{\chi}$ at H=0,

$$k^2\,\overset{\epsilon}{\chi}{}^{aa'}_{\mu\mu'}(k)\Big|_{H=0} = \pi^2\chi_\epsilon(\zeta_\epsilon,k)\,\delta_{aa'}\left(\delta_{\mu\mu'}-\hat{k}_\mu\hat{k}_{\mu'}\right) \qquad (28)$$

with

$$\chi_\epsilon(\zeta_\epsilon,k) = \frac{1}{N^2-1}\int_0^{\rho_c}\frac{d\rho}{\rho}\,n_0(\rho)\,x(\rho)\,\rho^4\,e^{\zeta_\epsilon}F^2(k\rho), \qquad (29)$$

both the ring and the chain terms will be strongly simplified. We will not go into the somewhat lengthy, but straightforward calculations. Only the following properties of R-integration should be mentioned:

$$\int[dR] = 1\,, \qquad \int[dR]\,R^{\alpha a}R^{\alpha' a'} = \frac{1}{N^2-1}\,\delta_{\alpha\alpha'}\delta_{aa'}$$
$$\int[dR]\,R^{\alpha a}R^{\beta b}R^{\gamma c} = \frac{1}{N(N^2-1)}\,f^{\alpha\beta\gamma}f^{abc}, \qquad (30)$$

where $f^{\alpha\beta\gamma}$ denotes the structure constants of SU(N)-Lie algebra. (We are interested here only in the cases N=2,3.) Using the identities fulfilled by the η symbols (see Ref. /8/) and fixing $\zeta_\pm = \zeta_\pm(\rho)$ we get the following contributions for $\frac{1}{V}\ln Z$.

With $\chi_\epsilon(\zeta_\epsilon) = \chi_\epsilon(\zeta_\epsilon, k=0)$ (compare equ. (29)),

$$\frac{1}{V}\ln Z_0[\zeta_\pm,H] = \sum_\epsilon\int\frac{d\rho}{\rho}\,n_0(\rho)\,e^{\zeta_\epsilon(\rho)} + \frac{1}{2}\sum_\epsilon\pi^2\chi_\epsilon(\zeta_\epsilon)H^a_{\mu\nu}\left(H^a_{\mu\nu}-\epsilon\tilde{H}^a_{\mu\nu}\right) \qquad (31)$$

is the noninteracting part corresponding to the "free" permeability $\mu = 1+4\pi^2\chi_0$ $\quad(\chi_0 = \chi_\pm(0))$.

Because of the $\delta(k)$ function in the expression(22') for $k\overset{\epsilon}{J}(k)$ the k-integration over chain contributions is reduced to an angular average around k=0, simplified further due to the structure (28) of $k^2\overset{\epsilon}{\chi}\big|_{H=0}$:

$$\left(\frac{1}{V}\ln Z[\zeta_\pm,H]\right)_{chains} = \frac{1}{2}\sum_\epsilon\pi^2\chi_\epsilon(\zeta_\epsilon)H^a_{\mu\nu}\left(H^a_{\mu\nu}-\epsilon\tilde{H}^a_{\mu\nu}\right)\frac{\pi^2\chi_+(\zeta_+)\,\pi^2\chi_-(\zeta_-)}{1-\pi^2\chi_+(\zeta_+)\,\pi^2\chi_-(\zeta_-)}, \qquad (32)$$

contributing to the permeability (1). The ring graphs give finally

$$\left(\frac{1}{V}\ln Z[\zeta_\pm,H]\right)_{rings} = \frac{1}{16\pi^2}\int dk\,k^3\Bigg\{-3(N^2-1)\ln\left(1-\pi^2\chi_+(\zeta_+,k)\,\pi^2\chi_-(\zeta_-,k)\right) +$$
$$+\frac{1}{2}\Bigg[\sum_\pm 3\pi^4K_\pm(\zeta_\pm,k)\,\frac{\pi^2\chi_\mp(\zeta_\mp,k)}{1-\pi^2\chi_+(\zeta_+,k)\,\pi^2\chi_-(\zeta_-,k)} + \qquad (33)$$
$$+\sum_\pm\frac{2\pi^6}{N}\,\Omega^2_\pm(\zeta_\pm,k)\left(\frac{\pi^2\chi_\mp(\zeta_\mp,k)}{1-\pi^2\chi_+(\zeta_+,k)\,\pi^2\chi_-(\zeta_-,k)}\right)^2\Bigg]H^a_{\mu\nu}\left(H^a_{\mu\nu}\mp\tilde{H}^a_{\mu\nu}\right)\Bigg\}.$$

Here, in addition to formula (29), the following definitions have been used

$$K_\pm(\zeta_\pm,k) = \frac{1}{N^2-1}\int\frac{d\rho}{\rho}\,n_0(\rho)\,x^2(\rho)\,\rho^8\,F^2(k\rho)\,e^{\zeta_\pm} \qquad (34)$$

$$\Omega_\pm(\zeta_\pm,k) = \frac{1}{N^2-1}\int\frac{d\rho}{\rho}\,n_0(\rho)\,x^{3/2}(\rho)\,\rho^6\,F^3(k\rho)\,e^{\zeta_\pm}\,. \qquad (34')$$

Later $K(\zeta)$ and $\Omega(\zeta)$ will denote these integrals for k=0.

In order to ensure the convergence of the sum of chain as well as ring terms, one has to require $\pi^2\chi_0 < 1$. This condition, if it is not automatically satisfied, defines the lowest x_c for either regularization scheme employed to define the instanton amplitude. It turns out to be somewhat more restrictive than the usually applied diluteness criterion (compare the Table). In the appendix will be seen that for the Gaussian approximation to be valid, $1-(\pi^2\chi_0)^2$ must be not too small. However, $\pi^2\chi_0$=0.95 will be an acceptable value.

The k-integrals in equ. (33) are convergent.Nevertheless, we decide to "regularize" them by cutting off at $k_c = \frac{1}{d_c} \sim O(\frac{1}{\rho_c})$. This procedure is equivalent to smoothing the interaction term (14) at small distances $|x_i-x_j| \lesssim d_c$. Such an assumption seems to us justifyable as long as there is yet a principal lack of knowledge how to deal with the problem of dense instanton configurations. The estimation of the correction term $O(h^4)$ to the Gaussian approximation for $Z[\tilde{H}=0]$ shows that this approximation becomes better the smaller k_c is chosen (cf. the Table).

Table

Numerical values for different quantities defined in this paper. For comparison they are shown for two regularization schemes applied in the calculation of the one-instanton amplitude (see Refs./1,4,8/). x_c and ρ_c are determined by the requirement $\pi^2\chi_0 \lesssim .95$ (which is automatically satisfied in the Pauli-Villars case even for $x_c=0$).

	Pauli-Villars regularization		dimensional regularization	
N	2	3	2	3
$C^R_{SU(N)}$	.01626	$1.518\cdot10^{-3}$	27.96	105.8
x_c	.001	.001	17.1	25.5
$f_0(x_c)$	.52	.98	.15	.28
$f(x_c)$ equ.(37)	$6.4\cdot10^{-2}\times(\rho_c^{PV}/d_c)^4$	$8.9\cdot10^{-2}\times(\rho_c^{PV}/d_c)^4$	$2.2\cdot10^{-2}\times(\rho_c^{PV}/d_c)^4$	$4.6\cdot10^{-2}\times(\rho_c^{PV}/d_c)^4$
$\pi^2\chi_0$	.88	.86	.94	.95
$\mu(\chi_0)$ equ.(1)	15.9	14.0	35.7	39.2
$\Delta\chi$ equ.(39)	$2.3\cdot10^{-2}\times(\rho_c^{PV}/d_c)^4$	$3.1\cdot10^{-2}\times(\rho_c^{PV}/d_c)^4$	$8.4\cdot10^{-3}\times(\rho_c^{PV}/d_c)^4$	$1.3\cdot10^{-2}\times(\rho_c^{PV}/d_c)^4$
$a(\rho_c,d_c)$ $\frac{1}{V}\ln Z[H=0]$ equ.(A4)	$4.2\cdot10^{-4}\times(\rho_c^{PV}/d_c)^8$	$1.7\cdot10^{-3}\times(\rho_c^{PV}/d_c)^8$	$1.3\cdot10^{-4}\cdot(\rho_c^{PV}/d_c)^8$	$8.2\cdot10^{-4}\times(\rho_c^{PV}/d_c)^8$

With the cutoff introduced the k-integrals can be estimated from above by putting $F(k\rho)\to1$.

$$\frac{1}{V}\ln Z[\zeta,H] \leq 2\int\frac{d\rho}{\rho}\,n_0(\rho)\,e^{\zeta(\rho)} - \frac{3(N^2-1)}{64\pi^2}\ln\left[1-(\bar{\pi}^2\chi(\zeta))^2\right]\frac{1}{d_c^4} +$$
$$+\left[\frac{\bar{\pi}^2\chi(\zeta)}{1-(\bar{\pi}^2\chi(\zeta))^2} + \frac{3\bar{\pi}^2}{64}K(\zeta)\,\frac{\bar{\pi}^2\chi(\zeta)}{1-(\bar{\pi}^2\chi(\zeta))^2}\,\frac{1}{d_c^4} +\right.$$

$$\left.+\frac{\bar{\pi}^4}{32N}\,\Omega^2(\zeta)\left(\frac{\bar{\pi}^2\chi(\zeta)}{1-(\bar{\pi}^2\chi(\zeta))^2}\right)^2\frac{1}{d_c^4}\right]H^2 + O(H^3) \qquad (35)$$

$$\left(\ \zeta(\rho) \equiv \zeta_+(\rho) \equiv \zeta_-(\rho)\right).$$

Consider first the case of vanishing external field. According to equ. (9) the mean density of instantons and antiinstantons is obtained

$$\langle n(\rho)\rangle \approx n_0(\rho)\left[2 + \frac{3}{32}\,\chi(\rho)\left(\frac{\rho}{d_c}\right)^4\frac{\bar{\pi}^2\chi_0}{1-(\bar{\pi}^2\chi_0)^2}\right]. \qquad (36)$$

To estimate the effect of interactions onto the density we calculate the fractional occupied space-time volume

$$f(\rho_c) = \int_0^{\rho_c}\frac{d\rho}{\rho}\langle n(\rho)\rangle\frac{\bar{\pi}^2}{2}\rho^4 = f_0(\rho_c)\left(1+\Delta f(\rho_c)\right)$$
$$\leq \bar{\pi}^2\int_0^{\rho_c}\frac{d\rho}{\rho}\,n_0(\rho)\,\rho^4 + \frac{3\bar{\pi}^2}{64}\int_0^{\rho_c}\frac{d\rho}{\rho}\,n_0(\rho)\,\chi(\rho)\,\frac{\rho^8}{d_c^4}. \qquad (37)$$

The numerical results given in the Table indicate that the classical dipole-dipole interaction has only an effect of order $O(10^{-2})\cdot(\frac{\rho_c}{d_c})^4$ on the fractional occupied volume. According to equs.(11) and (13) by differentiating equ.(35) with respect to H we obtain the permeability (at $\zeta=0$)

$$\mu = 1 + \frac{4\bar{\pi}^2\chi_0}{1-(\bar{\pi}^2\chi_0)^2}\left(1+\Delta\chi\right) \qquad (38)$$

where

$$\Delta\chi \leq \frac{3\bar{\pi}^2}{64}\,\frac{K(0)}{d_c^4} + \frac{\bar{\pi}^4}{32N}\,\frac{\Omega^2(0)}{d_c^4}\,\frac{\bar{\pi}^2\chi_0}{1-(\bar{\pi}^2\chi_0)^2}. \qquad (39)$$

To the relative correction $\Delta\chi$ contribute only terms coming from the ring expansion of the partition function which are of order $(\frac{\rho_c}{d_c})^4$ multiplied by some factors (see the Table), which are $O(10^{-2})$ and $O(1)$ in the case of Pauli-Villars and dimensional regularization, respectively. Adopting the view, that both d_c and the instanton density per 4-volume had to be fixed independently of the regularization scheme (i.e., $\rho_c^D \approx \frac{4}{3.7}\rho_c^{PV}$), we obtain formula (1) within a deviation of some percent in either case.

Up to order H^2 we get the density of instantons and antiinstantons (neglecting terms corresponding to $\Delta\chi$)

$$\langle n(\rho)\rangle \approx n_0(\rho)\left[2 + \frac{3}{32}\,\chi(\rho)\left(\frac{\rho}{d_c}\right)^4\frac{\bar{\pi}^2\chi_0}{1-(\bar{\pi}^2\chi_0)^2} + \frac{1+(\bar{\pi}^2\chi_0)^2}{(1-(\bar{\pi}^2\chi_0)^2)^2}\,\frac{\bar{\pi}^2}{N^2-1}\,\chi(\rho)\,\rho^4H^2\right]$$

$$\approx n_0(\rho)\left[2+\frac{3}{128}\,x(\rho)\left(\frac{\rho}{d_c}\right)^4(\mu-1)+\frac{\partial\mu}{\partial\chi_0}\,\frac{1}{N^2-1}\,x(\rho)\,\rho^4 H^2\right]. \tag{40}$$

Here the effects of reaction of the surrounding instanton medium and the external field (mediated by the medium) are separately exhibited.

4. Consequences for the phase transition and coupling constant renormalization

So far we have not calculated the partition function for arbitrary external fields. Nevertheless, we use the approximate expression for the permeability μ , derived for small external fields, in order to discuss again the phase transition problem in terms of an effective field approach. (We restrict ourselves here to the case of SU(2) as in Ref. /6/.) We specialize to electric type fields , including Wick rotation from Minkowskian fields E,D to Euclidean ones

$$\begin{aligned} H_{4k}^{a} &= iD_k^a = i\,\delta^{a3}\delta_{k3}\,D \;, \\ H_{4k}^{a} &= iP_k^a = i\,\delta^{a3}\delta_{k3}\,P \;, \\ F_{4k}^{a} &= iE_k^a = i\,\delta^{a3}\delta_{k3}\,E \;. \end{aligned} \tag{41}$$

We define an effective field D^{eff} such that it should determine the polarization as in the noninteracting case /6/

$$4\pi^2 P = 4\pi^2 P_{ideal}(D^{eff}) \equiv \varphi(D^{eff}) \tag{42}$$

where, with $\quad \xi = 4\pi^2(\rho^2/g(\rho))\,D^{eff}$

$$\varphi(D^{eff}) = 8\pi^2\int\frac{d\rho}{\rho}\,n_0(\rho)\,\frac{\rho^2}{g(\rho)}\left(\frac{\sin\xi}{\xi^2}-\frac{\cos\xi}{\xi}\right). \tag{43}$$

The susceptibility $\chi_0(D^{eff})$ will be defined as

$$4\pi^2\chi_0(D^{eff}) = \frac{\varphi(D^{eff})}{D^{eff}} \;. \tag{44}$$

Then the permeability $\mu(D^{eff})$ will be determined via $\mu(\chi_0(D^{eff}))$, according to our formula (1), to expression (2) as well as to the interaction-free permeability

$$\mu_0 = 1 + 4\pi^2\chi_0 \;, \tag{45}$$

for comparison. We obtain both

$$D = \frac{4\pi^2\,P(D^{eff})}{\mu(D^{eff})-1} \tag{46}$$

and

$$E = \mu(D^{eff})\,D \;, \tag{47}$$

i.e., the equation of state E(D) in either case.

We have studied the corresponding equation of state E(D), investigating (i) the dependence on the regularization scheme used to define the one-instanton amplitude, and (ii) the effect of including the instanton interaction in either way. Although we have performed these calculations only for SU(2) we notice here the dependence of $C_{SU(N)}$ on N and the regularization scheme (R) /4,10/

$$C^{R}_{SU(N)} = \frac{4.60}{\pi^2}\,\frac{e^{-1.68N}}{(N-1)!\,(N-2)!}\left(\frac{\Lambda^{PV}}{\Lambda^{R}}\right)^{\frac{11N}{3}}. \tag{48}$$

(It has been used in calculating the numerical values of the Table). Λ^R is the scale parameter of the one-loop running coupling constant $x_R(\rho) = (11N/3)\ln(1/\rho\Lambda_R)$, defined within R (PV refers to the Pauli-Villars scheme).

Consider first the case of PV regularization. As already noted in Ref./6/, the usual dilute gas criterion $f_0(x_c) < 1$ fails to provide any restriction beyond $x_c \geq 0$. Also the (generally more restrictive) condition necessary for convergence of (24), $\pi^2\chi_0(x_c) < 1$, does not specify x_c any further. Thus x_c can be put zero.

In Fig.1 the equation of state E(D) is shown for the effective field approach based on the formulae (45),(1),and (2), respectively. Qualitatively, the behaviour obtained in the noninteracting case (I) is reproduced in both ways of including the interaction; curve II, however, does more resemble the noninteracting case than curve III does. This is not surprising, since for the corresponding dependence $\chi_0(D^{eff})$ - in the transition region - the permeabilities $\mu(\chi_0)$ in either case do not differ appreciably.

In the case of dimensional regularization, because of the large factor $C^D_{SU(2)}$, the dilute gas criterion does restrict

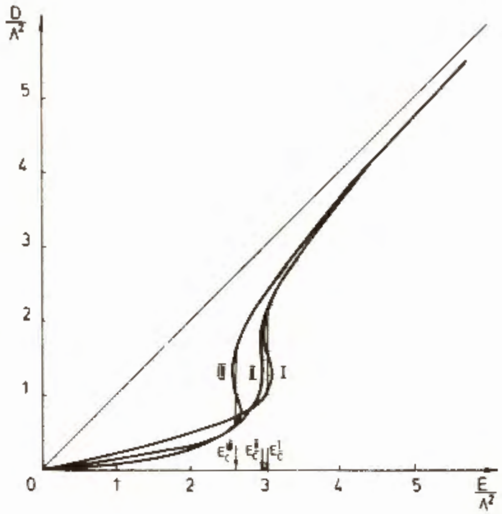

Fig. 1: D=D(E) in the case of Pauli-Villars regularization ($C^{PV}_{SU(2)}$ = 0.01626) for x_c = 0.01;
curve I: instanton gas without interaction,
curves II,III: interaction treated by an effective field acc. to equ. (1) and (2), resp.

$x_c \gtrsim 15$. Within this range relatively large χ_0 are attainable. This is illustrated in Fig. 2, where the equation of state D=D(E), produced by the noninteracting instanton gas, is shown for x_c=15, 17.5, and 20. Within this range of coupling constants the behaviour of the equation of state may vary drastically. By the condition of applicability of our expansion, $\Gamma^2\chi_0 < 1$, $x_c \gtrsim 17.5$ is selected. For the latter value we compare in Fig.3 the equation of state corresponding to both effective field methods with the noninteracting case. The general trend distinguishing the three curves is the same as in Fig. 1. (Notice also that the critical field strength values E_c in Fig. 3 are of the same order of magnitude as in Fig. 1, taking Λ^{PV}/Λ^{D}= 2.76 into account.)

Callan, Dashen, and Gross /4/ have proposed to consider the effective coupling constant $g^2(a)$, defined at length scale a, to be renormalized multiplicatively by instantons of smaller size, according to

$$g^2(a) = g^2_{AF}(a)\,\mu(a). \tag{49}$$

Here μ(a) is the permeability of the vacuum effected by instantons having size $\rho < \rho_c$, where the cutoff ρ_c coincides with a within a factor of order 1. $g^2_{AF}(a)$ is defined by the perturbative, running coupling constant at one-loop level

$$x(a) = \frac{8\pi^2}{g^2_{AF}(a)} = \frac{11N}{3}\ \ln\frac{1}{a\Lambda}\ .$$

This idea has been suggested by relating the effective action associated to a lattice of spacing a,

$$\mathcal{L}_{eff} = \sum_{\{\text{plaquettes } P\}} \frac{1}{g^2(a)}\left(\mathrm{tr}\, U(P) + \mathrm{h.c.}\right)\ ,\quad U(P) = P e^{\oint_P A_\mu dx_\mu} \tag{50}$$

to the constrained continuum functional integral, being saturated mainly by instantons having $\rho < \rho_c \approx a$. With a permeability μ calculated taking into account instanton interactions, it

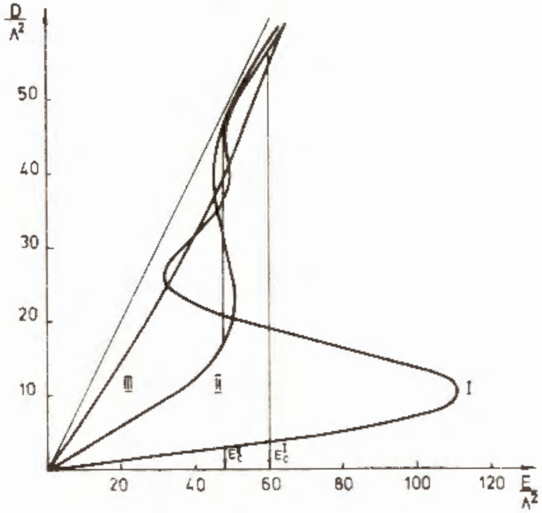

Fig. 2: D=D(E) in the case of dimensional regularization ($C^{D}_{SU(2)}$ = 27.965) without interactions;
curve I : x_c=15.0, curve II : x_c=17.5,
curve III: x_c=20.0.

seems worth to consider to what extent this influences the behaviour of $g(a)$ and of the β-function,

$$-\frac{\beta(g)}{g} = \frac{\partial \ln g(a)}{\partial \ln a}\Big|_{a=a(g)} \qquad (51)$$

Before comparing this with the strong coupling Euclidean lattice results /5/, one has to adopt a coupling constant definition according to the lattice regularization scheme /4/, for which $\Lambda^{PV}/\Lambda^{L} = 6.6$ and (for N=3) $C^{L}_{SU(3)}/C^{PV}_{SU(3)} = 1.04\cdot 10^{9}$.

Fig.4 represents the instanton effects on the β-function, compared both with the one-loop perturbative one (curve I) and the strong coupling β-function for Euclidean lattice theory /5/ (curve II). Shown are the different ways to define the permeability as a function of the dilute gas susceptibility. As far as the interpolation between the weak coupling and strong coupling behaviour is concerned occurring within the range g=1...2, the

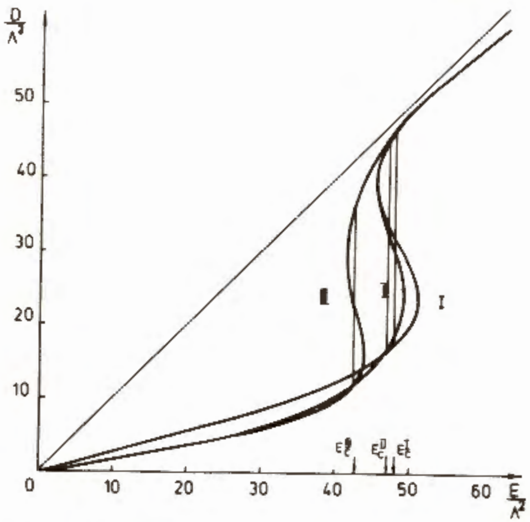

Fig. 3: D=D(E) in the case of dimensional regularization ($C^{D}_{SU(2)} = 27.965$) for $x_c = 17.5$;
curve I : instanton gas without interactions,
curves II,III: interaction treated by an effective field acc. to equ. (1) and (2), resp.

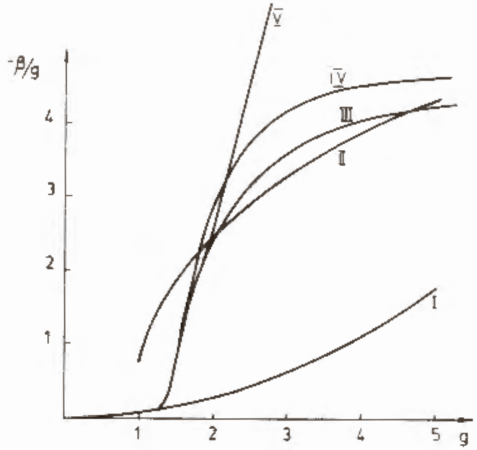

Fig. 4: β-function modified by instanton effects compared with the one-loop behaviour (curve I) and the strong coupling result (curve II);
curve III: instanton gas without interactions,
curve IV: interactions dealt with according to equ. (2),
curve V: interactions dealt with according to equ. (1).

different $\mu(\chi_0)$ do not differ appreciably. Within the transition region to the strong coupling regime, the latter being certainly described reliably by other than instanton physics, the both ways to include instanton interactions do not differ essentially from the free instanton gas. Therefore, the sudden rise of g(a) according to formula (1) as $\pi^2\chi_0 \to 1$ lies outside the range of interest of Fig. 4.

5. Summary

In this paper we have dealt with the instanton-antiinstanton (dipole-dipole) interaction from a microscopic point of view,i.e., starting from the partition function of the interacting (anti) instanton gas. The approach as a whole is based on the semiclassical approximation expanding around (anti)instanton superpositions, not being true solutions of the field equations /9/. This philosophy is complementary to the nowadays developing technique for dense instanton gases based on exact (anti)self-dual solutions /11/.

In this sense everything here is completely within the range of the dilute gas approximation, where infrared divergencies force us to introduce a cutoff ϱ_c (x_c). The quasiclassical approximation requires, strictly speaking, that the distances between (anti)instantons somehow tend to infinity with the coupling constant $g \to 0$. This could be guaranteed, e.g., by a "hard core" in the partition function ,however, this has not been done in this paper. We have summed a certain set of graphs representing instanton interactions, which corresponds to an expansion in powers of $\pi^2\chi$. For vanishing external field H=0 it resembles the ring approximation for the Coulomb gas, and for $H \neq 0$, it includes collective effects of the externally polarized medium onto any given instanton. Formally, this partial sum is obtained from the Gaussian approximation of the functional integral used to linearize the dipole-dipole interaction. Instead of imposing a sharp cutoff in the coordinate space ($|x_i - x_j| \geq d_c$) within the ring graphs we have cut off the corresponding k-integrals at $k_c = \frac{1}{d_c} = O(\frac{1}{\varrho})$ and obtained estimates of the contribution of the ring graphs to the density and susceptibility of the instanton gas.

We have not calculated the partition function for arbitrary large external fields H but only up to second order in H. This is sufficient for computing the permeability μ of the instanton gas. We obtained μ within an accuracy of a few percent as a function of the dilute gas susceptibility χ_0 (formula (1)), which differs from expression (2) used in Refs. /1,4/. On the basis of equ. (1) one confirms that, at least as long as $\pi^2\chi_0 < 1$, there is no"spontaneous magnetization", a possibility we could not exclude in Ref./6/.

We have studied the equation of state D=D(E), in the case of SU(2), for both the Pauli-Villars and dimensional regularization. To this aim we used an effective field method corresponding to either dependence $\mu(\chi_0)$. However, within the instability region of the equation of state between the dilute and dense phase the "dilute gas susceptibility" $\chi_0(D^{eff})$ is relatively small. Thus, concerning the instability as such and the critical field strength, the different expressions for $\mu(\chi_0)$ do not cause any essential differences.

We have also considered the effect of the different $\mu(\chi_0)$ onto the renormalization of the coupling constant by instanton

effects, equ. (49), as suggested in Ref. /4/. Unlike the other considered relations, equ. (1) results in a very strong rise of g(a) as $\pi^2\chi_0(a)$ approaches unity, i.e., at rather well defined distance a. Instantons are said to provide a "bridge"from weak to strong coupling/4/, well illustrated by the behaviour of the β-function. In this context instanton calculations should be reliable only in the intermediate region, to be replaced by strong coupling methods at larger lengths. Therefore one may perhaps conclude not to attribute too much significance to the way, how instanton interactions are taken into account.

Acknowledgements

We are grateful to Prof. D.V.Shirkov for his kind interest in this work and acknowledge useful discussions with D.I.Kazakov.

Appendix

We investigate the validity of the Gaussian approximation for the functional integral (17), in the case of vanishing external field H=0, that consists in neglecting higher order terms in the exponent of the integrand:

$$Z = Z_0 \left(\det \hat{\Pi}\right)^{-\frac{1}{2}} \int \delta h_M \exp\Big\{ -\frac{1}{2} h_M^T \cdot (\hat{\Pi}^{-1} + \hat{\chi}) \cdot h_M + \qquad \text{(A1)}$$

$$+ \frac{1}{3!} \sum_{\varepsilon} \int d^4x \int \frac{d\varrho}{\varrho} n_0(\varrho) \int [dR] \Big(-\frac{i}{\sqrt{2}} \int \overset{\varepsilon}{A}(y-x) \overset{\varepsilon}{h}(y) d^4y \Big)^3 + \ldots \Big\}.$$

We estimate the error, implied by this approximation, comparing the neglected terms (with $\prod_i \overset{\varepsilon}{h}(y_i)$ replaced by their Gaussian averages) with ln Z, as obtained in the Gaussian approximation. The lowest order nonvanishing contribution, e.g., is

$$\Delta\left(\frac{1}{V} \ln Z\right) = \frac{1}{4!} \sum_{\varepsilon} \int \frac{d\varrho}{\varrho} n_0(\varrho) \int [dR] \left(\frac{i}{\sqrt{2}}\right)^4 \int \prod_i d^4y_i \, \overset{\varepsilon}{A}(y_i - x) \left\langle \prod_i \overset{\varepsilon}{h}(y_i) \right\rangle$$

$$= \frac{3}{96} \sum_{\varepsilon} \int \frac{d\varrho}{\varrho} n_0(\varrho) \int [dR] \left\{ \sum_{n=0}^{\infty} \int \frac{d^4k}{(2\pi)^4} \overset{\varepsilon}{A}(-k) \left[\Pi \overset{-\varepsilon}{\chi} (\Pi \overset{+\varepsilon}{\chi})^n \Pi \overset{-\varepsilon}{\chi} \Pi \right](k) \overset{\varepsilon}{A}(k) \right\}^2. \qquad \text{(A2)}$$

Performing the calculation in the same way as shown in section 3 we get finally

$$\Delta\left(\frac{1}{V}\ln Z\right) \leq 9\cdot 2^{-12}(N^2-1)\frac{K(0)}{d_c^8}\left(\frac{\pi^2\chi_0}{1-(\pi^2\chi_0)^2}\right)^2 = a(\rho_c,d_c). \qquad (A3)$$

This expression should be much less than $\frac{1}{V}\ln Z(H=0)$, i.e., according to equ.(35)

$$\frac{a(\rho_c,d_c)}{\frac{1}{V}\ln Z(H=0)} = \frac{9\cdot 2^{-12}(N^2-1)\frac{K(0)}{\rho_c^4}\left(\frac{\pi^2\chi_0}{1-(\pi^2\chi_0)^2}\right)^2\left(\frac{\rho_c}{d_c}\right)^8}{2\int\frac{d\rho}{\rho}n_0(\rho)\,\rho_c^4-\frac{3(N^2-1)}{64\pi^2}\ln\left(1-(\pi^2\chi_0)^2\right)\left(\frac{\rho_c}{d_c}\right)^4} \ll 1. \qquad (A4)$$

The Table shows to what extent this is fulfilled in the different cases studied.

References

1. Callan C., Dashen R., Gross D. Phys. Rev., 1978, D17, p.2717; Phys. Rev., 1979, D19, p.1826.
2. Belavin A. et al. Phys. Letters, 1975, 59B, p.85.
3. Chodos A. et al. Phys. Rev., 1974, D9, p.3471.
4. Callan C., Dashen R., Gross D. Instantons as a Bridge Between Weak and Strong Coupling in QCD. Preprint, Princeton, 1979.
5. Kogut J., Pearson R., Shigemitsu J. Phys. Rev. Letters, 1979, 43, p.484.
6. Ilgenfritz E.-M., Kazakov D.I., Müller-Preussker M. Phys. Letters, 1979, 87B, p.242; JINR, E2-12628, Dubna, 1979.
7. Onsager L. J. Amer. Chem. Soc., 1936, 58, p.1486.
8. 't Hooft G. Phys. Rev., 1976, D14, p.3432; Phys. Rev., 1978, D18, p.2199, erratum.
9. Levine H., Yaffe L.G. Phys. Rev., 1979, D19, p.1225.
10. Bernard C. Phys. Rev., 1979, D19, p.3013.
11. cf. e.g. Berg B. DESY, 79/58, Hamburg, 1979.

Received by Publishing Department
on February 13 1980.