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ESSENTIALLY NONLINEAR FIELDS AND VACUUM POLARIZATION

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Существенно нелинейные поля и поляризация вакуума

В работе показано, что поведение плотности энергии взаимодействующих полей при больших градиентах находится в соответствии с классификацией по их перенормируемости. Рассмотрена также поляризация вакуума, приводящая к полям с ограниченными производными.

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Blokhintsev D.I. E2 - 7952

Essentially Nonlinear Fields and Vacuum Polarization

The behaviour of the energy density for interacting fields at large gradients is shown to be in agreement with the classification according to their renormalizability.

The vacuum polarization resulting in fields with limited derivatives is considered.

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1. FIELDS AT LARGE GRADIENTS

By the essentially nonlinear field I mean a field the weak discontinuity of which ("signal") propagates with a velocity depending on the field itself and/or its derivatives.

In this section a method, which will further be applied to the essentially nonlinear field, is at first explained by the example of simpler fields obeying the equations with constant coefficients for the highest derivatives.

It is shown that there arises a simple classification of fields which corresponds to the classification according to the field renormalizability.

The essence of the method consists in the study of the behaviour of the energy density of a classic (nonquantized) field concentrated in a small domain of dimension $\approx \ell$ and with the gradients $\frac{\partial}{\partial x}$, $\frac{1}{c}\frac{\partial}{\partial t}$ of the order $1/\ell$ at $\ell \to 0$ x/.

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x/ Note that the functional integration in the Feynman integral (defining the scattering matrix) covers all the possible values of the classical field, includes inevitably the mentioned field values too.

A. We first consider a scalar field $\Phi(x,t)$ with self-action. For this field the energy density is

$$H = \frac{1}{2}[\dot{\Phi}^2 + \nabla\Phi^2 + m^2\Phi^2] + g\Phi^n = 0. \qquad (1)$$

The expression in brackets is the energy density of a free field, the last term is the self-action energy and g is the self-action constant. Instead of Φ, g and H we introduce the dimensionless quantities $\tilde{\tilde{\Phi}}$, $\tilde{\tilde{g}}$ and $\tilde{\tilde{H}}$ according to the formulas:

$$\Phi = \frac{\sqrt{\hbar c}}{\ell}\tilde{\tilde{\Phi}}, \quad g = (\hbar c)^{1-\frac{n}{2}}\Lambda^{n-4}\tilde{\tilde{g}}, \quad H = \frac{\hbar c}{\ell^4}\tilde{\tilde{H}}. \qquad (2)$$

Here Λ is a certain length defining the self-action strength. The quantity $\frac{\hbar c}{\ell^4}$ is, in its order of magnitude, equal to the zero-energy density of a wave of length $\lambda = \ell$. It follows from eqs. (1) and (2)

$$\tilde{\tilde{H}} = \frac{1}{2}[\tilde{\tilde{\Phi}}'\tilde{\tilde{\Phi}}' + m^2\ell^2\tilde{\tilde{\Phi}}\tilde{\tilde{\Phi}}] + g\left(\frac{\ell}{\Lambda}\right)^{4-n}\tilde{\tilde{\Phi}}^n, \qquad (3)$$

$\tilde{\tilde{\Phi}}'$ being a dimensionless derivative. It is seen from eq. (3) that at $\ell \to 0$ the self-action disappears, provided $n < 4$. This is the so-called super-renormalizable theory. When $n = 4$ the self-action remains finite, that is the theory is renormalizable. Finally, at $n > 4$ the self-interaction grows infinitely, in this case the theory is nonrenormalizable.

B. We now turn to electrodynamics. The energy density is in this case

$$H = [\hbar c\bar{\Psi}\hat{\partial}\Psi + mc^2\bar{\Psi}\Psi] + e\bar{\Psi}\hat{A}\Psi + F_{\mu\nu}F^{\mu\nu}. \qquad (4)$$

The notation is obvious. We put

$$\Psi = \frac{1}{\ell^{3/2}}\tilde{\tilde{\Psi}}, \quad A_\mu = \frac{\sqrt{\hbar c}}{\ell}\tilde{\tilde{A}}_\mu, \quad F_{\mu\nu} = \frac{\sqrt{\hbar c}}{\ell^2}(\tilde{\tilde{A}}'_{\mu\nu} - \tilde{\tilde{A}}'_{\nu\mu}). \qquad (5)$$

From here and eq. (4) we get

$$\tilde{\tilde{H}} = \bar{\tilde{\tilde{\Psi}}}\tilde{\tilde{\Psi}}' + mc\ell\bar{\tilde{\tilde{\Psi}}}\tilde{\tilde{\Psi}} + \alpha^{1/2}\bar{\tilde{\tilde{\Psi}}}\hat{\tilde{\tilde{A}}}\tilde{\tilde{\Psi}} + \frac{1}{4}(\tilde{\tilde{A}}')^2. \qquad (6)$$

Here $\alpha = e^2/\hbar c$. It is seen from eq. (6) that at $\ell \to 0$ the density $\tilde{\tilde{H}}$ remains finite. Electrodynamics is known to be renormalizable. We consider

C. A weak interaction of two spinor fields Ψ and Φ via a contact interaction of currents. The expression for the energy density reads now

$$H = [\hbar c\bar{\Psi}\hat{\partial}\Psi + m_1c^2\bar{\Psi}\Psi] + [\hbar c\bar{\Phi}\hat{\partial}\Phi + m_2c^2\bar{\Phi}\Phi] +$$

$$+ G_F\bar{\Psi}O^\lambda\Psi \cdot \bar{\Phi}O_\lambda\Phi. \qquad (7)$$

Here m_1 and m_2 are the spinor field masses, G the Fermi-interaction constant, O_λ the current operator. Using the first formula of eqs. (5) we obtain

$$\tilde{\tilde{H}} = [\bar{\tilde{\tilde{\Psi}}}\tilde{\tilde{\Psi}}' + m_1c\ell\bar{\tilde{\tilde{\Psi}}}\tilde{\tilde{\Psi}}] + [\bar{\tilde{\tilde{\Phi}}}\tilde{\tilde{\Phi}}' + m_2c\ell\bar{\tilde{\tilde{\Phi}}}\tilde{\tilde{\Phi}}] +$$

$$+ \frac{\Lambda_F^2}{\ell^2}\bar{\tilde{\tilde{\Psi}}}O^\lambda\tilde{\tilde{\Psi}}\bar{\tilde{\tilde{\Phi}}}O_\lambda\tilde{\tilde{\Phi}}. \qquad (8)$$

$$\Lambda_F = (G_F\hbar c)^{1/2} = 0{,}67 \cdot 10^{-16}\ \text{cm}.$$

At $\ell \to 0$ the energy density increases infinitely and cannot be compensated by the kinetic energy density[x/]. The weak interaction theory is unrenormalizable. The situation changes if we replace the contact interaction by the interaction via an intermediate boson (B_μ field). Now the energy density takes on the form

$$H = T_1 + T_2 + \frac{1}{4}\left(\frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu}\right)^2 + M^2 B^\mu B_\mu + + g_1 \bar{\Psi} Q_\lambda \Psi B^\lambda + g_2 \bar{\Phi} O_\lambda \Phi B^\lambda, \qquad (9)$$

where by T_1 and T_2 we denote the density of free spinor fields (the two first terms in eq. (8)), the next two terms in eq. (9) is the energy density of the boson field, M is the intermediate boson mass. The two last terms are the interaction of the boson field with the fermion ones, in this case $\frac{g_1 g_2}{M^2} = G_F$. Putting $B_\mu = \sqrt{\frac{\hbar c}{\ell}}\,\tilde{\tilde{B}}_\mu$ it is not difficult to see that

$$\tilde{\tilde{H}} = \tilde{\tilde{T}}_1 + \tilde{\tilde{T}}_2 + \frac{1}{4}(B')^2 + M^2 \ell^2 \tilde{\tilde{B}}^2 + + \tilde{\tilde{g}}_1 \bar{\tilde{\tilde{\Psi}}} O_\lambda \tilde{\tilde{\Psi}} \tilde{\tilde{B}}^\lambda + \tilde{\tilde{g}}_2 \bar{\tilde{\tilde{\Phi}}} O_\lambda \Phi \tilde{\tilde{B}}^\lambda. \qquad (10)$$

At $\ell \to 0$ this expression remains finite. In this case $\tilde{\tilde{g}}_1 = \frac{g_1}{\sqrt{\hbar c}}$ and $\tilde{\tilde{g}}_2 = \frac{g_2}{\sqrt{\hbar c}}$. The theory remains renormalizable.

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x/ This conclusion was indicated by the author in 1957 in the article, devoted to weak interactions/1/.

D. The chiral-symmetric field Π_a ($a = 1,2,3$). In this case the energy density is

$$H = \frac{1}{2} g_{ab} (\dot{\Pi}_a \dot{\Pi}_b + \nabla \Pi_a \nabla \Pi_b) \qquad (11)$$

the chiral tensor g_{ab} being equal to/3/

$$g_{ab} = A \cdot \delta_{ab} + B \frac{\Pi_a \Pi_b}{\Pi^2}, \qquad (12)$$

$$B = 1 - A, \quad A = \left(\frac{\sin z}{z}\right)^2, \quad z = \frac{\Pi}{\Pi_0}, \qquad (13)$$

where Π_0 is a certain constant/2/. Putting $\Pi = \frac{\sqrt{\hbar c}}{\ell}\tilde{\tilde{\Pi}}$, $z = \frac{\sqrt{\hbar c}}{\ell \Pi_0}\tilde{\tilde{\Pi}}$ we make sure that the dimensionless density $\tilde{\tilde{H}}$ remains finite for $\ell \to 0$ x/. In this case infinities are removed by the superpropagator method/3/.

2. THE ESSENTIALLY NONLINEAR FIELD

The simplest example of the essentially nonlinear field is the scalar field Φ, the equation for which is determined by the Lagrangian

$$\mathcal{L} = \mathcal{L}(K), \quad K = \frac{1}{2}[\dot{\Phi}^2 - \nabla \Phi^2]. \qquad (14)$$

For the sake of simplification of the foregoing discussion we restrict ourselves to the case of the two dimensions (t, x). With this simplification the equation for the field takes the form

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x/ Note, that in spite of the dependence of the tensor g_{ab} on the field Π, the chiral-symmetric field has straight-line characteristics and therefore does not belong to the class of essentially nonlinear fields.

$$g_{00}\frac{\partial^2\Phi}{\partial t^2} + 2g_{01}\frac{\partial^2\Phi}{\partial t\partial x} + g_{11}\frac{\partial^2\Phi}{\partial x^2} = 0, \quad (15)$$

where

$$g_{00} = 1+\alpha p^2, \quad g_{01} = \alpha pq, \quad g_{11} = -(1-\alpha q^2) \quad (16)$$

and

$$p = \frac{\partial\Phi}{\partial t}, \quad q = \frac{\partial\Phi}{\partial x}, \quad \alpha = \frac{d^2\mathfrak{L}}{dK^2}\Big/\frac{d\mathfrak{L}}{kK}. \quad (17)$$

The velocity of the signal propagation (weak discontinuity) for such a field is/4,5/

$$u^{\pm} = \frac{1}{g_{00}}[g_{01} \pm \sqrt{-D}], \quad (18)$$

$$D = g_{00}g_{11} - g_{01}^2 = -[1+\alpha(p^2-q^2)]. \quad (19)$$

The velocities of opposite directions are obtained provided that $D<0$ and g_{00} and g_{11} have opposite signs (normal behaviour). If the signs of g_{00} and g_{11} are identical then both the velocities $u^{\pm}$ have the same direction. In this case there does not exist a reflected signal: a "collapse" for the field Φ comes. For $D>0$ the velocities become complex. The equation turns to an elliptic type equation. The time and space cease to be distinguishable with respect to the field Φ.

This phenomenon was called by the author the formation of a "lump" of events (see ref./5/)x/.

Thus, the region of the normal behaviour of a classic field is restricted by the conditions

$$D<0, \quad g_{00} \gtrless 0, \quad g_{11} \lessgtr 0. \quad (20)$$

According to eqs. (17) and (19) these conditions read

$$1+\alpha(p^2-q^2)>0, \quad 1+\alpha p^2 \gtrless 0, \quad 1-\alpha q^2 \gtrless 0. \quad (21)$$

The conditions (21) mean the restriction on the allowable values of the derivatives which is to be taken into account in the fundamental Feynman integral when integrating in the functional space $R\{\Phi(x,t)\}$.

Putting again $\Phi = \frac{\sqrt{\hbar c}}{\ell}\tilde{\Phi}$ and mentioning that the Lagrangian (14) is a function of the form $\mathfrak{L}(K) = b_0^2F(K/b_0^2)$ (b_0 is the field scale defining the nonlinearity of equations) we obtain for the measure of the derivatives the ratio

$$\gamma = \frac{\sqrt{\hbar c}}{\ell^2}\frac{1}{b_0}. \quad (22)$$

This ratio must not exceed the values of the order of unity. From here

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x/In this connection it is interesting to study in more detail the situation in the gravitational field during the collapse whether or not the collapse can result in the formation of a "lump" of events.

$$\ell > \left(\frac{\hbar c}{b_0^2}\right)^{1/4}. \qquad (23)$$

Thus, there arises a restriction on the magnitude of the field gradients, in other words, on the wave length. More exact condition (23) can be formulated only on the basis of the knowledge of the coefficient $a(K/b_0^2)$. In particular, for a Lagrangian of the Born-Infeld type

$$\mathcal{L} = b_0^2[\epsilon\sqrt{1 + 2\epsilon K/b_0^2} - \epsilon], \quad \epsilon = \pm 1. \qquad (24)$$

The limitations on the derivatives following from eq. (21) are

$$b_0^2 + \epsilon(p^2 - q^2) > 0, \; b_0^2 + \epsilon p^2 > 0, \; b_0^2 - \epsilon q^2 > 0. \qquad (25)$$

The class of fields described by the Lagrangian (24) includes the field on "one-dimensional" string studied in ref./6/. Recently the interest in the theory of such a string has increased due to papers/7,8/ in which one has pointed to the connection of this theory with the theory of the Veneziano dual amplitude.

3. VACUUM POLARIZATION

In this section we consider the vacuum polarization of a fermion field Ψ of a large mass M.

This field may be interpreted as a field of "partons", "quarks" or "maximons". By the notion of maximon I mean particles of large (extreme) mass which may, however, not

exist in the free state due to large decay constant Γ, comparable with the mass $\Gamma \cong M$.

We consider the polarization of such a vacuum induced by a convergent spherical wave of a certain boson field Φ the mass of which $m \ll M$. The vacuum polarization is defined by the field

$$b_0 = Mc^2/g\left(\frac{\hbar}{Mc}\right) = \frac{M^2c^3}{g\hbar}, \qquad (26)$$

having the simple meaning: the work of this field on the length $\hbar/Mc$ is equal to Mc^2 (comp./9/). As the measure of the field Φ, we take as before the quantity $\frac{\sqrt{\hbar c}}{\ell}$ so that $\Phi = \frac{\sqrt{\hbar c}}{\ell}\tilde{\tilde{\Phi}}$; $(\tilde{\tilde{\Phi}} \approx 1)$. The vacuum polarization induced by the wave Φ will be noticeable if γ (eq. (22)) is of the order of unity. On the other hand, at γ close to unity there arise the above-mentioned restrictions on the field derivatives. This limitation with the account of eqs. (22) and (23) reads

$$\ell \geq (g^2/\hbar c)^{1/4}\frac{\hbar}{Mc}. \qquad (27)$$

The vacuum polarization is appreciable when this relation is close to the equality.

For the field Φ to be efficiently described by a Lagrangian belonging to the class of essentially nonlinear fields (14) it is necessary that the condition of quasi-stationarity

$$\left|\frac{\hbar}{Mc}\frac{\partial\Phi}{\partial x}\right| = \left|\frac{\hbar}{Mc}\frac{1}{\ell}\Phi\right| \ll |\Phi| \qquad (28)$$

be fulfilled, that is $\ell \gg \frac{h}{Mc}$. For $g^2/hc \gg 1$ there exists the region of the values of the field and its derivatives for which both the condition (28) is fulfilled and the equality in eq. (27) is reached. This implies that in this situation when the boson field comes nearer to the maximon it becomes essentially nonlinear. In particular, an anomalous behaviour of the field (collapse) is also quite possible.

It follows from eqs. (27) and (28) that nothing of the kind can occur in electrodynamics since the equality in eq. (27) is reached in this case when $\ell \ll h/Mc$.

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