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ESSENTIALLY NONLINEAR ONE-DIMENSIONAL MODEL OF THE CLASSICAL FIELD THEORY

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Introduction

In this note we present a complete description of classical solutions of the equation

$$u_{tt} - u_{xx} + \sin u = 0, \quad -\infty < x, t < \infty \tag{1}$$

satisfying the boundary condition

$$u(x,t) \to 0 \pmod{2\pi} \quad \text{when} \quad |x| \to \infty . \tag{2}$$

In the field-theoretical approach eq. (1), with boundary condition (2), describes the chiral field $\chi(x,t)$ with the group $U(1)$, where $\chi(x,t) = \exp\{iu(x,t)\}$.

It turns out that eq. (1)-(2) can be solved through the inverse scattering method (for such a method see refs. /1-4/). Equation (1)-(2) is equivalent to the operator equation

$$\frac{\partial L}{\partial t} = [L, M],$$

where

$$L = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} A & B \\ B & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix},$$

and matrices I, J, A, B, C, D, are constructed from a solution $u(x,t)$ and function $w(x,t)=u_x(x,t)+u_t(x,t)$ in the following way:

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad A=\frac{i}{4}\begin{pmatrix}0 & w\\ w & 0\end{pmatrix},$$

$$B=\frac{1}{4}\begin{pmatrix}e^{iu/2} & 0\\ 0 & e^{-iu/2}\end{pmatrix},\ C=2JB,\quad D=2BJ.$$

The time dependence of the asymptotical characteristics of eigenfunctions of the operator L have been found earlier by us for the case when $u(x,t)$ obeys eq. (1). The present paper is devoted to the study of eq. (1) by means of the Hamiltonian formalism, following a scheme suggested in ref. /5/.

Let us first note that one can connect with eq. (1)-(2) the dynamical system with the Lagrangian

$$\mathcal{L}=\frac{1}{2\gamma}\int_{-\infty}^{\infty}(u_t^2-u_x^2+2(\cos u-1))\,dx\,.$$

Here the system of units $h=m=c=1$ is chosen, where m is the mass of field $u(x,t)$ and the dimensionless quantity γ stands for the coupling constant. The total energy and momentum of the field are as follows:

$$P_0=\frac{1}{2\gamma}\int_{-\infty}^{\infty}(w^2-2u_x w+2u_x^2+2(1-\cos u))\,dx, \tag{3}$$

$$P_1=\frac{1}{\gamma}\int_{-\infty}^{\infty}(u_x^2-u_x w)\,dx, \tag{4}$$

and the corresponding symplectic form Ω looks in the following way:

$$\Omega=\frac{1}{\gamma}\int_{-\infty}^{\infty}\left(dw(x)\wedge du(x)+du(x)\wedge d\frac{\partial u(x)}{\partial x}\right)dx. \tag{5}$$

We have assumed that one can take $u(x)$ and $\frac{1}{\gamma}u_t(x)$ as generalized coordinates and momenta. Note, that the form Ω and the Hamiltonian P_0 generate eq. (1) by the rules of the Hamiltonian mechanics. The aim of the paper is to

express these quantities through the scattering data for the operator L.

In Section I necessary information from the scattering theory is presented for the operator L. It should be noted that this operator is degenerate so that the main eigenvalue problem

$$L\Phi=\lambda\Phi$$

reduces to the equation

$$J\frac{d\Psi}{dx}+A\Psi+\frac{1}{\lambda}H\Psi=\lambda\Psi,\quad H=B^2, \tag{6}$$

where the spectral parameter λ appears also in the denominator. It is this equation that will be studied further. In Section 2 the symplectic form Ω will be expressed in terms of the scattering data. Here, we use the technique developed in /5,6/. In Section 3, on the basis of the so-called trace identities, the energy P_0 and total momentum P_1 of the field $u(x,t)$ will be expressed through the scattering data, and it will be shown that these depend only on canonical variables of the type of generalized momenta. In the same section we shall present an interpretation of the results obtained in terms of particles corresponding to the field $u(x,t)$.

I. Information from the Scattering Theory

Let us suppose that

$$\int_{-\infty}^{\infty}\left(|w(x)|+|u_x(x)|+\left|\sin\frac{u(x)}{2}\right|\right)dx<\infty\,.$$

Then eq. (6) has matrices-solutions $F(x,\lambda)$ and $G(x,\lambda)$ uniquely defined for real $\lambda\neq 0$ by the conditions

$$\begin{aligned}F(x,\lambda)&=E(x,\lambda)+o(1),\quad x\to\infty,\\ G(x,\lambda)&=E(x,\lambda)+o(1),\quad x\to-\infty,\end{aligned} \tag{7}$$

where

$$E(x,\lambda)=(e(x,\lambda),\overline{e(x,\lambda)}),\ e(x,\lambda)=\exp\{i(\lambda-\frac{1}{16\lambda})x\}\binom{1}{i}.$$

The first column $f(x,\lambda)$ of the matrix $F(x,\lambda)$ and the second one $g_2(x,\lambda)$ of the matrix $G(x,\lambda)$ allow analytical continuation to the half-plane $\mathrm{Im}\lambda>0$ and have the properties

$$\begin{aligned} &f_1(x,\lambda)\exp\{-i(\lambda-\frac{1}{16\lambda})x\}=\binom{1}{i}+o(1), \\ &\qquad\qquad\text{when } |\lambda|\to\infty, \\ &g_2(x,\lambda)\exp\{i(\lambda-\frac{1}{16\lambda})x\}=\binom{1}{-i}+o(1),\quad \mathrm{Im}\lambda\geq 0. \end{aligned} \tag{8}$$

Analogously, the column $f_2(x,\lambda)$ of the matrix $F(x,\lambda)$ and the column $g_1(x,\lambda)$ of $G(x,\lambda)$ also allow an analytic continuation to the half-plane $\mathrm{Im}\lambda<0$. For real $\lambda\neq 0$ the solutions $F(x,\lambda)$ and $G(x,\lambda)$ are the fundamental matrices of eq. (6), therefore there is such a matrix $T(\lambda)$ that

$$F(x,\lambda)=G(x,\lambda)T(\lambda). \tag{9}$$

The matrix $T(\lambda)$ is called the transition matrix of eq. (6). Since

$$\det F(x,\lambda)=\det G(x,\lambda)=-2i,$$

then

$$\det T(\lambda)=1. \tag{10}$$

Note, that the coefficients for eq. (6) obey the relations

$$J\,A(x)=\bar{A}(x)J,\quad JH(x)=\bar{H}(x)J,$$

and as

$$J\overline{E(x,\lambda)}=iE(x,\lambda)J,$$

we get that

$$J\overline{F(x,\lambda)}=iF(x,\lambda)J,\quad J\overline{G(x,\lambda)}=iG(x,\lambda)J,$$

whence, with the help of (9), we conclude that

$$T(\lambda)=J\overline{T(\lambda)}J^{-1},$$

or

$$T(\lambda)=\begin{pmatrix} a(\lambda) & -b(\lambda) \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix},$$

and the formula (10) takes the form

$$|a(\lambda)|^2+|b(\lambda)|^2=1. \tag{11}$$

Further, we note that the coefficients of eq. (6) and the matrix $E(x,\lambda)$ satisfy the relations

$$SA(x)=-\overline{A(x)}S,\ SH(x)=\overline{H(x)}S,\ S\overline{E(x,-\lambda)}=-iE(x,\lambda)R,$$

where

$$S=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad R=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

whence it follows that

$$S\overline{F(x,-\lambda)}=-iF(x,\lambda)R,\quad S\overline{G(x,-\lambda)}=-iG(x,\lambda)R$$

and for $T(\lambda)$ we have

$$T(\lambda)=R\overline{T(-\lambda)}R^{-1}, \tag{12}$$

or

$$a(\lambda)=\overline{a(-\lambda)},\quad b(\lambda)=-\overline{b(-\lambda)}.$$

Here

$$a(\lambda)=-\frac{1}{2i}\det(f_1(x,\lambda),g_2(x,\lambda)). \tag{13}$$

From the formula (13) it follows that the function $a(\lambda)$ may be analytically continued to the half-plane $\mathrm{Im}\lambda>0$ and

$$a(\lambda)=1+o(1),\qquad \text{when } |\lambda|\to\infty,\ \mathrm{Im}\lambda\geq 0. \tag{14}$$

Now we shall assume that the function $a(\lambda)$ has no zeros on the real axis. Then, from (14) we conclude that in the half-plane $\mathrm{Im}\,\lambda > 0$ the function $a(\lambda)$ can have only a finite number of zeros ζ_j, $j=1,\dots,N$; which will further be considered to be single zeros for more simple and clear presentation of the formulae derived. From (13) it follows that

$$f_1(x,\zeta_j) = b_j\, g_2(x,\zeta_j),\ j=1,\dots,N;$$

and on the basis of the formula (7) we conclude that zeros of the function $a(\lambda)$ are eigenvalues of the operator L. Making use of the relation (12) we find that the numbers ζ_j and also b_j, $j=1,\dots,N$; are all located symmetrically with respect to the imaginary axis.

The scattering data for the operator L by our definition will be the following set

$$S = \{ r(\lambda), \zeta_j, m_j, j=1,\dots,N \},$$

where

$$r(\lambda) = \frac{b(\lambda)}{a(\lambda)},\quad m_j = \frac{b_j}{i a(\zeta_j)}.$$

It should be noted that $a(\lambda)$ and $b(\lambda)$ are completely restored from the scattering data. Indeed, the following formulae

$$1+|r|^2 = \frac{|a|^2+|b|^2}{|a|^2} = \frac{1}{|a|^2},$$

$$a(\lambda) = \exp\left\{ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1+|r(\lambda')|^2)}{\lambda-\lambda'}\, d\lambda' \right\} \prod_{j=1}^{N} \frac{\lambda-\zeta_j}{\lambda-\bar\zeta_j},$$

$$\mathrm{Im}\,\lambda > 0; \tag{15}$$

$$a(\lambda) = \lim_{\epsilon\downarrow 0} a(\lambda+i\epsilon),\quad b(\lambda) = a(\lambda)\, r(\lambda),\quad \mathrm{Im}\,\lambda = 0.$$

are valid.

Now a few words on the inverse problem. Its solution is based on a system of integral equations of the Gelfand-Levitan-Marchenko type. We present here only the final formulation, without derivation, which rests on the existence of triangle representations for the solutions $F(x,\lambda)$ and $G(x,\lambda)$ (the so-called transformation operators) and on the theorem of expansion over eigenfunctions of the operator L.

Let s_1 and s_2 be the scattering data for the matrices $A_1(x)$, $H_1(x)$ and $A_2(x)$, $H_2(x)$ respectively, and $G^{(1)}(x,\lambda)$ - the solution of eq. (6), with the coefficients $A_1(x)$, $H_1(x)$ obeying condition (7).

Let us construct the kernels $\mathcal{F}_\ell(x,y)$, $\ell=1,2,3$

$$\begin{aligned}\mathcal{F}_\ell(x,y) = {} & \frac{1}{4\pi}\int_{-\infty}^{\infty}\Big[\frac{(r_2(\lambda)-r_1(\lambda))}{\lambda^{\ell-1}}\, g_2^{(1)}(x,\lambda)^{\tau} g_2^{(1)}(y,\lambda) - \\ & - \frac{(\bar r_2(\lambda)-\bar r_1(\lambda))}{\lambda^{\ell-1}}\, g_1^{(1)}(x,\lambda)^{\tau} g_1^{(1)}(y,\lambda)\Big]\, d\lambda + \\ & + \frac{1}{2}\sum_{j=1}^{N_2}\Big[\frac{1}{(\zeta_j^{(2)})^{\ell-1}}\, m_j^{(2)} g_2^{(1)}(x,\zeta_j^{(2)})^{\tau} g_2^{(1)}(y,\zeta_j^{(2)}) - \\ & - \frac{1}{(\bar\zeta_j^{(2)})^{\ell-1}}\, \bar m_j^{(2)} g_1^{(1)}(x,\bar\zeta_j^{(2)})^{\tau} g_1^{(1)}(y,\bar\zeta_j^{(2)})\Big] - \\ & - \frac{1}{2}\sum_{j=1}^{N_1}\Big[\frac{1}{(\zeta_j^{(1)})^{\ell-1}}\, m_j^{(1)} g_2^{(1)}(x,\zeta_j^{(1)})^{\tau} g_2^{(1)}(y,\zeta_j^{(1)}) - \\ & - \frac{1}{(\bar\zeta_j^{(1)})^{\ell-1}}\, \bar m_j^{(1)} g_1^{(1)}(x,\bar\zeta_j^{(1)})^{\tau} g_1^{(1)}(y,\bar\zeta_j^{(1)})\Big];\end{aligned} \tag{16}$$

where r_g is the row-vector transposed to the column-vector g.

Consider now the equation system for the kernels $K_1(x,y)$ and $K_2(x,y)$

$$K_1(x,y)+\mathcal{F}_1(x,y)+\int_{-\infty}^{x}K_1(x,u)\mathcal{F}_1(u,y)du+$$

$$+\int_{-\infty}^{x}K_2(x,u)\mathcal{F}_2(u,y)du=0, \tag{17}$$

$$K_2(x,y)B_1^{-1}(x)B_1^{-1}(y)+\mathcal{F}_2(x,y)+\int_{-\infty}^{x}K_1(x,u)\mathcal{F}_2(u,y)du+$$

$$+\int_{-\infty}^{x}K_2(x,u)\mathcal{F}_3(u,y)du=0,\quad \text{when } x>y,$$

where

$$K_1(x,y)=K_2(x,y)=0,\quad \text{when } x<y.$$

This system is uniquely solvable, and

$$A_2(x)-A_1(x)=K_1(x,x)J-JK_1(x,x), \tag{18}$$

$$JK_2(x,x)-H_2(x)K_2(x,x)H_1^{-1}(x)J+H_2(x)-H_1(x)=0.$$

At $s_1=0$, $H_1(x)=\frac{1}{16}I$, $G^{(1)}(x,\lambda)=E(x,\lambda)$ system (17) allows us to restore the coeffiients of the operator L from the scattering data s_2.

To conclude this section we note that using the methods developed in [7] it can be shown that to make the functions $1-e^{iu(x)}$ and $w(x)$ rapidly decreasing when $|x|\to\infty$ together with all their derivatives, it is necessary and sufficient that the function $r(\lambda)$ have this same property.

2. The Form Ω in Terms of Scattering Data

Here we obtain an expression for the form Ω in new coordinates using the transformation formulae for differential forms under change of coordinates. From (16), (17) and (18) we find the following expressions for infinite-dimensional analogs of the differentials, i.e., for the variations $du(x)$ and $dw(x)$

$$du(x)=-\frac{1}{i\pi}\int_{-\infty}^{\infty}g(x,\lambda)\,dr(\lambda)\,d\lambda-\frac{2}{i}\sum_{j=1}^{N}(g(x,\zeta_j)\,dm_j+m_j\dot{g}(x,\zeta_j)\,d\zeta_j),$$

where

$$g(x,\zeta)=\frac{g_{12}(x,\zeta)g_{22}(x,\zeta)}{\zeta};$$

and

$$dw(x)=\frac{2i}{\pi}\int_{-\infty}^{\infty}f(x,\lambda)\,dr(\lambda)\,d\lambda+4i\sum_{j=1}^{N}(f(x,\zeta_j)\,dm_j+m_j\dot{f}(x,\zeta_j)\,d\zeta_j),$$

where

$$f(x,\zeta)=g_{22}^{2}(x,\zeta)-g_{12}^{2}(x,\zeta).$$

Let us now insert these expressions into the definition (5) of Ω. Then we obtain

$$\Omega=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(\lambda,\mu)dr(\lambda)\wedge dr(\mu)\,d\lambda\,d\mu+\sum_{j=1}^{N}\int_{-\infty}^{\infty}B_j(\lambda)\,dr(\lambda)\wedge dm_j\,d\lambda+$$

$$+\sum_{j=1}^{N}\int_{-\infty}^{\infty}C_j(\lambda)\,dr(\lambda)\wedge d\zeta_j\,d\lambda+\sum_{\ell,j=1}^{N}D_{\ell j}\,dm_j\wedge dm_\ell+$$

$$+\sum_{\ell,j=1}^{N}E_{\ell j}\,d\zeta_j\wedge d\zeta_\ell+\sum_{\ell,j=1}^{N}F_{\ell j}\,dm_j\wedge dm_\ell;$$

where

$$A(\lambda,\mu)=\frac{1}{2\pi^2\gamma}\int_{-\infty}^{\infty}[g(x,\lambda)(2f(x,\mu)-g'(x,\mu))-(2f(x,\lambda)-g'(x,\lambda))\times$$

$$\times g(x,\mu)]\,dx,$$

and $B_j(\lambda)$, $C_j(\lambda)$, D_{ℓ_j}, E_{ℓ_j}, F_{ℓ_j} are expressed analogously. All the integrals, entering into definition of these coefficients, should be understood in the sense of the theory of distributions. It turns out that all the coefficients can be expressed only in terms of scattering data. Let us demonstrate the corresponding calculations by the example of $A(\lambda,\mu)$.

From eq. (6) it can easily be found that

$$\frac{d}{dx}\{\Psi(x,\lambda),\Psi(x,\mu)\}_1=\left(\frac{\lambda}{\mu}-\frac{\mu}{\lambda}\right)(\Psi_1(x,\lambda)\Psi_1(x,\mu)+$$

$$+\frac{e^{-iu(x)}}{16\lambda\mu}\Psi_2(x,\lambda)\Psi_2(x,\mu)), \qquad (19)$$

$$\frac{d}{dx}\{\Psi(x,\lambda),\Psi(x,\mu)\}_2=\left(\frac{\lambda}{\mu}-\frac{\mu}{\lambda}\right)\left(\frac{e^{iu(x)}}{16\lambda\mu}\Psi_1(x,\lambda)\Psi_2(x,\mu)+\right.$$

$$\left.+\Psi_2(x,\lambda)\Psi_2(x,\mu)\right);$$

where $\Psi(x,\zeta)$ is the solution of eq. (6) with $\lambda=\zeta$ and

$$\{\Psi(x,\lambda),\Psi(x,\mu)\}_1=\frac{\Psi_1(x,\lambda)\Psi_2(x,\mu)}{\lambda}-\frac{\Psi_2(x,\lambda)\Psi_1(x,\mu)}{\mu},$$

$$\{\Psi(x,\lambda),\Psi(x,\mu)\}_2=\frac{\Psi_1(x,\lambda)\Psi_2(x,\mu)}{\mu}-\frac{\Psi_2(x,\lambda)\Psi_1(x,\mu)}{\lambda}.$$

Now having the relation

$$\frac{d}{dx}(\Psi_1(x,\lambda)\Psi_2(x,\lambda))+\frac{1}{16\lambda}\left(e^{-iu(x)}\Psi_2^2(x,\lambda)-e^{iu(x)}\Psi_1^2(x,\lambda)\right)=$$

$$=\lambda\left(\Psi_2^2(x,\lambda)-\Psi_1^2(x,\lambda)\right),$$

which follows from eq. (6), and taking into account formulae (19), we obtain for $A(\lambda,\mu)$

$$A(\lambda,\mu)=\frac{1}{2\pi^2\gamma}\int_{-\infty}^{\infty}\left(\frac{\lambda}{\mu}-\frac{\mu}{\lambda}\right)^{-1}\frac{d}{dx}(\{g_2(x,\lambda),g_2(x,\mu)\}_1\times$$

$$\{g_2(x,\lambda),g_2(x,\mu)\}_2)\,dx.$$

Further, we employ formula (7), the identity $1+|a|^4-|b|^4=2|a|^2$, resulting from (11), and the known relation of the theory of generalized functions:

$$\lim_{N\to\infty}\mathcal{P}\frac{e^{ixN}}{x}=i\pi\delta(x)$$

and arrive at the finite expression for $A(\lambda,\mu)$

$$A(\lambda,\mu)=\frac{2}{i\pi\mu\gamma}|a(\mu)|^2\delta(\lambda+\mu)+\frac{4}{\pi^2\gamma}\mathcal{P}\frac{1}{\lambda^2-\mu^2}a(\lambda)a(\mu)b(-\lambda)b(-\mu), \qquad (20)$$

where the symbol $\mathcal{P}$ stands for principal value of $\frac{1}{x}$ and $\frac{1}{\lambda^2-\mu^2}$.

For other coefficicnts we have

$$B_j(\lambda)=0,\quad F_{\ell_j}=0$$

$$C_j(\lambda)=\frac{16}{i\pi\gamma}\frac{a(\lambda)b(-\lambda)}{\zeta_j^2-\lambda^2}, \qquad (21)$$

$$E_{\ell_j} = \frac{16}{\gamma(\zeta_\ell^2 - \zeta_j^2)} (1-\delta_{\ell_j}),$$

$$D_{\ell_j} = -\frac{8}{\gamma m_\ell \zeta_\ell}\delta_{\ell_j}, \qquad \ell, j=1,\dots,N. \tag{21}$$

From (20), (21) we find the following expression for Ω in terms of the scattering data:

$$\Omega = \frac{2}{i\pi\gamma}\int_{-\infty}^{\infty} \frac{|a(\lambda)|^2}{\lambda} dr(-\lambda)\wedge dr(\lambda)d\lambda + \frac{4}{\pi^2\gamma}\mathcal{P}\int\int_{-\infty}^{\infty} \frac{a(\lambda)a(\mu)\,b(-\lambda)\,b(-\mu)}{\lambda^2-\mu^2}\times$$

$$\times\, dr(\lambda)\wedge dr(\mu)d\lambda\, d\mu - \sum_{j=1}^{N}\frac{16}{i\pi\gamma}\int_{-\infty}^{\infty}\frac{a(\lambda)b(-\lambda)}{\lambda^2-\zeta_j^2} dr(\lambda)\wedge d\zeta_j\, d\lambda +$$

$$+\sum_{\ell,j=1}^{N}\frac{16}{\gamma(\zeta_\ell^2-\zeta_j^2)} d\zeta_j\wedge d\zeta_\ell - \sum_{\ell=1}^{N}\frac{8}{\gamma m_\ell\cdot\zeta_\ell} dm_\ell\wedge d\zeta_\ell ,$$

where we should take into account that the quantities in this expression are not independent but obey the relations

$a(-\lambda)=\overline{a(\lambda)}$, $b(-\lambda) = -\overline{b(\lambda)}$, $r(-\lambda) = -\overline{r(\lambda)}$ and

$\zeta_j = -\bar{\zeta}_j = i\kappa_j$, $\kappa_j > 0$, $m_j = -\bar{m}_j$ $\quad j=1,\dots,n_1$; and

$\zeta_k = -\bar{\zeta}_{k+1} = \lambda_k$, $\mathrm{Re}\lambda_k, \mathrm{Im}\lambda_k > 0$, $m_k = -\bar{m}_{k+1}$ at $k=1,\dots,n_2$

and $n_1 + 2n_2 = N$.

By use of the considerations, quite similar to the corresponding ones of papers /5-6/, it can be shown that the set of variables:

$$\rho(\lambda) = -\frac{8}{\pi\gamma\lambda}\ln|a(\lambda)|, \quad \phi(\lambda) = -\arg b(\lambda), \quad \lambda > 0;$$

$$p_\ell = \frac{1}{\gamma}\ln\kappa_\ell, \quad q_\ell = 8\ln|c_\ell|, \quad \ell=1,\dots,n_1;$$

$$\xi_k = \frac{4}{\gamma}\ln|\lambda_k|, \quad \eta_k = \frac{4}{\gamma}\ln|d_k|,$$

$$\theta_k = \arg\lambda_k, \quad \phi_k = -\frac{16}{\gamma}\arg d_k, \quad k=1,\dots,n_2;$$

$$c_\ell = m_\ell \dot{a}(i\kappa_\ell), \quad d_k = m_k \dot{a}(\lambda_k).$$

is canonical, i.e., the form Ω in these variables has the form

$$\Omega = \int_0^\infty d\rho(\lambda)\wedge d\phi(\lambda)d\lambda + \sum_{\ell=1}^{n_1} dp_\ell\wedge dq_\ell + \sum_{k=1}^{n_2}(d\xi_k\wedge d\eta_k + d\theta_k\wedge d\phi_k).$$

In virtue of the above remark these quantities form the complete set of the canonical variables.

Thus, we have expressed via the scattering data a certain set of variables canonical for the form Ω. In the next section it will be established that these variables play the role of variables of the type action-angle for the Hamiltonian P_0 and total momentum P_1.

3. Trace Identities

We will suppose that $1-e^{iu(x)}$ and $w(x)$ are functions of the Schwarz type. Then for $\ln a(\lambda)$ the following asymptotical expansions

$$\ln a_{.}(\lambda)=\sum_{n=1}^{\infty}\frac{C_n}{\lambda^n},\quad \operatorname{Im}\lambda>0,\quad |\lambda|\to\infty, \tag{22}$$

$$\ln a(\lambda)=\sum_{n=0}^{\infty}C_{-n}\lambda^n,\quad \operatorname{Im}\lambda>0,\quad |\lambda|\to 0 \tag{23}$$

are valid. The coefficients C_n are given by the formulae

$$C_{2n+1}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda^{2n}\ln(1+|r(\lambda)|^2)d\lambda-\sum_{j=1}^{N}\frac{1}{2n+1}(\zeta_j^{2n+1}-\bar{\zeta}_j^{2n+1}),$$

$$C_{2n+2}=0,\quad n=0,1,\ldots \tag{24}$$

$$C_{-2n-1}=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda^{-2n-2}\ln(1+|r(\lambda)|^2)d\lambda-\sum_{j=1}^{N}\frac{1}{2n+1}(\zeta_j^{-2n-1}-$$

$$-\bar{\zeta}_j^{-2n-1}),\qquad C_{-2n-2}=0,\quad n=0,1,\ldots. \tag{25}$$

and the number C_0 is fixed by the choice of branch of the logarithm in (23). These formulae can readily be derived using (15) and the relations (12). On the other hand, the coefficients for the expansion of $\ln a(\lambda)$ in powers of λ can be found with the help of eq. (6). The equalities thus obtained are called the trace identities.

First let us note that for $\operatorname{Im}\lambda>0$ the following relations

$$\begin{aligned}\ln f_{11}(x,\lambda)&=\ln a(\lambda)+i(\lambda-\frac{1}{16\lambda})x+o(1)\qquad x\to-\infty\\ \ln f_{11}(x,\lambda)&=i(\lambda-\frac{1}{16\lambda})x+o(1)\qquad x\to\infty\end{aligned} \tag{26}$$

hold, which opportunely fix also the logarithmic branch in (23). Next, we introduce the function

$$\sigma(x,\lambda)=\frac{d}{dx}\ln f_{11}(x,\lambda)-i(\lambda-\frac{1}{16\lambda}),$$

then

$$\ln a(\lambda)=-\int_{-\infty}^{\infty}\sigma(x,\lambda)dx. \tag{27}$$

This equality, having been derived for $\operatorname{Im}\lambda>0$, due to smoothness may be extended onto the real axis, also. On the other hand, for the function

$$w(x,\lambda)=\frac{f_{21}(x,\lambda)}{f_{11}(x,\lambda)}-i$$

from eq. (6) one can easily obtain that

$$\sigma=\lambda\omega-\frac{1}{16\lambda}e^{-iu}w-\frac{i}{16\lambda}(e^{-iu}-1)-q, \tag{28}$$

and the function $w(x,\lambda)$ satisfies the equation of the Rikatti type:

$$\begin{aligned}w_x=-2i\lambda\omega+\frac{1}{16\lambda}3e^{-iu}w^2-\lambda\omega^2+\frac{i}{8\lambda}e^{-iu}w+2qw+\\ +2qi+\frac{i}{8\lambda}\sin u\end{aligned} \tag{29}$$

and decreases as $x\to\infty$. Here $q(x)=\frac{i}{4}w(x)$. Using the differential equation (29) we see that $w(x,\lambda)$ and, consequently, $\sigma(x,\lambda)$ allow the asymptotical expansions

$$w(x,\lambda)=\sum_{n=1}^{\infty}\frac{f_n(x)}{(2i\lambda)^n},\quad \sigma(x,\lambda)=\sum_{n=1}^{\infty}\frac{g_n(x)}{(2i\lambda)^n},\quad |\lambda|\to\infty;$$

and

$$w(x,\lambda)=\sum_{n=0}^{\infty}r_n(x)\lambda^n,\quad \sigma(x,\lambda)=\sum_{n=0}^{\infty}\zeta_n(x)\lambda^n,\quad |\lambda|\to 0;$$

where the coefficints f_n, g_n, r_n and ζ_n are given by simple recurrent relations which are not presented here. We mention only that

$$f_1=2iq,\quad f_2=2i(q^2-q_x+\frac{i}{8}\sin u),\quad g_1=q^2-q_x+\frac{1}{8}(\cos u-1),$$

$$r_0 = i(e^{iu}-1), \quad r_1 = -16qe^{iu} + 8iu_x e^{iu},$$

$$\frac{i}{8} r_2 = e^{-2iu} + 1 + 16q^2 e^{iu} - 4u_x^2 e^{iu} - 16iu_x q e^{iu} + 16e^{iu}(-q_x + \frac{i}{2}u_{xx}),$$

$$\zeta_0 = -\frac{i}{2}u_x, \quad \zeta_1 = i(\cos u - 1) + 8iq^2 - 2iu_x^2 + 8u_x q + 16i(-q_x + \frac{i}{2}u_{xx}).$$

So, we have arrived at the set of relations

$$(2i)^n C_n = -\int_{-\infty}^{\infty} g_n(x)\,dx, \quad n = 1,2,\ldots,$$

$$C_{-n} = -\int_{-\infty}^{\infty} \zeta_n(x)\,dx, \quad n = 0,1,\ldots,$$

which are called the trace identities. In particular, turning to formulae (3), (4) we get

$$P_0 = \frac{1}{2i\gamma}(C_{-1} - 16C_1) = \frac{1}{4\pi\gamma}\int_{-\infty}^{\infty}(\frac{1}{\lambda^2} + 16)\ln(1+|r(\lambda)|^2)d\lambda + \frac{1}{2i\gamma}\sum_{j=1}^{N}(\frac{1}{\bar\zeta_j} - 16\bar\zeta_j - \frac{1}{\zeta_j} + 16\zeta_j), \tag{30}$$

and

$$P_1 = \frac{1}{2i\gamma}(C_{-1} + 16C_1) = \frac{1}{4\pi\gamma}\int_{-\infty}^{\infty}(\frac{1}{\lambda^2} - 16)\ln(1+|r(\lambda)|^2)d\lambda + \frac{1}{2i\gamma}\sum_{j=1}^{N}(\frac{1}{\bar\zeta_j} + 16\bar\zeta_j - \frac{1}{\zeta_j} - 16\zeta_j), \tag{31}$$

whence we obtain the following expressions for P_0 and P_1 in terms of the canonical variables:

$$P_0 = \int_0^{\infty}(\frac{1}{8\lambda} + 2\lambda)\rho(\lambda)d\lambda + \frac{1}{\gamma}\sum_{\ell=1}^{n_1}(\frac{1}{\kappa_\ell} + 16\kappa_\ell) + \sum_{k=1}^{n_2}\frac{\lambda_k - \bar\lambda_k}{i\gamma}(\frac{1}{|\lambda_k|^2} + 16), \tag{32}$$

$$P_1 = \int_0^{\infty}(\frac{1}{8\lambda} - 2\lambda)\rho(\lambda)d\lambda + \frac{1}{\gamma}\sum_{\ell=1}^{n_1}(\frac{1}{\kappa_\ell} - 16\kappa_\ell) + \sum_{k=1}^{n_2}\frac{\lambda_k - \bar\lambda_k}{i\gamma}(\frac{1}{|\lambda_k|^2} - 16). \tag{33}$$

From (32), (33) it thus follows that P_0 and P_1 depend only on the canonical variables of the type of generalized momenta that justifies our analogy with variables of the type of action-angle in classical mechanics. The Hamiltonian equations in these variables are solved trivially. The solution is as follows:

$$r(\lambda,t) = e^{-2i(\lambda + \frac{1}{16\lambda})t}\, r(\lambda,0),$$

$$\zeta_j(t) = \zeta_j(0), \quad m_j(t) = e^{-2i(\zeta_j + \frac{1}{16\zeta_j})t}\, m_j(0),$$

$$j = 1,\ldots,N.$$

Notice, that we have incidentally establishes the complete integrability of mechanical systems

$$\{\Omega, I_{2n+1}\}_{n=-\infty}^{n=\infty},$$

where

$$I_n = \int_{-\infty}^{\infty} g_n(x)\,dx, \quad n > 0;$$

and

$$I_n = \int_{-\infty}^{\infty} \zeta_{-n}(x)\,dx, \quad n < 0;$$

and the set $\{I_{2n+1}\}_{n=-\infty}^{n=\infty}$ is the complete set of the commuting integrals of motion for these systems.

Now let us turn back to formulae (32), (33). In terms of particles related to our dynamical system, these formulae have a rather simple interpretation. Indeed, the variables $\rho(\lambda)$ and $\phi(\lambda)$ for λ fixed, compose a pair of the canonical variables of the type of "number of particle-phase", so that $\rho(\lambda)$ can be interpreted as the particle density, $p(\lambda) = (\frac{1}{8\lambda} - 2\lambda)$ as the momentum and $h(\lambda) = (\frac{1}{8\lambda} + 2\lambda) = (p^2(\lambda)+1)^{1/2}$ as the corresponding energy. In other words, the first terms in right-hand side of formulae (32), (33) represent the contribution to energy and momentum from the particles of the mass $= 1$. Just in the same way it can be shown that the second terms in (32), (33) represent the contribution from the particles of the mass $8/\gamma$. Now we proceed to the last terms in (32), (33). Unlike the previous cases, the phase space of the corresponding elementary object is four-dimensional, so that it can be represented as a particle with internal degree of freedom. Its energy and momentum

$$h_k = \frac{\lambda_k - \bar{\lambda}_k}{i\gamma}\left(\frac{1}{|\lambda_k|^2} + 16\right),$$

$$p_k = \frac{\lambda_k - \bar{\lambda}_k}{i\gamma}\left(\frac{1}{|\lambda_k|^2} - 16\right),$$

are linked by the relation

$$h_k^2 = p_k^2 + M_k^2,$$

where

$$M = M(\theta) = \frac{16}{\gamma}\sin\theta,$$

i.e., its mass varies from 0 to $\frac{16}{\gamma}$ depending on the internal state. It is possible to write explicitly the solutions to eqs. (1), (2) which describe the motion of finite number of the particles of the second and third types. The equality $\rho(\lambda) = 0$ corresponds to this situation. Under this condition the kernels of system of the type (17) appear to be degenerate and the system can be solved explicitly (see [8]). We write here the final formula

$$u(x,t) = 2i\,\ln\frac{\det(I + V(x,t))}{\det(I - V(x,t))},$$

$$V_{jk}(x,t) = i m_j [\zeta_j + \zeta_k]^{-1} \exp\{-i[\zeta_j - (16\zeta_j)^{-1} + \zeta_k - (16\zeta_k)^{-1}]\,x - 2i[\zeta_j + (16\zeta_j)^{-1}]\,t\}, \; j,k = 1,\dots,N.$$

In the particular case, when $n_1 = 1$, $n_2 = 0$

$$u(x,t) = 4\,\mathrm{arctg}\left(\exp\left\{\frac{\epsilon}{\sqrt{1-v^2}}(x - vt + x_0)\right\}\right),$$

where

$$\epsilon = \mathrm{sign}\, b, \; v = \frac{1-16\kappa^2}{1+16\kappa^2}, \; x_0 = \ln\frac{|b|}{2\kappa}\sqrt{1-v^2}, \; m = ib.$$

For this solution one has

$$\frac{1}{2\pi}\left(u(\infty,t) - u(-\infty,t)\right) = \epsilon$$

and this additional characteristic of the second-type particles can be treated as some sort of charge (see refs. [9,10]). An analogous quantity for the solution with $n_1 = 0$, $n_2 = 1$ equals zero, so it can be said that a particle of the third type represents the relativistic bound state of two second-type particles with opposite charges.

Thus, we have shown that within the classical field theory our dynamical system can have elementary excitations of the following three types:

1. The neutral particle with mass 1;
2. The charged particle with mass $8/\gamma$;
3. The bound state of two particles of the second type with opposite charges, with mass varying from 0 to $16/\gamma$ depending on the internal state. In the standard approach, based on the perturbation expansion in the parameter γ , we would find that with the field $u(x,t)$ there is connected only one sort of particles - excitations of the first type.

At present we are studying the question as to what extent the obtained results will hold in the quantum version of the considered model.

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