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1. Introduction

To explain the experimental data on the pion form factor $F_\pi(t)$ already for $t > 1$ GeV^2 [1] one must somehow take into account the inelastic contributions. There are developed several indirect methods [2,3] how to do it. But all of them are model-dependent and their correctness can be verified only through the comparison of obtained formulas with the existing experimental data. So, with an appearance of new experimental data the theoretical problem faces us of an extension of $F_\pi(t)$ to still higher energies. Moreover, although at present we have experimental information about the pion form factor up to $t = 9.0$ GeV^2, the data for values of 2.5 GeV$^2 < t <$ < 9.0 GeV^2 cannot be (without some theoretical and model-dependent assumptions) unambiguously identified as being pionic events [4,5] and we can doubtless confide only in the data for $t < 2.5$ GeV^2. For that reason, even if some explicit formula of $F_\pi(t)$ is consistent with the existing experimental data in some restricted region of t we are not allowed to believe very much in its predictable ability values of t.

In this paper we would like to demonstrate the method of prediction of the time-like region form factor tail which is independent of the afore-mentioned methods of the inclusion of inelastic contributions. More concretely, combining the real and imaginary parts of $F_\pi(t)$ from resonant region (which are given by explicit formulas obtained by the dispersion method) with sum rules derived for them we interpolate in a definite sense the pion form factor between the elastic and asymptotic regions.

Because, owing to the previous method, $|F_\pi(t)|$ is now known along the whole cut (in some reasonable approximation, of course) the upper bound for the space-like region values of $F_\pi(t)$ and the support for existence of form factor zeros from the so-called modulus representation are obtained.

Finally, the space-like region behaviour of $F_\pi(t)$ is predicted using the unsubtracted dispersion relation and combined knowledge of $\operatorname{Im} F_\pi(t)$ from the explicit formula valid in resonant region and from sum rules.

2. Behaviour of the Real and Imaginary Parts of Form Factor in Resonant Region

It is well known that the electromagnetic pion form factor can be written in the following phase representation (for more details see ref. /1/)

$$F_\pi(t) = P_n(t) \exp\left\{ \frac{t}{\pi} \int_4^\infty \frac{\arg F_\pi(t')}{t'(t'-t)} dt' \right\}, \qquad (1)$$

where $P_n(t)$ (normalized to $P_n(0) = 1$) is an arbitrary polynomial and its degree can be determined only by some physical requirements. Commonly, it is taken not to destroy the asymptotic behaviour of $F_\pi(t)$ assumed in deriving (1). So, in our case $P_n(t)$ takes the following concrete form

$$P_1(t) = 1 + A \cdot t, \qquad (2)$$

where A is an unknown constant which can be determined only through the comparison of $F_\pi(t)$ with experimental data.

To obtain the explicit form of $F_\pi(t)$ from (1) we shall choose some concrete parametrization for $\arg F_\pi(t)$. There are some instructions how to do it. Namely, in elastic region $\arg F_\pi(t)$ is identical with the isovector P-wave $\pi\pi$ scattering phase shift δ_1^1 and therefore our parametrization must possess all its basic properties.

One can see immediately that the following form

$$\arg F_\pi(t) = \frac{1}{2i} \ln \frac{(1+q^2)(q_\rho^2-q^2)+iaq^3}{(1+q^2)(q_\rho^2-q^2)-iaq^3} \qquad (3)$$

has correct threshold and also (due to the ρ -meson) resonant behaviour. $q = \frac{1}{2}\sqrt{t-4}$ is the c.m. momentum, $q_\rho = \frac{1}{2}\sqrt{m_\rho^2-4}$ and a is a constant which, following the requirement

$$\lim_{t\to m_\rho^2} \frac{\frac{m_\rho \Gamma_\rho}{m_\rho^2-t}}{\operatorname{tg}\{\arg F_\pi(t)\}} = 1, \qquad (4)$$

can be expressed through ρ -meson parameters (m_ρ, Γ_ρ) by means of the following relation

$$a = \frac{\Gamma_\rho}{2}\left(1 + \frac{1}{q_\rho^2}\right)^{3/2}. \qquad (5)$$

Now inserting (3) into (1) and calculating the integral by means of the theory of residues we obtain the following explicit form of the pion form factor

$$F_\pi(t) = P_1(t) \frac{(q-q_1)}{(q+q_2)(q+q_3)(q+q_4)} \frac{(i+q_2)(i+q_3)(i+q_4)}{(i-q_1)}, \qquad (6)$$

where q_i $(i=1,\dots,4)$ are the positions of the branch points of the integrand in (1) and they are roots of the numerator of the logarithm in (3). The connection between q and the ρ -meson parameters looks as follows

$$q_1 = \frac{i}{-\sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3}}, \quad q_3 = \frac{i}{-\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}},$$

$$q_2 = \frac{i}{\sqrt{z_1}-\sqrt{z_2}+\sqrt{z_3}}, \quad q_4 = \frac{i}{\sqrt{z_1}+\sqrt{z_2}-\sqrt{z_3}}, \qquad (7)$$

where z_ν $(\nu = 1,\dots,3)$ are solutions of the cubic equation

$$z^3 - \frac{(q_\rho^2-1)}{2q_\rho^2} z^2 + \frac{(q_\rho^2+1)^2}{16q_\rho^4} z - \frac{a^2}{64q_\rho^4} = 0 \qquad (8)$$

and the signs of $\sqrt{z_\rho}$ in (7) are taken to satisfy the following condition

$$\sqrt{z_1}\sqrt{z_2}\sqrt{z_3} = -\frac{a}{8q_\rho^2}. \qquad (9)$$

The best fit of the existing experimental data in space-like and time-like regions simultaneously by our formula (6) (leaving m_ρ, Γ_ρ, A as free parameters) gives

$$m_\rho = 778 \pm 4 \text{ MeV}$$

$$\Gamma_\rho = 152 \pm 4 \text{ MeV}$$

$$A = 0.0027 \pm 0.0003 \, [\mu^{-2}] \qquad (10)$$

and graphically is shown in fig. 1.

Then numerically, using the values of parameters given by (10), from (7) we get

$$q_1 = -i0.960504 \, [\mu]$$

$$q_2 = -2.565913 + i0.289811 \, [\mu]$$

$$q_3 = +i1.048006 \, [\mu]$$

$$q_4 = +2.565913 + i0.289811 \, [\mu]. \qquad (11)$$

The behaviour of the real and imaginary parts of the pion form factor (resulted from (6)) is given by the expressions

$$\mathrm{Re}\, F_\pi(t) = \frac{P_1(t)R}{D(q)} \{ q^4 + (2\beta_1\beta_2 + \beta_1\beta_3 - 2\beta_2\beta_3 - \alpha_2^2 - \beta_2^2) q^2 -$$

$$- \beta_1\beta_3(\alpha_2^2 + \beta_2^2) \}, \qquad (12a)$$

$$\mathrm{Im}\, F_\pi(t) = \frac{P_1(t)R}{D(q)} \{ (\beta_1 - 2\beta_2 - \beta_3) q^3 -$$

$$-[(\alpha_2^2+\beta_2^2)(\beta_1-\beta_3)+2\beta_1\beta_2\beta_3]q\}, \qquad (12b)$$

where

$$R=[-\alpha_2^2-(1+\beta_2^2)^2]\,\frac{(1+\beta_3)}{(1+\beta_1)}$$

$$D(q)=[(q^2+\alpha_2^2+\beta_2^2)-2\alpha_2 q][(q^2+\alpha_2^2+\beta_2^2)+2\alpha_2 q](q^2+\beta_3^2)$$

and for the sake of simplicity we have used the redefinition

$$\beta_1=-\mathrm{Im}\,q_1$$

$$\alpha_2=-\mathrm{Re}\,q_2=\mathrm{Re}\,q_4$$

$$\beta_2=\mathrm{Im}\,q_2=\mathrm{Im}\,q_4$$

$$\beta_3=\mathrm{Im}\,q_3\ .$$

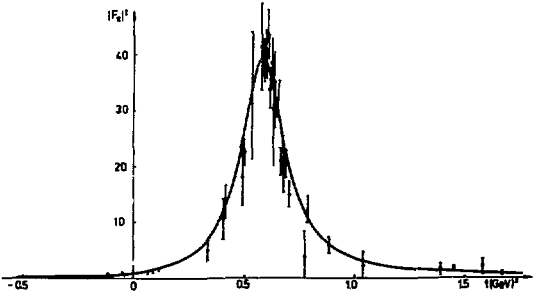

Fig. 1. Theoretical predictions for $|F_\pi(t)|^2$ by means of (6) with (10) in resonant region.

The pion charge radius calculated by means of (6) with (10) takes the value

$$<r^2>^{1/2} = 0.68 \pm 0.01\,F. \tag{13}$$

Here we would like to note that we did not take into account $\rho-\omega$ interference in our considerations because in such a way, as it was used to fit the ORSAY-data [6], it is inconsistent with reality condition

$$F_\pi^*(t) = F_\pi(t^*). \tag{14}$$

However, we believe that this neglect of $\rho-\omega$ mixing in our scheme has no an extraordinary effect for our eventual results in which we are interested in this paper further.

3. Prediction of Time-Like Region Form Factor Tail

Taking into account theoretical [7-9] and also experimental [5] (at least in a finite energy interval) indications that

$$\lim_{t\to\infty} F_\pi(t) = 0 \tag{15}$$

one can write for $F_\pi(t)$ the unsubtracted dispersion relation

$$F_\pi(t) = \frac{1}{\pi}\int_4^\infty \frac{\mathrm{Im}\,F_\pi(t')}{t'-t}\,dt' \tag{16}$$

from which the following sum rules can be obtained

$$1 = \frac{1}{\pi}\int_4^\infty \frac{\mathrm{Im}\,F_\pi(t')}{t'}\,dt', \tag{17a}$$

$$\frac{<r^2>}{6} = \frac{1}{\pi}\int_4^\infty \frac{\mathrm{Im}\,F_\pi(t')}{t'^2}\,dt'. \tag{17b}$$

Further, from (15) one can see immediately that the real and imaginary parts of the pion form factor have the following asymptotic behaviour

$$\lim_{t\to\infty} | t^M \mathrm{Re}\, F_\pi(t)| < \infty , \tag{18a}$$

$$\lim_{t\to\infty} | t^N \mathrm{Im}\, F_\pi(t)| < \infty , \tag{18b}$$

where in principle M can be different from N (then the asymptotic behaviour of $F_\pi(t)$ is determined by means of that which takes the smaller value) and both of them are positive.

Next, the real and imaginary parts of $F_\pi(t)$ for $t > t_c$ are represented by the following formulas

$$\mathrm{Re}\, F_\pi(t) = \mathrm{Re}\, F_\pi(t_c)\left(\frac{t_c}{t}\right)^M, \tag{19a}$$

$$\mathrm{Im}\, F_\pi(t) = \mathrm{Im}\, F_\pi(t_c)\left(\frac{t_c}{t}\right)^N, \tag{19b}$$

where constants $\mathrm{Re}\, F_\pi(t_c)$ and $\mathrm{Im}\, F_\pi(t_c)$ equal expressions (12a,b) taken at $t = t_c$, respectively, and t_c, M, N are unknown parameters.

Of course, in principle, one could choose more complicated forms for the real and imaginary parts of $F_\pi(t)$ that (19a,b). However, as it will be seen later, all the existing data for $t > t_c$ can be described already by the (12b) with (17a,b) we get two equations for the determination of t_c and N from which we obtain numerically

$$t_c = 36.0\, [\mu^2] \qquad N = 6.24 . \tag{20}$$

To predict the tail $|F_\pi(t)|$ we are in need of $\mathrm{Re}\, F_\pi(t)$ also. For that reason we must determine the parameter M in (19a). Applying the Cauchy theorem to the function $\frac{F_\pi(t)}{t\sqrt{t-4}}$ one can get the following sum rule for $\mathrm{Re} F_\pi(t)$

$$\int_4^\infty \frac{\mathrm{Re}\, F_\pi(t')}{t'\sqrt{t'-4}}\, dt' = \frac{\pi}{2} \tag{21}$$

from which, using (12a), (19a) and taking t_c already given by (20) we obtain the equation for M . Numerically, we get

$$M = 1.10 . \tag{22}$$

By means of (19a,b), (20) and (22) one can express the time-like region form factor tail in the following definitive form

$$|F_\pi(t)| = \{[-2.695 (\frac{36.0}{t})^{1.10}]^2 + [3.509 (\frac{36.0}{t})^{6.24}]^2\}^{1/2} \tag{23}$$

from which it is straightforward to see that the pion form factor has $1/t$ – behaviour for $t \to \infty$. The comparison of (23) with experimental data is shown in fig. 2.

4. Existence of Zeros and Upper Bound on Space-Like Region Form Factor Values

We suppose that the pion form factor has only a finite number of zeros. Further, let us consider the new function

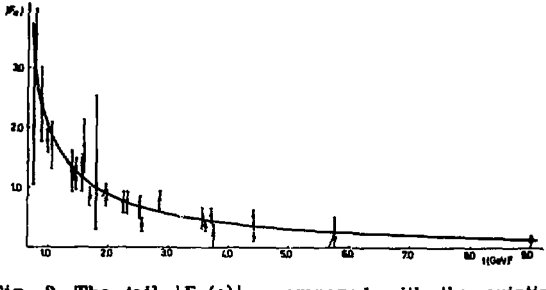

Fig. 2. The tail $|F_\pi(t)|$ compared with the existing experimental data. $\bar{I}''$ denote FRASCATI-data /5/.

$$\frac{\ln F_\pi(t)}{\sqrt{t-4}} \tag{24}$$

which is analytic with the same cut as $F_\pi(t)$ and in addition it has cuts due to the zeros of the form factor. Then an application of the *Cauchy formula* to (24) yields the following well-known modulus representation[10,11]

$$F_\pi(t) = B(t)\exp\{\frac{\sqrt{4-t}}{\pi}\int_4^\infty \frac{\ln|F_\pi(t')|}{(t'-t)\sqrt{t'-4}}dt'\} \tag{25}$$

with

$$|B(t)| \le 1 . \tag{26}$$

Then extracting $B(t)$ from (25) we obtain the upper bound for space-like region values of the pion form factor (see fig. 3)

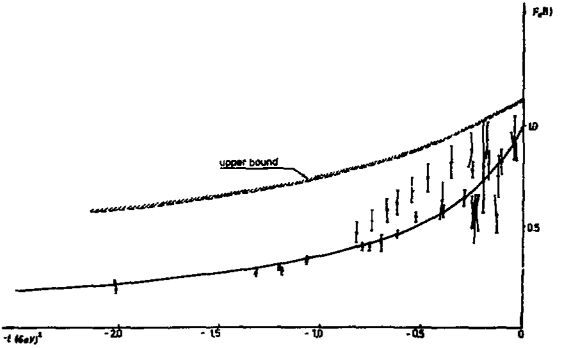

Fig. 3. Upper bound on space-like region values of form factor from (27) and the predictions of $F_\pi(t)$ from (16) with (12b) and (19b). ⌶ denote new data from ref.[12] .

$$|F_\pi(t)| \leq \exp\{\frac{\sqrt{4-t}}{\pi}\int_4^\infty \frac{\ln|F_\pi(t')|}{(t'-t)\sqrt{t'-4}}dt'\} \qquad (27)$$

and for $t=0$ the following sum rule

$$I_1 = \frac{2}{\pi}\int_4^\infty \frac{\ln|F_\pi(t')|}{t'\sqrt{t'-4}}dt' \geq 0, \qquad (28)$$

where the equality sign holds if and only if $F_\pi(t)$ has no zeros.

There is another sum rule for testing of existence of the form factor zeros

$$I_2 = \int_4^\infty \frac{\ln|\frac{F_\pi(t')}{F_\pi(4)}|}{(t'-4)^{3/2}}dt' \geq 0 \qquad (29)$$

which is also obtained [11] from the modulus representation (25) by using the identity of the form factor phase for $4 \leq t \leq 16$ with the $\pi\pi$ scattering phase shift δ_1^1 and its threshold behaviour. The equality sign in (29) again holds if and only if $F_\pi(t)$ has no zeros in the cut t -plane.

Now using the values of $|F_\pi(t)|$ for $4 \leq t \leq t_c$ given by (6) with (10) and for $t > t_c$ using the tail (23), from (28) and (29) we get

$$I_1 = 1.7092 - 0.9300 \simeq 0.78 \qquad (30)$$

$$I_2 = 1.1284 + 0.0016 = 1.13\ [2\mu]^{-1}, \qquad (31)$$

where in the calculation of integral (29) we have used the dimensionless variable $x = \frac{t}{4\mu^2}$ to compare our value with that ($2\mu I_2 \simeq 0.2$) roughly estimated in ref. [11] .

Both results (30) and (31) indicate that the zeros of the electromagnetic pion form factor do exist.

5. Prediction of Space-Like Region Behaviour

The space-like region behaviour of the pion form factor can in principle be predicted from the modulus representation (25). Before doing so, the number of form factor zeros and their positions must be found to specify the explicit form of the *function* $B(t)$.

Although, by means of the so-called logarithmic residue and its modified version (taking into account (6) for $4 \leq t \leq t_c$ *and for* $t > t_c$ the complex function consisting of (19a,b) together with the reality condition (14)), we can, in principle, carry out this non-trivial program, we know beforehand that results obtained in such a way must not be identical with those of the real pion form factor and for that reason we shall not realize it.

There is another simpler way how to do it which also allows one to verify the correctness of our model to a certain extent.

We know the behaviour of $\mathrm{Im}\, F_\pi(t)$ for $4 \leq t \leq t_c$ from (12b) and for $t > t_c$ it is given by (19b) with the values of parameters (20) determined by means of the sum rules. Now inserting this combined knowledge about the imaginary part of the pion form factor into the unsubtracted dispersion relation (16) we predict the space-like region behaviour of $F_\pi(t)$ as it is shown in fig. 3.

The evident agreement of our predictions with the existing *experimental data (see fig. 3)* confirms that $\mathrm{Im}\, F_\pi(t)$ given by (12b) and (19b) with (20) is a good approximation *of the imaginary part of the real pion form factor.*

6. Conclusions

In this section we summarize the main results of our work. Using the phase representation and a reasonable parametrization *of the form* factor phase we have found the explicit formula for $F_\pi(t)$ possessing all the basic properties and describing the data *in finite* time-like and space-like regions simultaneously. Moreover, from this explicit formula we have obtained the behaviour of the real and imaginary parts of the pion form factor for $t < 36.0\ [\mu^2]$ and the value of the pion charge radius. Further, combining this result with special assumptions in the asymptotical region and sum rules derived for $\mathrm{Im}\,F_\pi(t)$ and $\mathrm{Re}\,F_\pi(t)$ we have predicted the behaviour of $|F_\pi(t)|$ for $t_c < t < +\infty$.

By means of the modulus representation we have tested the existence of form factor zeros, and an upper bound on the space-like region values of the form factor has been found.

Finally, using combined knowledge of $\mathrm{Im}\,F_\pi(t)$ and *unsubtracted dispersion relation* we have *predicted the* space-like region behaviour of $F_\pi(t)$ which is in a good agreement with the existing experimental data.

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