СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E2 - 7977

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ON CLEBSH-GORDAN EXPANSION FOR O(2h= 1,1)

1974

ЛАБОРАТОРИЯ
ТЕОРЕТИЧЕСНОЙ ФИЗИКИ

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1. INTRODUCTION

A. Physical motivation

The problem of the direct product decomposition of two spin 0 unitary representations of the supplementary series of SO(2h+1,1) (h - positive integer) arose in the study of a conformal covariant model of a self interacting (quantized) scalar field (see [M 1]). The model originates from the Green function formulation of a (renormalizable) ϕ^3 theory in six-dimensions given by Symanzik [S 1] some 13 years ago; the discussion of conformal covariance was facilitated by a modification of the equation for the propagator involving the stress energy tensor [M 3]. It is the desire to incorporate this 6-dimensional model along with the physical 4-dimensional case that led us to work from the outset with the general case of 2h-space-time dimensions. This generality has also the advantage that we are able to check our formulas for $h=1$ with the known case of SO(3,1) (see [N 1]). We are concerned throughout with the Euclidean formulation of the theory (see, e.g., [S 2]) in which the Lorentz group is replaced by SO(2h) and the covariance under infinitesimal conformal transformations of $SO(2h,2)/\mathbb{Z}_2$ is extended to global $O^{\uparrow}(2h+1,1)$ covariance (the arrow $\uparrow$ indicates that we do not consider transformations which change the sign of the 2h + 2nd axis).

It is shown in [M 1] that the conformal expansion of the Euclidean Green functions allows one to diagonalize and solve the infinite set of integral equations for these functions. As a result one obtains conformal covariant operator product expansions which have been suggested in a number of papers (see [F1], [M5] and references therein). It is also related to the problem of duality considered in [P1] .

For background and further references on the conformal group and its applications to quantum field theory the reader may consult the recent reviews [M2],[T1] and [R1] . Some preliminary results of this paper are quoted in Appendix A of [T2].

B. A two-parameter family of infinite dimensional representations of the Euclidean conformal group

In order to fix notation and terminology we start with a brief description of a two-parameter family of (irreducible) representations $\chi = [\ell, c]$ of $O^{\uparrow}(2h+1,1)$ ($\ell = 0,1,2, \ldots$; c is an arbitrary complex number). (A complete classification of the unitary irreducible representations of SO (n , 1) is given in [H 1] [O 2] . In the case of $n=2h+1$ these representations are labelled by one continuous and h discrete parameters.)

We shall introduce a space C_χ of infinitely differentiable (symmetric, traceless)tensor-valued functions

$$f(x) = f_{\mu_1 \ldots \mu_\ell}(x_1, \ldots, x_{2h}) \qquad (\mu_i = 1, \ldots, 2h)$$

on $\mathbb{R}^{2h}$, whose behaviour at infinity is dictated by conformal covariance. In order to reveal the meaning of the latter statement we shall first display the action of the representation χ on C_χ.

The Euclidean conformal group $O^{\uparrow}(2h+1,1)$ acts transitively on the compactification S^{2h} of $\mathbb{R}^{2h}$. Here S^{2h} is the unit sphere in 2h+1 dimensions related to $\mathbb{R}^{2h}$ through the stereographic projection

$$\hat{\xi}_\mu = \frac{2x_\mu}{1+x^2}, \quad \mu = 1,\dots,2h, \qquad x^2 = x_1^2 + \dots + x_{2h}^2,$$

$$\hat{\xi}_{2h+1} = \frac{1-x^2}{1+x^2} \qquad (\hat{\xi}_\mu \hat{\xi}_\mu + \hat{\xi}_{2h+1}^2 = 1). \tag{I.1}$$

Its action is generated by the following transformations in $\mathbb{R}^{2h}$:

a) translations and Euclidean rotations: $x' = a + \Lambda x$

$a = (a_1, \dots, a_{2h})$, $\Lambda \in O(2h)$;

b) dilatations $x' = \rho x$, $\rho > 0$;

c) conformal inversion

$$Rx = -\frac{x}{x^2} \tag{I.2}$$

The special conformal transformations are given by

$$x' = R T_b R x = \frac{x - x^2 b}{1 - 2bx + b^2 x^2}, \tag{I.3}$$

where T_b is a translation: $T_b x = x + b$.

We shall define the representation $\chi = [\ell, c]$ of the generating transformations a), b), c) of $O(2h+1,1)$ in C_χ in the following way (cf. [T1]):

$$[U(a,\Lambda) f](x) = \Lambda^{\otimes \ell} f(\Lambda^{-1}(x-a)) \tag{I.4}$$

$$[U(\rho) f](x) = \rho^{-h-c} f\left(\frac{x}{\rho}\right) \tag{I.5}$$

$$[U(R)f](x) = \frac{\tau(x)^{\otimes \ell}}{(x^2)^{c+h}} f(Rx), \tag{I.6}$$

where

$$\tau(x)_{\mu\nu} = -\delta_{\mu\nu} + 2\frac{x_\mu x_\nu}{x^2} , \quad \tau^2 = 1 \tag{I.7}$$

and

$$[B^{\otimes \ell} f(x)]_{\mu_1 \ldots \mu_\ell} = B_{\mu_1\nu_1} \ldots B_{\mu_\ell \nu_\ell} f_{\nu_1 \ldots \nu_\ell}(x) \tag{I.8}$$

($B = \Lambda, \tau$). [The exponent $h+c$ in (I.5) is often denoted by d and called (scale) dimension of f .]

Now we are in a position to determine the behaviour at infinity of the tensor functions $f(x) \in C_\chi$. Conformal covariance implies that if $f \in C_\chi$ then also $U(R)f \in C_\chi$. Using (I.6) and the involution property (I.7) of $\tau(x)$, we find that

$$(1+x^2)^{h+c} \tau(x)^{\otimes \ell} [U(R)f](x) \underset{x\to\infty}{\longrightarrow} f(0).$$

Accordingly, we shall postulate that for any $f(x) \in C_\chi$ there exists a finite (tensor-valued) limit

$$\lim_{x\to\infty} (1+x^2)^{h+c} \tau(x)^{\otimes \ell} f(x) = A^f . \tag{I.9}$$

We shall see in Sec.II that the representations χ , so defined, can be extended (by an appropriate completion of C_χ) to unitary (irreducible) representations of $0^\uparrow(2h+1,1)$ for the following values of c and ℓ:

c - pure imaginary ($c = i\sigma$), ℓ - arbitrary (principal series) (I.10a)

$$\left.\begin{array}{lll} -h<c<h & \ell=0 & (h\geq 1) \text{ and} \\ 1-h<c<h-1 & \ell=1,2,\dots & (h\geq 2) \\ c\neq 0 & & \end{array}\right\} \text{(supplementary series)} \quad \text{(I.10b)}$$

(Note that our terminology follows the analogy with the Lorentz group $O^{\uparrow}(3,1)$ and differs from the terminology adopted in [K1] [O2]).

C. Outline of results

We consider the problem of decomposition of the direct product of two unitary representations of the supplementary series

$$\chi_{01} \otimes \chi_{02}, \quad \chi_{0a} = [0, c_a], \quad a = 1, 2 \tag{I.11}$$

into irreducible unitary representations.

In other words we would like to expand each

$$f(x_1, x_2) \in C_{\chi_{01}} \otimes C_{\chi_{02}} \tag{I.12}$$

in functions $F_{\chi}(x)$ transforming according to the unitary representation χ of $O^{\uparrow}(2h+1,1)$.

For $|c_1|+|c_2| \leq h$ the direct product (I.11) is expanded in representations $\chi = [\ell, c=i\sigma]$ of the principal series (cf. [N1] and Appendix A to [M1]):

$$f(x_1, x_2) = \oint d\chi \int dx \; V(x_1 c_1, x_2 c_2; x\, \tilde{\chi})\, F_{\chi}(x). \tag{I.13}$$

Here $\tilde{\chi}$ is the representation dual to $\chi = [\ell, c]$. x);

x) For $h = 1$ the representation $[\ell, 0]$ ($\ell > 0$) should be identified with the irreducible representation $[\ell, c] \oplus [\ell, -c]$ of the full Lorentz group.

$$\chi = [l, c] \Rightarrow \tilde{\chi} = [l, -c] \quad ; \tag{I.14}$$

$$\oint d\chi = \sum_{l=0}^{\infty} \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \rho_l(\sigma) = \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} \rho_l(\sigma) , \tag{I.15}$$

where

$$\rho_l(\sigma) = \frac{2(l+h-1)!}{(2\pi)^h\, l!} \left| \frac{\Gamma(h-1+i\sigma)}{2\Gamma(i\sigma)} \right|^2 [\sigma^2 + (h+l-1)^2] \tag{I.16}$$

is the Plancherel measure (cf [H2]); finally, $V(x_1c_1, x_2c_2; x\tilde{\chi})$ are the Clebsh-Gordan kernels.

We start in Sec. II by defining an invariant bi-linear form of the type

$$(f,g)_\chi = \int dx_1 \int dx_2\, f(x_1)\, G_{\tilde{\chi}}(x_1 - x_2)\, g(x_2) \tag{I.17}$$

on $C_\chi \times C_\chi$.

The 2-point function $G_{\tilde{\chi}}$ and the kernels V are determined uniquely(up to a constant factor) from conformal invariance . In writing down explicit expressions for the Green functions G_χ and the kernels V it is convenient to use technique of homogeneous polynomials(see, e.g.,[T3] [O1] [Z1]) instead of multiple tensor indices. We write

$$V(x_1c_1, x_2c_2; x\tilde{\chi}z) = \frac{1}{\sqrt{l!}} V_{\mu_1\ldots\mu_l}(x_1c_1, x_2c_2, x\tilde{\chi})\, z^{\mu_1}\ldots z^{\mu_l}$$

$$\equiv V(x_1c_1, x_2c_2, x\tilde{\chi}) \frac{z^{\otimes l}}{\sqrt{l!}} , \tag{I.18}$$

$$G_{\tilde{\chi}}(x_1 - x_2; z_1, z_2) = \frac{1}{l!} z_1^{\otimes l} G_{\tilde{\chi}}(x_1 - x_2)\, z_2^{\otimes l} \tag{I.19}$$

$$= \frac{1}{l!} z_1^{\mu_1}\ldots z_1^{\mu_l} G_{\tilde{\chi}}(x_1 - x_2)_{\mu_1\ldots\mu_l\nu_1\ldots\nu_l}\, z_2^{\nu_1}\ldots z_2^{\nu_l} ,$$

where z, z_1, z_2 are (complex) isotropic vectors:

$$z^2 = z_1^2 + \ldots + z_{2h}^2 = 0 \, . \tag{I.20}$$

It is easily seen, for instance, that the homogeneous polynomial (I.18) is in one-to-one correspondence with the symmetric traceless tensors $V_{\mu_1 \ldots \mu_\ell}$. Indeed, each polynomial function $f(z)$ on the cone (I.20) can be extended in a unique way to a harmonic polynomial in $\zeta \in \mathbb{C}^{2h}$ by setting

$$f(\zeta) = \frac{2^\ell \ell! \, \Gamma(2h+\ell-2)}{\Gamma(2h+2\ell-2)} \left(\zeta^2 \partial_z^2\right)^{\ell/2} P_\ell^{(h-\frac{3}{2}, h-\frac{3}{2})} \left(\frac{\zeta \partial_z}{\sqrt{\zeta^2 \partial_z^2}} \right) f(z), \tag{I.21}$$

$$(\Delta_\zeta f(\zeta) = 0)$$

where $P_\ell^{(\alpha,\beta)}(t)$ is the Jacobi polynomial satisfying the differential equation

$$\left[(1-t^2)\frac{d^2}{dt^2} + \left[\beta - \alpha - (\alpha+\beta+2)t\right]\frac{d}{dt} + \ell(\ell+\alpha+\beta+1)\right] P_\ell^{(\alpha,\beta)}(t) = 0$$

and the normalization condition

$$\frac{d^\ell}{dt^\ell} P_\ell^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+\beta+2\ell+1)}{2^\ell \, \Gamma(\alpha+\beta+\ell+1)} \, .$$

Then the inverse formula to (I.18) is

$$V_{\mu_1 \ldots \mu_\ell}(\ldots \, z \tilde{z}) = \frac{1}{\sqrt{\ell!}} \frac{\partial}{\partial \zeta_{\mu_1}} \cdots \frac{\partial}{\partial \zeta_{\mu_\ell}} V(\ldots \, z \tilde{z} \zeta) \, .$$

The contraction of two tensors f and g can be written in terms of the polynomials (I.21) as

$$f_{\mu_1 \ldots \mu_\ell} \, g_{\mu_1 \ldots \mu_\ell} = f(\partial_z) \, g(z) = g(\partial_z) f(z) \, .$$

Further, in Sec. II we study the implications of the positivity condition G_{χ} and establish the unitarity of the representations of the supplementary series (I.10b) . The method used-the $SO(2h-1)_p$ expansion of $\tilde{G}_{\tilde{\chi}}(p)$ allows also to find restrictions on c for Minkowski space positivity condition of the corresponding Wightman functions. One has to use different normalization conventions for the two alternative interpretations of G_{χ} : first, when G_{χ} is regarded as the kernel of a $O(2h+1,1)$ invariant bi-linear form; second, when it is considered as analytic continuation to the Euclidean region of the τ - function of two tensor fields. In the first case, the adopted convention [see Eqs. (II.24) (II.25) below] implies

$$\int dy\, G_{\chi}(x_1-y)\, G_{\tilde{\chi}}(y-x_2) = 1\!\!1\,\delta(x_1-x_2), \tag{I.22}$$

where $1\!\!1$ stands for the unit operator in the space of symmetric traceless tensors of rank ℓ . In the second case, the two-point function can be normalized in such a way that Wightman positivity is satisfied for

$$\ell=0,\ c\geq -1;\quad \ell=1,2,\ldots\quad c\geq \ell+h-2\,. \tag{I.23}$$

The choice of normalization [given by Eqs. (II.40), (II.41)] guarantees also the validity of a number of other desirable properties of the two-point Wightman function listed in Sec. II.D.

Sec. III is devoted to the evaluation of the normalization factor N_ℓ of the invariant 3-point function V (III.4), which plays the role of a Clebsh-Gordan kernel. The normalization of V is determined by requiring a symmetry property with respect to "amputation of external legs" [Eqs. (III.7-8)] and from the Plancherel formula, which implies that Eq. (I.13) can be inverted in the form (III.12).

II. Invariant bi-linear forms. Supplementary series of unitary representations of $O^\uparrow(2h+1,1)$

A. Covariant two-point kernels and their Fourier transforms

According to (I.19) we are looking for a function $G(x_1,x_2;z_1,z_2)$ which is a homogeneous polynomial in each of the isotropic 2h-vectors z_1 and z_2 of degree ℓ and transforms covariantly under the representation $\chi=[\ell,c]$ of $O^\uparrow(2h+1,1)$. In other words we require that

$$G(\Lambda x_1+a, \Lambda x_2+a; \Lambda z_1, \Lambda z_2) = G(x_1,x_2;z_1,z_2) \quad \text{for } \Lambda\in O(2h),\ a=(a_1,\dots,a_{2h}); \qquad (II.1)$$

$$\rho^{2(h+c)} G(\rho x_1,\rho x_2;z_1,z_2) = G(x_1,x_2;z_1,z_2) \text{ for } \rho>0; \qquad (II.2)$$

$$\frac{1}{(x_1^2x_2^2)^{h+c}} G(Rx_1, Rx_2;\ r(x_1)z_1,\ r(x_2)z_2) = G(x_1,x_2;z_1,z_2), \qquad (II.3)$$

where the conformal inversion R and its vector representation $r(x)$ are defined by (I.2) and (I.7).

The general form of G, satisfying the above condition is

$$G = G_\ell(x_{12}; z_1, z_2) = \frac{n(\ell)}{(2\pi)^h}\left(\frac{2}{x_{12}^2}\right)^{h+\ell}\frac{1}{\ell!}\left(z_1\, r(x_{12})\, z_2\right)^\ell . \tag{II.4}$$

where

$$x_{12} = x_1 - x_2 , \quad z_1\, r(x_{12})\, z_2 = z_1 z_2 - 2\frac{(z_1 x_{12})(x_{12} z_2)}{x_{12}^2} \tag{II.5}$$

and $n(\ell)$ is a normalization constant. The Euclidean and dilatation invariance of (II.4) are obvious. The verification of its R-invariance (i.e. of property (II.3)) is based on the identity

$$r(x_1)\, r(Rx_1 - Rx_2)\, r(x_2) = r(x_1 - x_2) . \tag{II.6}$$

The homogeneity property of G_ℓ with respect to x_{12} is a consequence of dilatation invariance alone; the tensor structure of G_ℓ is fixed by R-invariance (cf.[T1]).

Using the integral formula

$$\frac{\Gamma(d)}{(2\pi)^h}\int\left(\frac{2}{x^2}\right)^d e^{-ipx}\,dx = \frac{1}{(2\pi)^h}\int_0^\infty d\alpha\, \alpha^{d-1}\int dx\, e^{-\frac{\alpha x^2}{2} - ipx} =$$

$$= \int_0^\infty d\alpha\, \alpha^{d-h-1} e^{-\frac{1}{2\alpha}p^2} = \Gamma(h-d)\left(\frac{2}{p^2}\right)^{h-d} \tag{II.7}$$

(valid for $d < h$) we obtain the following expression for the Fourier transform of (II.4):

$$\tilde{G}_\chi(p; z_1, z_2) = \int G_\chi(x; z_1, z_2) e^{-ipx} dx$$

$$= \frac{n(\chi)}{(2\pi)^h} \sum_{\kappa=0}^{\ell} \frac{(z_1 z_2)^{\ell-\kappa}}{(\ell-\kappa)!} \frac{[(z_1\partial_p)(z_2\partial_p)]^\kappa}{\kappa!} \int \left(\frac{2}{x^2}\right)^{h+c+\kappa} e^{-ipx} dx$$

$$= \frac{n(\chi)}{\Gamma(c+h+\ell)\Gamma(c+h-1)} \left(\frac{p^2}{2}\right)^c \sum_{\kappa=0}^{\ell} \frac{\Gamma(\ell-\kappa-c)\Gamma(h+\kappa+c-1)}{\kappa!\,(\ell-\kappa)!} \left[\frac{(pz_1)(pz_2)}{\frac{1}{2}p^2}\right]^{\ell-\kappa} (z_1 z_2)^\kappa \qquad \text{(II.8)}$$

$$= n(\chi) \frac{(-1)^\ell \Gamma(-c)}{\Gamma(c+h+\ell)} \left[\frac{(pz_1)(pz_2)}{\frac{1}{2}p^2}\right]^\ell \left(\frac{1}{2}p^2\right)^c P_\ell^{(c-\ell,\, h-2)}(\omega),$$

where

$$\omega = \cos\theta = 1 - \frac{p^2(z_1 z_2)}{(pz_1)(pz_2)} \qquad \text{(II.9)}$$

and we have used the following expansion formula for the Jacobi polynomial

$$(-1)^\ell P_\ell^{(c-\ell,\, h-2)}(\omega) = \frac{1}{\Gamma(-c)\Gamma(c+h-1)} \sum_{\kappa=0}^{\ell} \frac{\Gamma(\ell-\kappa-c)\Gamma(h+\kappa+c-1)}{\kappa!\,(\ell-\kappa)!} \left(\frac{1-\omega}{2}\right)^\kappa .$$

[For comparing different representations of $P_\ell^{(\alpha,\beta)}$ the identity

$$\sin\pi x\,\Gamma(x)\Gamma(1-x) = \pi$$

is useful. It implies, in particular, that

$$(-1)^\ell \frac{\Gamma(\ell-\kappa-c)}{\Gamma(-c)} = (-1)^\kappa \frac{\Gamma(c+1)}{\Gamma(c+\kappa-\ell+1)} \quad].$$

Note that θ in (II.9) is the angle between the vector $\underline{z}_1$ and $\underline{z}_2$ in the rest frame of p (in Minkowski space).

For real c we can assume that the space C_χ (see Sec. I.B) consists of real-valued functions. Then, the bi-linear form (I.17), defined by $G_{\tilde{\chi}}$ is real and symmetric. Its p-space picture

$$(f,g)_\chi = \int \overline{\tilde{f}}(p)\, \tilde{G}_\chi(p)\, \tilde{g}(p)\, (dp)\,, \quad (dp) = \frac{d^{2h}p}{(2\pi)^{2h}} \tag{II.10}$$

could be regarded as a hermitian form on the set of Fourier transforms $\tilde{C}_\chi$. To be sure, the reality of $f(x)$ implies that $\overline{\tilde{f}}(p) = \tilde{f}(-p)$, and $\tilde{C}_\chi\ (= \mathcal{F}C_\chi)$ has to be considered as a vector space over the reals.

B. Expansion in projection operators. Positivity and normalization

The representation χ belongs to the supplementary series of unitary representations of $O^\uparrow(2h+1,1)$ iff the Hermittian form (II.10) is positive-definite and thus defines an invariant scalar product. The operators U (I.4–6) would be unitary in the (real) Hilbert space $\mathcal{H}_\chi$ obtained from C_χ by completion with respect to the scalar product (I.17) [or (II.10)].

For fixed p the kernel $\tilde{G}_\chi(p)$ is an operator in the finite dimensional space $\mathcal{T}^{\ell}_{2h}$ of SO(2h)-irreducible (symmetric, traceless) tensors of rank ℓ . A straightforward way to investigate the restrictions on χ imposed by positivity is to expand $\tilde{G}$ in projection operators $\Pi^{\ell s}(p)$ defined as follows. $\Pi^{\ell s}(p)$ project onto the subspaces $\mathcal{T}^{s}_{2h-1}(p)$ of $\mathcal{T}^{\ell}_{2h}$ irreducible with respect to the stability subgroup $SO(2h-1)_p \subset SO(2h)$ of the vector p . Note that the dimensions of the space $\mathcal{T}^{\ell}_{2h}$ and of its subspaces $\mathcal{T}^{s}_{2h-1}$ ($s = 0,1,\dots,\ell$) are given by

$$\dim \mathcal{T}^{\ell}_{2h} = \frac{(2h+\ell-3)!}{\ell!\,(2h-2)!}(2h+2\ell-2) = \sum_{s=0}^{\ell} \dim \mathcal{T}^{s}_{2h-1}\,,$$

$$\dim \mathcal{T}^{s}_{2h-1} = \mathrm{Tr}\, \Pi^{\ell s} = \frac{(2h+s-4)!}{(2h-3)!\, s!}(2h+2s-3)\,. \tag{II.11}$$

In 4-dimensional space-time (i.e.,for 2h=4) the number s plays the role of spin. In terms of such an expansion positivity of $\widetilde{G}$ is expressed as positivity of the scalar coefficients to Π^{ls}.

Let

$$S^{(a)}_{\mu\nu} = -i\left(z_{a\mu}\frac{\partial}{\partial z_{a\nu}} - z_{a\nu}\frac{\partial}{\partial z_{a\mu}}\right) \quad , \; a=1,2 \tag{II.12}$$

be the (hermitian) generators of the index part of 2h-rotations. Then the functions

$$\Pi^{ls}(p; z_1, z_2) = \frac{1}{l!}\, z_1^{\otimes l}\, \Pi^{ls}(p)\, z_2^{\otimes l} \tag{II.13}$$

can be found up to a normalization factor as solutions of the equation

$$\left[\tfrac{1}{2}\, S^{(a)}_{\mu\nu} S^{(a)}_{\mu\nu} - S^{(a)}_{\sigma\mu} S^{(a)}_{\sigma\nu} \frac{p_\mu p_\nu}{p^2} \right] \Pi^{ls} = s(s+2h-3)\,\Pi^{ls} \tag{II.14}$$

(valid for both $a=1$ and $a=2$ - Cf.[T3]). The result is

$$\Pi^{ls}(p, z_1, z_2) = A_{ls}(-1)^s \left[\frac{(pz_1)(pz_2)}{\frac{1}{2}p^2}\right]^l P_s^{(h-2,h-2)}(\omega) . \tag{II.15}$$

The normalization constant A_{ls} is determined from the condition that Π^{ls} are (orthogonal) projectors

$$\Pi^{ls}(p)\Pi^{ls'}(p) = \delta_{ss'}\, \Pi^{ls}(p). \tag{II.16}$$

In order to evaluate A_{ls} we use the completeness relation

$$\sum_{s=0}^{l} \Pi^{ls}(p) = \mathbb{1} . \tag{II.17}$$

According to (II.15) its z - picture expression is

$$\sum_{s=0}^{\ell} (-1)^s A_{\ell s} P_s^{(h-2,h-2)}(\omega) = \frac{1}{\ell!}\left(\frac{1-\omega}{2}\right)^{\ell}, \tag{II.18}$$

where ω is given by (II.9). We multiply both sides by $(1-\omega^2)^{h-2}$ and integrate over ω in the interval $[-1,1]$, using the ortho-normalization property

$$\int_{-1}^{1}(1-\omega^2)^{h-2} P_s^{(h-2,h-2)}(\omega)\; P_{s'}^{(h-2,h-2)}(\omega)\, d\omega = \frac{\delta_{ss'}\, 2^{2h-s}\,[(h+s-2)!]^2}{s!\,(2h+2s-3)(2h+s-4)!} \tag{II.19}$$

and the integral formula (see, e.g., [GJ] Eq. 7.391.4)

$$\frac{1}{2^{\ell}\ell!}\int_{-1}^{1}(1-\omega)^{\ell+h-2}(1+\omega)^{h-2} P_s^{(h-2,h-2)}(\omega)\, d\omega = \frac{(-1)^s 2^{2h-s}(\ell+h-2)!\,(h+s-2)!}{s!\,(\ell-s)!\,(2h+\ell+s-3)!}. \tag{II.20}$$

The result is

$$A_{\ell s} = \frac{(2h+2s-3)(h+\ell-2)!\,(2h+s-4)!}{(\ell-s)!\,(h+s-2)!\,(2h+\ell+s-3)!}. \tag{II.21}$$

In order to expand the right-hand side of (II.8) in the projection kernels (II.15) we use the formula

$$P_{\ell}^{(c-\ell,\,h-2)}(\omega) = \frac{(\ell+h-2)!}{\Gamma(c+h-1)\Gamma(c-h-\ell+2)}\sum_{s=0}^{\ell}\frac{(2s+2h-3)(s+2h-4)!\,\Gamma(c+h+s-1)\Gamma(c-h-s+2)}{(\ell-s)!\,(s+\ell+2h-3)!\,(h-2+s)!}P_s^{(h-2,h-2)}(\omega). \tag{II.22}$$

Combining (II.15), (II.21) with (II.22) we find

$$\left[\frac{(pz_1)(pz_2)}{-\frac{1}{2}p^2}\right]^{\ell} P_{\ell}^{(c-\ell,h-2)}(\omega)$$

$$=\sum_{s=0}^{\ell}(-1)^{\ell-s}\frac{\Gamma(c+h+s-1)\Gamma(c-h-s+2)}{\Gamma(c+h-1)\,\Gamma(c-h-\ell+2)}\,\Pi^{\ell s}(p;z_1,z_2). \qquad \text{(II.23)}$$

We shall fix the normalization constant $n(\chi)$ in such a way that the coefficient to $\Pi^{\ell 0}(p)$ in the spin-expansion of $\tilde{G}_{\chi}$ to be just $\left(\frac{p^2}{2}\right)^c$:

$$\tilde{G}_{\chi}(p)=\left[\Pi^{\ell 0}(p)+\alpha_{\ell 1}(c)\Pi^{\ell 1}(p)+\ldots+\alpha_{\ell\ell}(c)\Pi^{\ell\ell}(p)\right]\left(\frac{p^2}{2}\right)^c. \qquad \text{(II.24)}$$

This gives x)

$$n(\chi)=\frac{(-1)^{\ell}\Gamma(c+h+\ell)\Gamma(c-h-\ell+2)}{\Gamma(-c)\,\Gamma(c-h+2)}=\frac{\Gamma(c+h+\ell)\Gamma(h-c-1)}{\Gamma(-c)\,\Gamma(\ell+h-c-1)}. \qquad (\text{II}.25)$$

We shall discuss the advantage (and peculiarities) of this choice in the next subsection. With such a normalization we obtain

$$\alpha_{\ell s}(c)\equiv\alpha_s(c)=(-1)^s\frac{\Gamma(c+h+s-1)\Gamma(c-h-s+2)}{\Gamma(c+h-1)\,\Gamma(c-h+2)} \qquad \text{(II.26)}$$

$$=\frac{\Gamma(c+h+s-1)\Gamma(h-1-c)}{\Gamma(h+s-1-c)\,\Gamma(h-1+c)}=\frac{(c+h-1)\ldots(c+h+s-2)}{(h-c-1)\ldots(h+s-c-2)}.$$

The scalar distribution $\left(\frac{p^2}{2}\right)^c$ is a positive measure on the space of fast decreasing functions of p for all $c>-h$. However, the scalar product $(f,g)_{\tilde{\chi}}$ [see (I.17)] can be

x) We note that with this normalization the Plancherel measure (I.16) is given by

$$\rho_{\ell}(\sigma)=\frac{(\ell+h-1)!}{2(2\pi)^h\ell!}\,n(\chi)\,n(\tilde{\chi}). \qquad (\text{I}.16')$$

transformed to its p-space form (see (II.10) with $\chi \rightleftarrows \tilde{\chi}$) without recourse to analytic regularization only for $c<0$ [since Eq. (II.7) (with $d = h n$) can be derived using ordinary convergent integrals only in that domain]. Combining this with (II.24) (II.26) we see that $G_\chi(x_1-x_2)$ is a positive distribution in $C_{\tilde{\chi}} \times C_{\tilde{\chi}}$ for

$$\begin{aligned} -h < c < 0 \qquad & \text{if} \quad \ell = 0 \\ -(h-1) < c < 0 \qquad & \text{if} \quad \ell \geq 1, \; h > 1 \end{aligned} \tag{II.27}$$

We shall see in the next subsection, that the scalar product $(f,g)_{\tilde{\chi}}$ defined for $c>0$ via analytic regularization (cf. [G1]) is positive also in the wider region (I.10b).

C. Equivalent representations and intertwining operators

Similarly to the special case of the Lorentz group obtained for $h=1$ (see [G2]), the representations

$$\chi = [\ell, c] \qquad \text{and} \qquad \tilde{\chi} = [\ell, -c] \tag{II.28}$$

are equivalent. The intertwining operators for these representations are integral operators with kernels $G_\chi(x_{12})$ and $G_{\tilde{\chi}}(x_{12})$. We have

$$\mathcal{U}_\chi G_\chi = G_\chi \mathcal{U}_{\tilde{\chi}}, \qquad G_{\tilde{\chi}} \mathcal{U}_\chi = \mathcal{U}_{\tilde{\chi}} G_{\tilde{\chi}} \tag{II.29}$$

$$G_\chi G_{\tilde{\chi}} = \mathbb{1} \tag{II.30}$$

or explicitly

$$\int G_\chi(x_1-y; z_1, \partial_z) \, G_{\tilde{\chi}}(y-x_2, z, z_2)\, dy = \delta(x_1-x_2) \frac{(z_1 z_2)^\ell}{\ell!}. \tag{II.31}$$

The last equation is obviously a consequence of (II.24) because of (II.16) and the property

$$\alpha_s(c)\alpha_s(-c) = 1, \tag{II.32}$$

which follows from (II.26). (That is one reason for our choice of normalization .) We leave it to the reader to verify that if, for instance, $f_\chi(x) \in C_\chi$ then

$$\int G_{\tilde\chi}(x-y) f_\chi(y)dy \in C_{\tilde\chi}.$$

In the previous subsection we have established the unitarity of the representation $\tilde\chi$ for negative c, satisfying (I.10b). It follows from the equivalence of χ and $\tilde\chi$ that the representation χ is also unitary for such c. Hence, $\tilde\chi$ (or χ) is unitary for both positive and negative c in the domain (I.10b).

The coefficients $\alpha_s(c)$ (II.26) become zero or infinite for $s \gtrsim 1$ and integer c such that $|c| \geqslant h-1$. We could have reversed the places of zeros and infinities by a different choice of normalization. With our choice $G_{\tilde\chi}$ is well defined for all positive c and that is precisely what we need in the physical applications (cf. [M1] and [T2]).

The integer points with $|c| \geqslant h+l$ correspond to reducible, but non-decomposable representations of $0^+(2h+1,1)$. To see that, we consider first the case of a representation $\tilde\chi_{ln} = [l, -c_{ln}]$ with $c_{ln} = h+l+n$. In this case $C_{\tilde\chi_{ln}}$ contains a finite dimensional invariant subspace: the space E_{ln} of all polynomials of degree $2(n+l) = 2(c_{ln}-h)$ (or less). But E_{ln} does not have an invariant complement in $C_{\tilde\chi_{ln}}$. The factor space $C_{\tilde\chi_{ln}}/E_{ln}$

is isomorphic to an infinite dimensional invariant subspace $F_{\ell n}$ of $C_{\chi_{\ell n}}$ that consists of all tensor functions $f(x) \in C_{\chi_{\ell n}}$ which satisfy the condition

$$\int f(x)\, P_k(x)\, dx = 0 \qquad \text{for} \quad k \leq 2(n+\ell), \tag{II.33}$$

where $P_k(x)$ is an arbitrary polynomial of x of degree k.

According to (II.24) (II.25) the momentum space Green function $\tilde{G}_{\chi_{\ell n}}(p)$ is a homogeneous polynomial in p of degree $2(\ell+n+h)$. Therefore $G_{\chi_{\ell n}}$ acts as a differential operator on $C_{\tilde{\chi}_{\ell n}}$ which annihilates the finite dimensional invariant subspace $E_{\ell n}$. In this case the representations $\chi_{\ell n}$ and $\tilde{\chi}_{\ell n}$ are not equivalent. The map $G_{\chi_{\ell n}}: \ C_{\tilde{\chi}_{\ell n}} \to C_{\chi_{\ell n}}$ only establishes equivalence between the irreducible representations realized in $C_{\tilde{\chi}_{\ell n}}/E_{\ell n}$ and $F_{\ell n} \subset C_{\chi_{\ell n}}$.

D. Wightman positivity

Functions with the properties of G_{χ} (for real c) arise not only in studying invariant bi-linear forms, but also in considering analytic continuation of Wightman functions

$$W(x_1 - x_2 ;\, z_1, z_2) = \langle O(x_1, z_1)\; O(x_2, z_2) \rangle_0 \tag{II.34}$$

(or τ - functions) to Euclidean points (for which $x_0 = i x_{2h}$) in a conformal invariant quantum field theory (in the sense of [M4] [T1]). Here $O(x, z)$ is a (local) tensor field

$$O(x, z) = \frac{1}{\sqrt{\ell!}}\, O_{\mu_1 \dots \mu_\ell}(x)\, z^{\mu_1} \dots z^{\mu_\ell} \equiv O(x)\, \frac{z^{\otimes \ell}}{\sqrt{\ell!}}\,. \tag{II.35}$$

Wightman positivity for the two-point function implies that

$$\tilde{w}(p;\ z, \bar{z}) \geqslant 0 \tag{II.36}$$

in Minkowski space.

The Fourier transform of the Wightman function can be obtained from $\tilde{G}_\chi(p)$ (II.8) by the following procedure. First of all, using (II.8) and (II.23) we find the following expression for the M-space τ - function

$$\tau_\chi(p) = \frac{n_w(\chi)\Gamma(-c)}{\Gamma(c+h+\ell)}\left(\frac{1}{2}p^2 - i0\right)^{c-\ell}\sum_{s=0}^{\ell}\frac{\Gamma(c+h+s-1)\Gamma(c-h-s+2)}{\Gamma(c+h-1)\Gamma(c-h+\ell+2)}(-1)^{\ell-s}\left(\frac{p^2}{2}\right)^{\ell}\Pi^{\ell s}(p) \tag{II.37}$$

($p^2 = \underline{p}^2 - p_0^2$) (We are writing $n_w(\chi)$ instead of $n(\chi)$, since we have to use a different normalization in the new interpretation of the 2-point function.) Then the p-space Wightman function is given by

$$\tilde{w}(p) = -i\,\theta(p_0)\left[\tau_\chi(p) - \bar{\tau}_\chi(p)\right]$$

$$= -\frac{2\sin\pi(c-\ell)\,n_w(\chi)\,\Gamma(-c)}{\Gamma(c+h+\ell)\,\Gamma(c+h-1)\Gamma(c-h+\ell+2)}\theta(p_0)\left(-\frac{1}{2}p^2\right)_+^{c-\ell}\sum_{s=0}^{\ell}\Gamma(c+h+s-1)\Gamma(c-h-s+2)(-1)^{\ell-s}\left(\frac{1}{2}p^2\right)^{\ell}\Pi^{\ell s}(p), \tag{II.38}$$

where $t_+^\lambda \equiv \theta(t)t^\lambda$ (cf. [G1]); in deriving the last equality we have used the identity

$$(Q+i0)^\lambda - (Q-i0)^\lambda = 2i\sin\pi\lambda\,(-Q)_+^\lambda. \tag{II.39}$$

In this case we shall use the normalization

$$\begin{aligned} n_w(\chi) &\equiv 2^c(c+h+\ell-1)\frac{\Gamma(c+h-1)\Gamma(c-h+\ell+2)}{\Gamma(c-h-\ell+2)} \\ &= 2^c\,\frac{\Gamma(c+h-1)\Gamma(-c)\Gamma(c-h+2)\Gamma(c-h+\ell+2)}{\Gamma(c+h+\ell-1)\,\Gamma^2(c-h-\ell+2)}\,n(\chi)(-1)^\ell. \end{aligned} \tag{II.40}$$

which gives

$$\tilde{w}(p) = \frac{2\pi\theta(p_0)}{\Gamma(c+1)}(-p^2)_+^{c-\ell}\sum_{s=0}^{\ell}\frac{\Gamma(c+h+s-1)\Gamma(c-h-s+2)}{\Gamma(c+h+\ell-1)\Gamma(c-h-\ell+2)}\cdots(-1)^s(p^2)^\ell \Pi^{\ell s}(p) \qquad \text{(II.41)}$$

In order to establish when the right-hand side of (II.41) is positive, we notice that the operator

$$(-1)^s(p^2)^\ell\,\Pi^{\ell s}(p)$$

is positive, since, according to (II.15) (II.21)

$$(p^2)^\ell(-1)^s\,\Pi^{\ell s}(p,z,\bar{z}) = A_{\ell s}\left[2|pz|^2\right]^\ell P_s^{(\ell-s,\ell-s)}(\omega) \geq 0 \qquad \text{(II.42)}$$

for $\omega = 1 - \frac{p^2 z\bar{z}}{|pz|^2} \geq 1$. The last inequality $(\omega \geq 1)$ is fulfilled because, for $z^2=0$ we have $z\bar{z} = \underline{z}\underline{\bar{z}} - z_0\bar{z}_0 \leq 0$. Therefore, $\tilde{w}(p)$ is positive for (II.43)

$$c \geq -1 \quad \text{if } \ell=0\,; \quad c \geq h+\ell-2 \quad \text{for } \ell=1,2,\ldots .$$

This result was obtained by different methods also in [R2] and [F2].

Our choice of normalization (II.40) ensures the following additional properties of $\tilde{w}(p)$.

(i) For $\ell=0$, $c=-1$ (II.41) goes into the conventional expression for the two-point function of a free zero-mass field

(ii) For canonical dimensions

$$c = \ell+h-2 \qquad (\ell > 0) \qquad \text{(II.44)}$$

we recover the two-point functions of conserved (tensor) currents (while the expression (II.26) for $\alpha_\ell(c)$ is going to infinity for such a c).

III. Direct product expansions and Clebsh-Gordan kernels

A. General form of the expansion. Normalisation conditions

We consider now the direct product space $C_{\chi_{01}} \otimes C_{\chi_{02}}$ ($\chi_{0a} = [0, c_a]$) of infinitely smooth functions $f(x_1, x_2)$ satisfying the asymptotic conditions

$$\lim_{x_a^2 \to \infty} (x_a^2)^{h+c_a} f(x_1, x_2) = f_a(x_b) \in C_{\chi_{0b}} \qquad \text{(III.1)}$$

where $(a\ b)$ stands for (1,2) or (2,1). For c_a in range (I.10b) we can expand $f(x_1, x_2)$ in irreducible (unitary) representations of $O^{\uparrow}(2h+1,1)$ as follows

$$f(x_1, x_2) = \sum\!\!\!\!\!\!\int d\chi \int dx\, V(x_1 c_1, x_2 c_2, x \tilde{\chi})\, F_\chi(x) + D.T. \qquad \text{(III.2)}$$

where D.T. indicates (possible) discrete terms and the summation and integration is spread over the principal series of unitary representations (see (I.15) (I.16)).

The conformal "Fourier transform" $F_\chi(x)$ satisfies the symmetry condition

$$F_{\tilde{\chi}}(x) = \int G_{\tilde{\chi}}(x-y)\, F_\chi(y)\, dy. \qquad \text{(III.3)}$$

Conformal invariance implies that the Clebsh-Gordan kernel is given by

$$V(x_1 c_1,\ x_2 c_2;\ x_3 \chi_\ell, z)$$

$$= \frac{N_\ell(c_+, c_-, \delta) \qquad (\lambda z)^\ell}{(2\pi)^h \left(\frac{1}{2}x_{12}^2\right)^{h+c_+-\delta+\frac{1}{2}\ell} \left(\frac{1}{2}x_{13}^2\right)^{\delta+c_--\frac{\ell}{2}} \left(\frac{1}{2}x_{23}^2\right)^{\delta-c_-\frac{\ell}{2}} \sqrt{\ell!}} \qquad \text{(III.4)}$$

(cf. [T1] Sec. IV.2C) . Here we have used the definition (I.20) and the following notation:

$$c_{\pm} = \frac{1}{2}(c_1 \pm c_2) \ , \qquad \chi_\ell = [\ell, 2\delta - h] \tag{III.5}$$

$$\lambda_\mu = 2\frac{(x_{13})_\mu}{x_{13}^2} - 2\frac{(x_{23})_\mu}{x_{23}^2}. \tag{III.6}$$

The normalization constant N_ℓ will be fixed by the following conditions:

$$\int dx_1' \, V(x_1' c_1, x_2 c_2 ; x_3 \chi) \, G_{-c_1}(x_1' - x_1) = V(x_1 -c_1, x_2 c_2 ; x_3 \chi) ; \tag{III.7a}$$

$$\int dx_2' \, V(x_1 c_1, x_2' c_2 ; x_3 \chi) \, G_{-c_2}(x_2' - x_2) = V(x_1 c_1, x_2 -c_2 ; x_3 \chi) ; \tag{III.7b}$$

(G_c is a shorthand for $G_{[0,c]}$);

$$\int dx_3' \, V(x_1 c_1, x_2 c_2 ; x_3' \chi) \, G_{\tilde{\chi}}(x_3' - x_3) = V(x_1 c_1, x_2 c_2 ; x_3 \tilde{\chi}) ; \tag{III.8}$$

$$\begin{aligned} \frac{1}{2}\int dx_1 \int dx_2 V(x_1 -c_1, x_2 -c_2 ; x_3 \chi) \otimes V(x_1 c_1, x_2 c_2 ; x_3' \tilde{\chi}') \\ = \frac{1}{2}\left[\mathbb{1} \delta(\chi, \chi') + G_\chi \, \delta(\chi, \tilde{\chi}') \right] , \end{aligned} \tag{III.9}$$

where

$$\delta(\chi, \chi') = \frac{\delta_{\ell\ell'}}{\rho_\ell(\sigma)} \delta(\sigma - \sigma') 2\pi \tag{III.10}$$

($\rho_\ell(\sigma)$ is the Plancherel measure (I.16)) and the unit operator is defined in the (x, z) - picture as follows

$$\frac{1}{\ell!}\, z^{\otimes \ell}\, \mathbb{1}\, \bar{z}^{\otimes \ell} = \delta(x_3 - x_3')\, \frac{(\bar{z}\, z')^{\ell}}{\ell!}\,. \tag{III.11}$$

Eq. (III.9) along with the symmetry property (III.3) implies that the expansion (III.2) can be inverted and the conformal Fourier transform of $f(x_1, x_2)$ is given by

$$F_\chi(x) = \int dx_1 \int dx_2\, V(x_1 - c_1,\, x_2 - c_2\, ;\, x\, \lambda)\, f(x_1, x_2)\,. \tag{III.12}$$

We shall see in what follows that conditions (III.7) and (III.9) are sufficient to determine the normalization constant N_ℓ. Eq. (III.8) then can be derived as a consequence.

B. Amputation of scalar lines

We start with the exploitation of the symmetry property (III.7).

The calculation is based on the integral formula

$$I(x_1 \delta_1,\, x_2 \delta_2,\, x_3 \delta_3) = \frac{1}{(2\pi)^h} \int \frac{\Gamma(\delta_1)}{[\frac{1}{2}(x_1 - x)^2]^{\delta_1}}\, \frac{\Gamma(\delta_2)}{[\frac{1}{2}(x_2 - x)^2]^{\delta_2}}\, \frac{\Gamma(\delta_3)}{[\frac{1}{2}(x_3 - x)^2]^{\delta_3}}\, dx$$

$$= \int_0^\infty \frac{d\alpha_1}{\alpha_1} \int_0^\infty \frac{d\alpha_2}{\alpha_2} \int_0^\infty \frac{d\alpha_3}{\alpha_3}\, \frac{\alpha_1^{\delta_1} \alpha_2^{\delta_2} \alpha_3^{\delta_3}}{(k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3)^h}\, \exp\left\{ -\frac{\alpha_1\alpha_2 x_{12}^2 + \alpha_1\alpha_3 x_{13}^2 + \alpha_2\alpha_3 x_{23}^2}{2(k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3)} \right\}$$

$$(k_i \geq 0\,,\ \sum k_i > 0)$$

$$= \frac{\Gamma(h-\delta_1)\quad\Gamma(h-\delta_2)\quad\Gamma(h-\delta_3)}{\left(\frac{1}{2}x_{23}^2\right)^{h-\delta_1}\left(\frac{1}{2}x_{13}^2\right)^{h-\delta_2}\left(\frac{1}{2}x_{12}^2\right)^{h-\delta_3}} \qquad \text{for} \quad \delta_1+\delta_2+\delta_3 = 2h \tag{III.13}$$

(see [D1] [S3]) and on the identity

$$\frac{(\lambda z)^{\ell}}{\left(\frac{1}{2}x_{13}^2\right)^{\delta+c_- -\frac{\ell}{2}}\left(\frac{1}{2}x_{23}^2\right)^{\delta-c_- -\frac{\ell}{2}}} = \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{\Gamma(\delta-c_- -\frac{\ell}{2})}{\Gamma(\delta-c_- -\frac{\ell}{2}+k)}\left(\frac{2}{x_{13}^2}\right)^{\delta+c_- +\frac{\ell}{2}-k}(z x_{13})^{\ell-k}(z\partial_3)^k\left(\frac{2}{x_{23}^2}\right)^{\delta-c_- -\frac{\ell}{2}}$$

$$= \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{\Gamma(\delta+c_- -\frac{\ell}{2})\,\Gamma(\delta-c_- -\frac{\ell}{2})}{\Gamma(\delta+c_- +\frac{\ell}{2}-k)\,\Gamma(\delta-c_- -\frac{\ell}{2}+k)}\left(\frac{2}{x_{13}^2}\right)^{\delta+c_- -\frac{\ell}{2}}(z\overleftarrow{\partial}_3)^{\ell-k}(z\overrightarrow{\partial}_3)^k\left(\frac{2}{x_{23}^2}\right)^{\delta-c_- -\frac{\ell}{2}}, \tag{III.14}$$

where $\overleftarrow{\partial}_3\ (\overrightarrow{\partial}_3)$ differentiates with respect to x_3 to the left (to the right). Using the first equation (III.14) and (III.13) we find

$$\int dx_2'\, V(x_1c_1, x_2'c_2; x_3\lambda z)\; G_{-c_2}(x_2'-x_2)$$

$$= \frac{N_\ell(c_+,c_-,\delta)}{\sqrt{\ell!}\,(2\pi)^h}\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{\Gamma(\delta-c_+ -\frac{\ell}{2})\,\Gamma(h-\delta+c_- +\frac{\ell}{2})}{\Gamma(h+c_+ -\delta+\frac{\ell}{2})\,\Gamma(\delta-c_- -\frac{\ell}{2}+k)}\left(\frac{2}{x_{12}^2}\right)^{h-\delta+c_- -\frac{\ell}{2}}$$

$$\times\left(\frac{2}{x_{13}^2}\right)^{\delta+c_- -\frac{\ell}{2}}\left(\frac{2z x_{13}}{x_{13}^2}\right)^{\ell-k}(z\partial_3)^k\left(\frac{2}{x_{13}^2}\right)^{c_2}\left(\frac{2}{x_{23}^2}\right)^{\delta-c_+ -\frac{\ell}{2}}$$

$$= \frac{N_\ell(c_+,c_-,\delta)\,\Gamma(h-\delta+c_- +\frac{\ell}{2})}{\sqrt{\ell!}\,(2\pi)^h\,\Gamma(c_2)\,\Gamma(h+c_+ -\delta+\frac{\ell}{2})}\left(\frac{2}{x_{12}^2}\right)^{h-\delta+c_- +\frac{\ell}{2}}\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{1}{\Gamma(\delta-c_- -\frac{\ell}{2}+k)}$$

$$\times\sum_{j=0}^{k}\binom{k}{j}\frac{(z x_{13})^{\ell-j}\,(z x_{23})^j\,\Gamma(c_2+k-j)\,\Gamma(\delta-c_+ -\frac{\ell}{2}+j)}{\left(\frac{1}{2}x_{13}^2\right)^{\delta+c_- +\frac{\ell}{2}-j}\left(\frac{1}{2}x_{23}^2\right)^{\delta-c_+ -\frac{\ell}{2}+j}}\;;$$

Changing the order of summation and using the sum rule

$$\sum_{k=j}^{\ell} (-1)^k \binom{\ell}{k}\binom{k}{j} \frac{\Gamma(\alpha+k)}{\Gamma(\beta+k)} = (-1)^j \binom{\ell}{j} \frac{\Gamma(\beta-\alpha+\ell-j)\,\Gamma(\alpha+j)}{\Gamma(\beta+\ell)\,\Gamma(\beta-\alpha)} \qquad \text{(III.15)}$$

(see Appendix A) for $\alpha = c_2 - j$, $\beta = \delta - c_- - \frac{\ell}{2}$

we obtain

$$\int dx_2' \, V(x_1 c_1, x_2' c_2 ; x_3 \lambda z) \, G_{-c_2}(x_2' - x_2)$$

$$= \frac{N_\ell(c_+, c_-, \delta)\,\Gamma(\delta - c_+ + \frac{\ell}{2})\,\Gamma(h-\delta+c_-+\frac{\ell}{2}) \qquad (\lambda z)^\ell}{\sqrt{\ell!}\,(2\pi)^h\,\Gamma(h+c_+-\delta+\frac{\ell}{2})\,\Gamma(\delta-c_-+\frac{\ell}{2})\,(\frac{1}{2}x_{12}^2)^{h-\delta+c_-+\frac{\ell}{2}}\,(\frac{1}{2}x_{13}^2)^{\delta+c_+-\frac{\ell}{2}}\,(\frac{1}{2}x_{23}^2)^{\delta-c_+-\frac{\ell}{2}}} \qquad \text{(III.16)}$$

$$= \frac{N_\ell(c_+, c_-, \delta)}{N_\ell(c_-, c_+, \delta)} \frac{\Gamma(-c_+ + \delta + \frac{\ell}{2})\,\Gamma(h-\delta+c_-+\frac{\ell}{2})}{\Gamma(h+c_+-\delta+\frac{\ell}{2})\,\Gamma(\delta-c_-+\frac{\ell}{2})} V(x_1 c_1, x_2 c_2 ; x_3 \lambda z) .$$

Applying the obvious symmetry property

$$V(x_1 c_1, x_2 c_2 , x_3 \lambda z) = (-1)^\ell \frac{N_\ell(c_+, c_-, \delta)}{N_\ell(c_+, -c_-, \delta)} V(x_2 c_2 , x_1 c_1 ; x_3 \lambda z) \qquad \text{(III.17)}$$

we can derive from (III.16) another relation of that type, involving integration over the first argument of V (say x_1').

Combining these two equations and comparing with (III.7) we obtain

$$\frac{N_\ell(c_+, c_-, \delta)}{N_\ell(-c_+, -c_-, \delta)} = \frac{\Gamma(h+c_+-\delta+\frac{\ell}{2})\,\Gamma(\delta+c_+ +\frac{\ell}{2})}{\Gamma(h-c_+-\delta+\frac{\ell}{2})\,\Gamma(\delta-c_+ +\frac{\ell}{2})} . \qquad \text{(III.18)}$$

C. The Plancherel formula

In order to give a precise meaning of the singular equation (III.9) we start with the following regularization of the left-hand side:

$$I_\varepsilon (x_3 \ell \delta z,\ x_3' \ell' \delta' z') = \frac{\sqrt{\ell!}\,\sqrt{\ell'!}}{N_\ell(-c_+, -c_-, \delta) N_{\ell'}(c_+, c_-, h-\delta')} \times \qquad \text{(III.19)}$$

$$\times \int dx_1 \int dx_2\, V(x_1 - c_1,\ x_2 - c_2\,;\ x_3 \tilde{\ell} z)\, V(x_1\, c_1 - 2\varepsilon,\ x_2 c_2\,;\ x_3' \tilde{\ell}' z')$$

and shall go to the limit $\varepsilon \to +0$ only after smearing with an analytic test function of δ'.

Setting

$$x_i - x_3' = x_{i3'} \qquad i = 1, 2, 3 \qquad \text{(III.20)}$$

and performing the integration in (III.19) over x_2 we obtain:

$$I_\varepsilon = \frac{1}{(2\pi)^h} \sum_{k=0}^{\ell} \sum_{k'=0}^{\ell'} (-1)^{k+k'} \binom{\ell}{k}\binom{\ell'}{k'} \frac{\Gamma(\delta-\delta'-\frac{\ell+\ell'}{2}+\varepsilon)\,\Gamma(h-\delta'-c_-+\frac{\ell}{2})\,\Gamma(\delta'+c_-+\frac{\ell'}{2}-\varepsilon)}{\Gamma(h-\delta+\delta'+\frac{\ell+\ell'}{2}-\varepsilon)\,\Gamma(\delta+c_--\frac{\ell}{2}+k)\,\Gamma(h-\delta'-c_-+k'-\frac{\ell'}{2}+\varepsilon)} \times$$

$$\int dx_1 \frac{(z x_{13})^{\ell-k}\,(z' x_{13'})^{\ell'-k'}}{(\frac{1}{2}x_{13}^2)^{\delta-c_-+\frac{\ell}{2}-k}\,(\frac{1}{2}x_{13'}^2)^{h-\delta'+c_-+\frac{\ell'}{2}-k'-\varepsilon}} (z\partial_3)^k (z'\partial_{3'})^{k'} \left[\left(\frac{2}{x_{33'}^2}\right)^{\delta-\delta'-\frac{\ell+\ell'}{2}+\varepsilon} \left(\frac{2}{x_{13'}^2}\right)^{h-c_--\delta'+\frac{\ell}{2}} \left(\frac{2}{x_{13}^2}\right)^{c_-+\delta'+\frac{\ell'}{2}-\varepsilon}\right]$$

Here we have used again (III.14) and (III.13). Further, we apply the binomial formula:

$$\frac{(z x_{13})^{\ell-k}(z' x_{13'})^{\ell'-k'}(z\partial_3)^k(z'\partial_{3'})^{k'}}{(\frac{1}{2}x_{13}^2)^{\delta-c_-+\frac{\ell}{2}-k}\,(\frac{1}{2}x_{13'}^2)^{h-\delta'+c_-+\frac{\ell'}{2}-k'-\varepsilon}} \left[\left(\frac{2}{x_{33'}^2}\right)^{\delta-\delta'-\frac{\ell+\ell'}{2}+\varepsilon} \left(\frac{2}{x_{13'}^2}\right)^{h-\delta'-c_-+\frac{\ell}{2}} \left(\frac{2}{x_{13}^2}\right)^{\delta'+c_-+\frac{\ell'}{2}-\varepsilon}\right] =$$

$$= \sum_{j=0}^{k} \sum_{j'=0}^{k'} \binom{k}{j}\binom{k'}{j'} \frac{\Gamma(h-\delta-c_- + \frac{\ell}{2} + k' - j')\,\Gamma(\delta' + c_- + \frac{\ell'}{2} + k - j - \varepsilon)}{\Gamma(h-\delta-c_- + \frac{\ell}{2})\,\Gamma(\delta' + c_- + \frac{\ell'}{2} - \varepsilon)} \times$$

$$\times \frac{(z x_{13})^{\ell-j}\,(z' x_{13'})^{\ell'-j'}}{(\frac{1}{2}x_{13}^2)^{\delta+\delta'+\frac{\ell+\ell'}{2}-j-\varepsilon}\,(\frac{1}{2}x_{13'}^2)^{2h-\delta-\delta'+\frac{\ell+\ell'}{2}-j'-\varepsilon}} (z\partial_3)^{j} (z'\partial_{3'})^{j'} \left(\frac{2}{x_{33'}^2}\right)^{\delta-\delta'-\frac{\ell+\ell'}{2}+\varepsilon}.$$

then change the order of summation in k and j, and in k' and j', and use twice the sum rule (III.15) with $\alpha = \delta' + c_- + \frac{\ell'}{2} - j - \varepsilon$, $\beta = \delta + c_- - \frac{\ell}{2}$ (and $\alpha' = h - \delta - c_- + \frac{\ell}{2} - j'$, $\beta' = h - \delta' - c_- - \frac{\ell'}{2} + \varepsilon$).

The result is

$$I_\varepsilon(x_3 \ell \delta z, x_{3'} \ell' \delta' z') = (2\pi)^h \sum_{j=0}^{\ell} \sum_{j'=0}^{\ell'} (-1)^{j+j'} \binom{\ell}{j}\binom{\ell'}{j'} \frac{\Gamma(\delta-\delta'+\frac{\ell-\ell'}{2}+\varepsilon)\,\Gamma(\delta-\delta'-\frac{\ell-\ell'}{2}+\varepsilon)}{\Gamma(\delta-\delta'-\frac{\ell+\ell'}{2}+j+\varepsilon)\,\Gamma(\delta-\delta'-\frac{\ell+\ell'}{2}+j'+\varepsilon)} \times$$

$$\times \frac{\Gamma(\delta-\delta'-\frac{\ell+\ell'}{2}+\varepsilon)\,\Gamma(\delta'+\frac{\ell'}{2}+c_- - \varepsilon)\,\Gamma(h-\delta+\frac{\ell}{2}-c_-)}{\Gamma(h-\delta+\delta'+\frac{\ell+\ell'}{2}-\varepsilon)\,\Gamma(\delta+\frac{\ell}{2}+c_-)\,\Gamma(h-\delta'+\frac{\ell'}{2}-c_-+\varepsilon)} \left[(z\partial_3)^{j}(z'\partial_{3'})^{j'}\left(\frac{2}{x_{33'}^2}\right)^{\delta-\delta'-\frac{\ell+\ell'}{2}+\varepsilon}\right] \times$$

$$\times \frac{\Gamma(\delta+\delta'+\frac{\ell'-\ell}{2}-h+\varepsilon)\,\Gamma(h-\delta-\delta'-\frac{\ell'-\ell}{2}+\varepsilon)}{\Gamma(\delta+\delta'+\frac{\ell+\ell'}{2}-j-\varepsilon)\,\Gamma(2h-\delta-\delta'+\frac{\ell+\ell'}{2}-j'-\varepsilon)} (z\partial_3)^{\ell-j}(z'\partial_{3'})^{\ell'-j'}\,\delta_\varepsilon(x_{33'}), \tag{III.21}$$

where

$$\delta_\varepsilon(x_{33'}) = \frac{1}{(2\pi)^{2h}} \frac{\Gamma(\delta+\delta'+\frac{\ell'-\ell}{2}-\varepsilon)\,\Gamma(2h-\delta-\delta'+\frac{\ell-\ell'}{2}-\varepsilon)}{\Gamma(h-\delta-\delta'+\frac{\ell-\ell'}{2}+\varepsilon)\,\Gamma(\delta+\delta'+\frac{\ell'-\ell}{2}-h+\varepsilon)} \int \frac{dx_1}{\left(\frac{x_{13}^2}{2}\right)^{\delta+\delta'+\frac{\ell'-\ell}{2}-\varepsilon}\left(\frac{x_{13'}^2}{2}\right)^{2h-\delta-\delta'+\frac{\ell-\ell'}{2}-\varepsilon}}$$

$$= \int \frac{dp}{(2\pi)^{2h}} \left(\frac{2}{p^2}\right)^{2\varepsilon} e^{ipx_{33'}} = \frac{\Gamma(h-2\varepsilon)}{(2\pi)^h\,\Gamma(2\varepsilon)\left(\frac{x_{33'}^2}{2}\right)^{h-2\varepsilon}} \left(\xrightarrow[\varepsilon\to+0]{} \delta(x)\right). \tag{III.22}$$

Because of the distribution character of the limit $\varepsilon \to +0$ [as is already suggested by Eq. (III.9)] we shall first smear the right-hand side of (III.21) by a suitable test function of the representation label

$$c' = 2\delta' - h \, . \tag{III.23}$$

Let

$$2\delta = h + c \, , \quad c = i\sigma \quad (\sigma - \text{real}), \tag{III.24}$$

and let $f(c')$ be an analytic function in some finite strip

$$0 \leq \operatorname{Re} c' < a \, , \tag{III.25}$$

fast decreasing at infinity inside the strip. We shall evaluate the integral

$$I_\varepsilon^f(x_{33'}, \sigma, \ell) = \int_{-i\infty}^{i\infty} \frac{dc'}{4\pi i} \, \rho_\ell(-ic') \, I_\varepsilon\left(x_3 \, \ell \, \tfrac{h+i\sigma}{2} \, z, \; x_{3'} \, \ell \, \tfrac{h+c'}{2} \, z'\right) f(c') \tag{III.26}$$

in the limit $\varepsilon \to +0$ by closing the contour of integration in the strip (III.25). (In order to simplify the calculation we have set $\ell' = \ell$.) The result is

$$\lim_{\varepsilon \to +0} I_\varepsilon^f = -\frac{1}{2} \rho_\ell(\sigma) \lim_{\varepsilon \to +0} \left[f(i\sigma) \operatorname*{Res}_{c' = i\sigma} I_\varepsilon + f(-i\sigma) \operatorname*{Res}_{c' = -i\sigma} I_\varepsilon \right] . \tag{III.27}$$

For the first residue we obtain from (III.21) (for $\ell = \ell'$)

$$- \lim_{\varepsilon \to +0} \operatorname*{Res}_{c' = i\sigma + 2\varepsilon} I_\varepsilon = 2(2\pi)^h \sum_{j,j'=0}^{\ell} (-1)^{j+j'} \frac{\ell! \, \ell! \, \Gamma(i\sigma) \Gamma(-i\sigma) \left[(z\partial_z)^\ell (z'\partial_{z'})^\ell \left(\frac{x_{33'}}{2}\right)^\ell \right]}{j! \, j'! \, (h+\ell-1)! \, \Gamma(h+\ell-j+i\sigma) \Gamma(h+\ell-j'-i\sigma)} \delta(x_{33'})$$

$$= 2(2\pi)^h \frac{\ell!}{(h+\ell-1)!} \left| \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{\ell-j} \frac{(\ell-j)! \, \Gamma(i\sigma)}{\Gamma(h+\ell-j+i\sigma)} \right|^2 \delta(x_{33'}) \, (z z')^\ell . \tag{III.28}$$

In writing down the first equality (III.28) we used that for any test function $u(x_3)$:

$$\int [(z\partial_3)^{l-j}(z'\partial_{3'})^{l-j'}\delta(x_{33'})]\, u(x_3')\, [(z\partial_3)^{j}(z'\partial_{3'})^{j'}] \left(\frac{x_{33'}^2}{2}\right)^l dx_3'$$

$$= (-1)^{l-j'}\int \delta(x_{33'})\,(z\partial_{3'})^{l-j}\,(z'\partial_{3'})^{l-j'}\{u(x_3')(z\partial_3)^{j}(z'\partial_{3'})^{j'}\}\left(\frac{x_{33'}^2}{2}\right)^l dx_3' \cdot$$

$$= (-1)^{j+j'}\int \delta(x_{33'})\, u(x_3')\,(z\partial_3)^{l}(z'\partial_{3'})^{l}\left(\frac{x_{33'}^2}{2}\right)^l dx_3'$$

(since the terms containing derivatives of $u(x_3')$ vanish); we also applied the identity

$$\frac{\Gamma(\varepsilon-l)\,\Gamma(\varepsilon)}{\Gamma(\varepsilon+j'-l)\,\Gamma(\varepsilon+j-l)} = (-1)^{j+j'}\frac{\Gamma(1+l-j'-\varepsilon)\,\Gamma(1+l-j-\varepsilon)}{\Gamma(1+l-\varepsilon)\,\Gamma(1-\varepsilon)} \,.$$

Next we take into account that

$$\sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}\frac{(l-j)!}{\Gamma(h+l-j+i\sigma)} = \frac{1}{(i\sigma+h+l-1)\,\Gamma(h+i\sigma-1)} \qquad \text{(III.29)}$$

(see Appendix A). Inserting it in(III.28)and(III.27) and recalling the notation (III.20) and the expression for the Plancherel measure (I.16) we obtain:

$$-\rho_l(\sigma)\,\underset{l'=i\sigma}{\mathrm{Res}}\, I_{\varepsilon=0} = \delta(x_{33'})\,(zz')^l. \qquad \text{(III.30)}$$

The orthogonality relation for $\ell \neq \ell'$ implied by (III.9), (III.10) can also be verified by a slightly more complicated calculation along the same lines.

Now we proceed to the evaluation of the second residue term in (III.27). Setting $z' = z$ ($\ell' = \ell$) for δ given by (III.24) and using (II.4) (II.25) we find

$$
\begin{aligned}
&-\lim_{\varepsilon \to +0} \operatorname*{Res}_{c' = -i\sigma + 2\varepsilon} I_\varepsilon \left(x_3\, \ell\, \delta\, z,\; x_{3'}\, \ell'\, \tfrac{h+c'}{2}\, z \right) \\
&= 2(h-1)! \sum_{j,j'=0}^{\ell} (-1)^{j+j'} \binom{\ell}{j} \binom{\ell}{j'} \frac{[\Gamma(2\delta-h)]^2\, \Gamma(2\delta-h-\ell)\, \Gamma(h-\delta+c_-+\frac{\ell}{2})\, \Gamma(h-\delta-c_-+\frac{\ell}{2})}{\Gamma(2h-2\delta+\ell)\, \Gamma(\delta+c_-+\frac{\ell}{2})\, \Gamma(\delta-c_-+\frac{\ell}{2})\, \Gamma(2\delta-h-\ell+j)\, \Gamma(2\delta-h-\ell+j')} \times \\
&\quad \times \frac{1}{\Gamma(h+\ell-j)\, \Gamma(h+\ell-j')} \left[(\partial_3 z)^{\ell-j} (z\partial_{3'})^{\ell-j'} \left(\frac{2}{x_{33'}^2} \right)^h \right] (z\partial_3)^j (z\partial_{3'})^{j'} \left(\frac{2}{x_{33'}^2} \right)^{2\delta-h-\ell} \\
&= (2\pi)^h \sum_{j,j'=0}^{\ell} (-1)^{j+j'} \binom{\ell}{j} \binom{\ell}{j'} \frac{[\Gamma(2\delta-h)]^2\, \Gamma(2\delta-h-\ell+j+j')\, \Gamma(2\ell+h-j-j')\, i}{\Gamma(2\delta-\ell-h+j)\, \Gamma(2\delta-\ell-h+j')\, (\ell-j+h-1)!\, (\ell-j'+h-1)!} \times \\
&\quad \times \frac{\Gamma(h-\delta-c_-+\frac{\ell}{2})\, \Gamma(h-\delta+c_-+\frac{\ell}{2})\, \Gamma(h-2\delta)\, \Gamma(2h-2\delta+\ell-1)}{\Gamma(\delta-c_-+\frac{\ell}{2})\, \Gamma(\delta+c_-+\frac{\ell}{2})\, \Gamma(2h-2\delta+\ell)\, \Gamma(2\delta+\ell)\, \Gamma(2h-2\delta-1)} \times \\
&\quad \times G_\chi (x_{33'};\, z, z) .
\end{aligned}
$$

(III.31)

On the other hand, as is shown in Appendix B,

$$
\sum_{j,j'=0}^{\ell} (-1)^{j+j'} \binom{\ell}{j} \binom{\ell}{j'} \frac{\Gamma(i\sigma-\ell+j+j')\; (2\ell-j-j'+h-1)!}{\Gamma(i\sigma+j-\ell)\, \Gamma(i\sigma+j'-\ell)\, (\ell-j+h-1)!\, (\ell-j'+h-1)!}
$$

$$
= \frac{\ell!\, \Gamma(i\sigma+h+\ell-1)}{(h+\ell-1)!\, \Gamma(i\sigma)\, \Gamma(i\sigma+h-1)} . \qquad \text{(III.32)}
$$

Inserting (III.32) in (III.31) we obtain

$$- \rho_l(\sigma) \lim_{\varepsilon \to +0} \operatorname*{Res}_{c' = -i\sigma + 2\varepsilon} I_l\left(x_3\, l\, \frac{h+i\sigma}{2} z,\ x_3'\, l\, \frac{h+c'}{2} z\right)$$

$$= l!\, \frac{\Gamma(h-c_- -\delta+\frac{l}{2})\,\Gamma(h+c_- -\delta+\frac{l}{2})}{\Gamma(\delta-c_- +\frac{l}{2})\,\Gamma(\delta+c_- +\frac{l}{2})}\, G_l(x_{33'}, z, z) . \tag{III.33}$$

Comparing (III.27) (III.30) (III.33) with the normalization condition (III.9-11) we find that for imaginary $2\delta-h$

$$N_l(-c_+, -c_-, \delta)\, N_l(c_+, c_-, h-\delta) = 1, \tag{III.34}$$

$$N_l(-c_+, -c_-, \delta)\, N_l(c_+, c_-, \delta) = \frac{\Gamma(\delta-c_- +\frac{l}{2})\,\Gamma(\delta+c_- +\frac{l}{2})}{\Gamma(h-\delta-c_- +\frac{l}{2})\,\Gamma(h-\delta+c_- +\frac{l}{2})} \tag{III.35}$$

Multiplying Eqs. (III.18) with (III.35) (side by side) we obtain

$$N_l(c_+, c_-, \delta) = \left\{ \frac{\Gamma(h+c_+ -\delta+\frac{l}{2})\,\Gamma(c_+ +\delta+\frac{l}{2})\,\Gamma(\delta-c_- +\frac{l}{2})\,\Gamma(\delta+c_- +\frac{l}{2})}{\Gamma(h-c_+ -\delta+\frac{l}{2})\,\Gamma(-c_+ +\delta+\frac{l}{2})\,\Gamma(h-\delta-c_- +\frac{l}{2})\,\Gamma(h-\delta+c_- +\frac{l}{2})} \right\}^{1/2} . \tag{III.36}$$

The sign of the square root can be fixed, requiring that for real dimensions and $-1 \leq c_+ \leq 0$, $|c_-| < \delta < h + c_+$ the factors N_l be positive. Notice that Eq. (III.34) (which was not used in the derivation of (III.36)) is satisfied automatically by this expression.

The authors are grateful to Dr.R.P.Zaikov for acquainting them with his results prior to publication.

APPENDIX A

A summation formula involving ratios of Γ- functions

Eqs. (III.15) and (III.29) used in Sec. III can both be derived from the following known formula for the value of the hypergeometric function $F = {}_2F_1$ at the point $x=1$:

$$F(a,b;c;1) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b+m)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+m)\,m!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (A.1)$$

(see e.g. [G3] Eq. 9.122.1).

In order to reduce Eq. (III.15) to the form (A.1) we set $k-j=m$, $l-j=n$ and continue to non-integer n , writing (III.22) in the form

$$\sum_{m=0}^{\infty} \frac{\Gamma(m-n)\,\Gamma(\alpha+j+m)}{\Gamma(-n)\,m!\,\Gamma(\beta+j+m)} = \frac{\Gamma(\beta-\alpha+n)\,\Gamma(\alpha+j)}{\Gamma(\beta+j+n)\,\Gamma(\beta-\alpha)} \qquad (A.2)$$

Here, we have used the identity

$$(-1)^m \frac{\Gamma(n+1)}{\Gamma(n-m+1)} = \frac{\Gamma(m-n)}{\Gamma(-n)} \; . \qquad (A.3)$$

Eq. (III.29) is established in a similar way.

There exists also a direct elementary proof of Eqs. (III.15) and (III.29) which exploits their similarity to the Newton binomial formula.

In the above notation, Eq. (III.15) assumes the form

$$f_n(a,b) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (a)_m\,(b+m)_{n-m} = (b-a)_n \qquad (A.4)$$

where $a=\alpha+j$, $b=\beta+j$ ($n=l-j$) and

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)\ldots(x+k-1) \tag{A.5}$$

is the finite-difference counter part of the power x^k.

In order to prove (A.4), we evaluate the finite difference $f_n(a,b) - f_n(a,b-1)$ using

$$(x)_k - (x-1)_k = [x+k-1-(x-1)](x)_{k-1} = k(x)_{k-1}\ . \tag{A.6}$$

Thus, we find the recurrence relation

$$f_n(a,b) - f_n(a,b-1) = n f_{n-1}(a,b) \tag{A.7}$$

with the initial condition

$$f_1(a,b) = b-a\ . \tag{A.8}$$

In order to fix $f_n(a,b)$ uniquely we have to evaluate it for a particular value of b. For $b=a$ we have

$$f_n(a,a) = (a)_n \sum_{m=0}^{n} (-1)^m \binom{n}{m} = (a)_n (1-1)^n = 0. \tag{A.9}$$

It is easily seen that the only polynomial solution of (A.7–9) is given by the right-hand side of (A.4).

Eq. (III.29) can also be reduced to the form (A.4). Indeed multiplying both sides by $\Gamma(h+i\sigma+l)$ and substituting the summation index $j \to k = l-j$ we obtain

$$\sum_{k=0}^{l} (-1)^k \binom{l}{k} (1)_k (h+i\sigma+k)_{l-k} = (h+i\sigma-1)_l \tag{A.10}$$

which is a special case of (A.4) (with $a=1,\ b=h+i\sigma$).

Appendix B. Evaluation of a double sum involving Γ - functions

We wish to evaluate the expression

$$S \equiv \sum_{j,j'=0}^{\ell} (-1)^{j+j'} \binom{\ell}{j}\binom{\ell}{j'} \frac{\Gamma(i\sigma-\ell+j+j')\,(2\ell-j-j'+k-1)!}{\Gamma(i\sigma+j-\ell)\,\Gamma(i\sigma+j'-\ell)\,(\ell-j+k-1)!\,(\ell-j'+k-1)!} \tag{B.1}$$

First we make a change of summation variables to $\tau=\ell-j,\ \tau'=\ell-j'$.
Using the familiar identity $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$
and the definition of the Euler Beta-function B we may then rewrite (B.1) as

$$S = \pi^{-2}\sin^2 i\pi\sigma \, \frac{\Gamma(i\sigma+k+\ell)}{\Gamma(i\sigma+k-1)^2}\, S', \qquad \text{with} \tag{B.2}$$

$$S' = \sum_{\tau,\tau'=0}^{\ell} \binom{\ell}{\tau}\binom{\ell}{\tau'} B(1-i\sigma+\tau,\, i\sigma+k-1)\, B(1-i\sigma+\tau',\, i\sigma+k-1)\, B(i\sigma+\ell-\tau-\tau',\, k+\tau+\tau').$$

We now insert the standard integral representation of the B-functions,

$$B(a,b) = \int_0^1 dx\, x^{b-1}(1-x)^{a-1}$$

The result is

$$S' = \sum_{\tau,\tau'=0}^{\ell} \binom{\ell}{\tau}\binom{\ell}{\tau'} \int_0^1\!\!\int_0^1\!\!\int_0^1 dx\,dy\,dz \left\{\frac{z(1-x)}{1-z}\right\}^{\tau} \left\{\frac{z(1-y)}{1-z}\right\}^{\tau'} \{(1-x)(1-y)\}^{-i\sigma}$$
$$\cdot\,(xy)^{i\sigma+k-2}\, z^{k-1}\,(1-z)^{i\sigma+\ell-1} .$$

Both summations can now be performed with the help of the binomial theorem, this gives

$$S' = \int_0^1\!\!\int_0^1\!\!\int_0^1 dx\,dy\,dz\,(xy)^{i\sigma+k-2}\{(1-x)(1-y)\}^{-i\sigma}(1-zx)^{\ell}(1-zy)^{\ell} z^{k-1}(1-z)^{i\sigma-\ell-1}.$$

The x and y integrations can be performed, each of them produces a Jacobi-polynomial op. [G3] , formulae 8.962.1 and 9.111.

This gives

$$S' = \left\{ \frac{\ell!\,\Gamma(1-i\sigma)\Gamma(i\sigma+h-1)}{\Gamma(h+\ell)} \right\}^2 \frac{1}{2} \int_{-1}^{1} dt \left(\frac{1-t}{2}\right)^{i\sigma-\ell-1} \left(\frac{1+t}{2}\right)^{h-1} \left\{ P_\ell^{(i\sigma-\ell-1,\,h-1)}(t) \right\}^2 .$$

We have introduced $t = 2z-1$ as a new variable of integration. Finally, the t-integration can also be performed with the help of the standard orthonormality relation of Jacobi-polynomials, cp. [GJ], formula 7.391.1. One obtains

$$S' = \ell! \, \frac{\Gamma(i\sigma)\Gamma(i\sigma+h-1)}{\Gamma(h+\ell)} \, \frac{\Gamma(1-i\sigma)^2}{i\sigma+h+\ell-1} .$$

This has to be inserted into Eq. (B2). Splitting the $\sin^2$ into Γ- functions again, we obtain after some cancellations the final result

$$S = \frac{\ell!}{(h+\ell-1)!} \, \frac{\Gamma(i\sigma+h+\ell-1)}{\Gamma(i\sigma)\,\Gamma(i\sigma+h-1)} . \tag{B.3}$$

We remark that result and derivation are equally valid when h is not an integer. The factorials $(h+\ell-1)!$, etc., have to be read as Γ- functions in this case.

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Received by Publishing Department
on May 24, 1973.