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TRANSLATION INVARIANT QUANTUM FIELD THEORY WITH DE-SITTER MOMENTUM SPACE OFF THE MASS SHELL

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Summary

In the framework of Bogolubov's axiomatic approach problems connected with the extension of the scattering matrix off the mass shell are considered. A specific point for the standard extension procedure is the assumption that the four dimensional space of virtual momenta in which the extended objects (fields, currents, S-matrix coefficient functions, etc) are defined is flat Minkowski space. However, such a choice of the geometry of the virtual momentum space does not follow from the basic axioms of the theory and in fact is an independent postulate. In our opinion the pseudoeuclidean momentum space is not adequate for the description of the phenomena at high energies (short distances). We suppose that the use of Minkowski p-space is actually responsible for the known difficulties of the local quantum field theory connected with the problem of multiplying of distributions with coinciding singularities on the light cone. As an alternative we propose to use in the extension of the S-matrix a 4-momentum space of constant curvature (De-Sitter space) with curvature radius $1/\ell_0$, where ℓ_0 is a fundamental length. The interaction laws of the elementary particles at large momenta are completely different in the new scheme.

The off-mass-shell S-matrix extension in the spirit of De-Sitter p-space geometry is consistent with the requirements of Poincaré invariance, unitarity, spectrality, completeness of the system of asymptotic states. With the help of a Fourier transformation in De-Sitter momentum space a new configuration ξ -space is introduced, whose geometry for small distances $\lesssim \ell_0$ is essentially different from the pseudoeuclidean one. The causality condition which is direct generalization of Bogolubov's causality condition, going to it in the limit $\ell_0 \to 0$, is formulated in terms of this ξ -space. It is demonstrated that in the developed theory the problem of distribution products loses its acuteness. In particular the commutation functions and propagators in the new scheme are usual (not generalized) functions and there is no arbitrariness in any their powers and products.

§ 1. As it is well known the dynamical description in quantum field theory requires the use of quantities off the mass shell. Such quantities are the Green's functions, the Heisenberg fields and currents, the scattering matrix depending on external fields and sources, etc. Some physical requirements and in particular the Bogolubov's causality condition can be formulated only off the mass shell.

The extension of the S-matrix off the mass shell obeying besides the causality condition the standard set of axioms:

1) *Poincaré invariance*,

2) Unitarity,

3) Completeness of the system of states with positive energy,

4) Uniqueness of the vacuum state,

5) Stability of the vacuum and one particle states,

leads to most general formulation of the present local quantum field theory/1/. If we restrict ourselfs in considering only one-component scalar field φ with mass m, then the physical S-matrix may be represented as the following decomposition:

$$S = \sum_{n} \int d^4p_1 \dots d^4p_n \, S_n(p_1, \dots, p_n) : \varphi(p_1) \dots \varphi(p_n) : \qquad (1)$$

Here we deliberately use p-representation. By definition:

$$\varphi(p) = \frac{1}{(2\pi)^{3/2}} \int e^{-ipx} \varphi(x) \, d^4x \qquad (2)$$

$$\varphi^+(p) = \varphi(-p) \qquad (3)$$

and

$$(m^2 - p^2)\varphi(p) = 0 \qquad (4a)$$

$$\varphi(p) = \delta(m^2 - p^2)\tilde{\varphi}(p). \qquad (4b)$$

From (4b) and the invariance of S under translation transformations

$$\varphi(p) \to e^{iPa}\varphi(p) \tag{5}$$

it follows that the coefficient function (c.f.) $S_n(p_1, \dots, p_n)$ in the decomposition (1) is defined only on the surface:

$$\begin{cases} (p_1 + \dots + p_n)_\mu = 0 \quad , \quad \mu = 0,1,2,3 \\ p_1^2 = m_1^2, \dots, p_n^2 = m_n^2 \quad , \end{cases} \tag{6}$$

which further on we shall call mass shell.

§ 2. One of the ways to extend the S-matrix off the mass shell is to add a classical field to the operator $\varphi(p)$. The resulting extended operator $\phi(p)$ does not satisfy more the Klein Gordon equation (4a) and the 4-momenta p_i (i = 1, 2, ... n), on which depend the extended c.f. $S_n(p_1, \dots, p_n)$ are off the hyperboloids:

$$p_i^2 - m_i^2 = 0 \ , \tag{7}$$

i.e., become virtual.

It is usually considered as obvious that the virtual 4-momenta form a pseudoeuclidean Minkowski space (M). At least it is supposed in the present quantum field theory. This is the reason why, for example, the region in which the c.f. $S_n(p_1, \dots, p_n)$ are defined in translation invariant way is (c.f. eq. (6)):

$$\begin{cases} (p_1 + \dots + p_n)_\mu = 0 \quad , \quad \mu = 0,1,2,3 \\ p_1 \in M, \dots, p_n \in M \ . \end{cases} \tag{8}$$

Analyzing the axioms of quantum field theory we concluded [2] that the pseudoeuclidean character of the 4-dimensional space of virtual momenta does not follow from these axioms and is <u>essentially new</u> postulate of the theory.

It is well known that in local field theory one has to work with products of generalized functions with coinciding singularities on the light cone:

$$(x_1 - x_2)^2 \equiv \xi^2 = 0 \tag{9}$$

These products are not defined in a unique way and as a result in the theory appear arbitrary constants, divergences, etc.

In momentum representation these difficulties are associated with the region of large virtual momenta. But, as we said above, the choice of the geometry of the virtual momentum space is not firmly connected with the basic requirements of the theory and is in fact in our hands. We think that in the usual theory this choice is unsuccessful, i.e., pseudoeuclidean character of the 4-dimensional momentum space is actually responsible for the mentioned difficulties of the local theory.

<u>§ 3.</u> As an alternative in order to describe the virtual 4-momenta, we propose to use one of the De-Sitter spaces:

$$p_0^2 - \vec{p}^2 + M^2 p_4^2 = M^2 = \frac{1}{\ell_0^2}, \tag{10}$$

$$p_0^2 - \vec{p}^2 - M^2 p_4^2 = -M^2 = -\frac{1}{\ell_0^2}. \tag{11}$$

Here ℓ_0 is a new universal constant with dimension of length ("fundamental length"). M is the corresponding "fundamental mass". Later on we shall put $\hbar = c = M = \ell_0 = 1$.

When $|p| \ll 1$ both De-Sitter spaces (10) and (11) coincide with the Minkowski space. If $|p| \gtrsim 1$ the curvature effects become essential. Therefore in a field theory using De-Sitter p-space the large virtual momenta are described in a completely different manner in comparison with the usual theory.

Presently we can not definitively choose one of the possibilities (10)-(11). Here we shall consider only the case (10). Therefore our basic idea may be formulated in the following way: in the extension off the mass shell the virtual 4-momenta p_μ ($\mu = 0,1,2,3$) become arbitrary vectors in De-Sitter space

$$p_0^2 - \vec{p}^2 + p_4^2 = 1 \,. \tag{12}$$

Doing that we consider that the axioms 1-5 have to be satisfied like they were before.

It is clear that in the new scheme the region in which the extended c.f. $S_n(p_1, \ldots, p_n)$ are defined have to be (instead of (8)):

$$\begin{cases} (p_1 + \cdots + p_n)_\mu = 0 \\ p_{10}^2 - \vec{p}_1^2 + p_{14}^2 = 1, \ldots, \; p_{n0}^2 - \vec{p}_n^2 + p_{n4} = 1 \,. \end{cases} \tag{13}$$

Let $\Phi(p, p_4)$ be the operator of the extended field, defined on De-Sitter space (12). Like before this quantity depends on four variables, for instance (p_0, p_1, p_2, p_3). It follows from (13) that in the new scheme the extended S-matrix is invariant under the "gauge" transformation (compare with eq. (5)):[*)]

$$\Phi(p, p_4) \to e^{ipa}\, \Phi(p, p_4) \,, \; pa = p_0 a_0 - \vec{p}\cdot\vec{a} \,. \tag{14}$$

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*) In the proposed theory transformations of the type (14) can be considered as primary, completely forgeting that in the usual approach they correspond to translations $x \to x + a$ in the space time. The important point is that the invariance under the group (14) leads to the usual 4-momentum conservation law in the c.f: $(p_1 + \cdots + p_n)_\mu = 0$.

It is extremely important to understand that the curvature of the p-space and the requirement of invariance of the theory under transformations (14) are <u>completely compatible with each other</u>. This is seen when analyzing relations (13) and comparing them with (8).

<u>§ 4.</u> It can be easily seen that the hyperboloide (7) can be embedded in De-Sitter p-space only if the condition

$$m^2 \leq 1 \tag{15}$$

is satisfied.

We shall suppose that the restriction (15) is always fulfilled for the masses of these objects, which are described by quantized φ -fields. Then (7) is equivalent to the relation:

$$(p_4 - m_4)(p_4 + m_4) = 0 \tag{16}$$

where by definition $m_4 = \sqrt{1-m^2} \geq 0$. Since on the surface (12) to any fixed value of p there correspond two different just by sign values of p_4, then each of the brackets in (16) can vanish:

$$p_4 - m_4 = 0, \tag{17a}$$

$$p_4 + m_4 = 0. \tag{17b}$$

Let us now make an important physical assumption: for the free field $\varphi(p,p_4)$ defined in De-Sitter p-space only the condition (17a) is satisfied. In other words:

$$2(p_4 - m_4)\varphi(p,p_4) = 0. \tag{18}$$

We introduced the factor 2 in order eq. (18) to coincide exactly with eq.(4a) in the "flat" limit, $m, |p| \ll 1$ *).

From (18) it follows (compare with (4b)) that:

$$\varphi(p,p_4) = \delta(2p_4 - 2m_4)\tilde{\varphi}(p,p_4), \tag{19}$$

where $\tilde{\varphi}(p,p_4)$ is operator without singularities on the surface (17a).

The invariant volume element in the space (12) may be written in different ways:

1). $$d\Omega_p = d\omega\, d^3\vec{p} \tag{20}$$

$$\begin{cases} p_4 = \cos\omega\sqrt{1+\vec{p}^2} \\ p_0 = \sin\omega\sqrt{1+\vec{p}^2} \end{cases} \tag{21}$$

2). $$d\Omega_p = 2\,\delta(1-p_0^2+\vec{p}^2-p_4^2)\,d^5p = \tag{22a}$$

$$= 2\,\underset{\lambda=-1}{\mathrm{Res}}\,(1-p_0^2+\vec{p}^2-p_4^2)_+^\lambda\, d^5p = \tag{22b}$$

$$= 2\lim_{\lambda\to-1}\frac{1}{\Gamma(\lambda+1)}(1-p_0^2+\vec{p}^2-p_4^2)_+^\lambda\, d^5p\,, \tag{22c}$$

where

$$(x)_+^\lambda = \begin{cases} x^\lambda, & x \geqslant 0 \\ 0, & x<0\,. \end{cases} \tag{23}$$

Last formulae show that the integration over De-Sitter p-space can be reduced to standard operations with analytical functionals. In fact this is equivalent to some natural way of regularization of the integrals in this space (see for instance /3/).

*) The equation based on relation (17b) has no formally correct flat limit. Let us note, however, that from an optimistic point of view on the theory developed here we have not to exclude the possibility, that particle states with $p_4<0$ can have for the new theory such a fundamental meaning as, for instance, the states with negative energies in Dirac's theory of the electron.

The four-dimensional δ-function in De-Sitter p-space can be represented by the relation:

$$\int d^4\Omega_p\, \delta^{(4)}(p',p)\, \Phi(p,p_4) = \Phi(p',p'_4) \tag{24a}$$

$$\frac{\delta\Phi(p',p'_4)}{\delta\Phi(p,p_4)} = \delta^{(4)}(p',p) \tag{24b}$$

In the decomposition of the scattering matrix in terms of φ-fields every field operator appears accompanied by "its own" volume element:

$$\int \dots d\Omega_p\, \varphi(p,p_4) \dots \tag{25}$$

(the dots substitute other operators, volume elements, c.f., etc.). Now using (19) and (20) expression (25) can be written in the following way:

$$\int \dots d\Omega_p \varphi(p,p_4) \dots = \int \dots 2\delta(p_0^2-\vec{p}^2+p_4^2-1)d^5p\,\delta(2p_4-2m_4)\tilde{\varphi}(p,m_4) \dots =$$
$$= \int \dots \delta(p^2-m^2)\tilde{\varphi}(p,m_4)\, d^4p \dots \tag{26}$$

In the "flat" limit taking into account (4b) we should have instead of (26):

$$\int \dots d^4p\, \varphi(p) \dots = \int \dots \delta(p^2-m^2)\tilde{\varphi}(p)\, d^4p \dots \tag{27}$$

Comparing (26) and (27) we conclude that on the surface (7) the following equality should be satisfied:

$$\tilde{\varphi}(p,m_4) = \tilde{\varphi}(p). \tag{28}$$

Relation (28) plays the role of a specific "correspondence principle". With its help the commutation relations which should be satisfied by the solutions of equation (19) can be determined. Simple calculations give:

$$[\varphi(p_1,p_{14}),\varphi(p_2,p_{24})] = \delta^{(4)}(p_1-p_2)\delta(p_2^2)\delta(2p_{24}-2m_4). \tag{29}$$

The notion of a normal product of field operators and the correspondent Wick's theorem can be formulated in the new scheme without changes in principle. In such a way the S-matrix, like before, may be represented in the form of a decomposition, just making formally the substitution:

$$d^4p \to d\Omega_p \quad , \quad \varphi(p) \to \varphi(p, p_4) . \tag{30}$$

On the mass shell, because of (28), the substitution (30) is reduced only into introduction of new notations. However in the extension of the field off the shell $p_4 = m_4$, i.e., in the transition from the operator $\varphi(p, p_4)$ to the operator $\phi(p, p_4)$ a new extended S-matrix appears. Its coefficient functions, as we already mentioned are defined in regions of the type (13) and therefore the behaviour of these functions at large virtual momenta $|p| \gtrsim 1$ is much different to all with which we are familiar in the conventional theory.

Let us further note that because of the translation invariance the total 4-momentum in the new scheme is conserved in any transition and all the properties of this quantity, in particular the character of the spectrum remain unchanged too. For instance for a system of two free particles with 4-momenta p_1 and p_2 we have:

$$4m^2 \leq (p_1 + p_2)^2 < \infty . \tag{31}$$

However the curvature of the p-space inevitably affects the 4-momenta which are not fixed by the total momentum conservation law. These 4-momenta in the usual theory are proportional to the differences of the particles momenta (real or virtual) and may be conventionally called "relative" momenta.

§ 5. The "distortion" of the relative momentum of the system of two particles in the new scheme can be illustrated by the following reasoning. Let (p_1, p_{14}) and (p_2, p_{24}) are two 5-vectors of the space (12). If $p_{14} = p_{24} = m_4$ then we have real particles; in the general case, which we shall consider, the corresponding particles are virtual. Let us pass from eight independent variables (p_1, p_{14}) and (p_2, p_{24}) to new variables among which we shall obligatory want the total energy-momentum vector to be:

$$P_\mu = (p_1 + p_2)_\mu \ , \ \mu = 0, 1, 2, 3. \tag{32}$$

In the usual theory the second independent 4-vector is usually taken to be the "relative" momentum q, defined by the relations:

$$\begin{aligned} p_1 &= q + \frac{P}{2} \\ p_2 &= -q + \frac{P}{2} \end{aligned} \quad , \quad q = p_1 - \frac{P}{2} = \frac{p_1 - p_2}{2} \tag{33}$$

In De-Sitter p-space direct analogues of the formulae (33) exist:

$$\begin{cases} p_1 = q (+) \vec{U} \ , \quad p_2 = -q (+) \vec{U} \\ U_\mu = \dfrac{P_\mu}{2\sqrt{1-q^2}} \ , \quad U_4 = \sqrt{1-U^2} \end{cases} \tag{34}$$

$$\begin{cases} q = p_1 (-) \vec{U} = \dfrac{\mu_2 p_1 - \mu_1 p_2}{\mu_1 + \mu_2} \\ q_4 = \sqrt{1-q^2} \end{cases} , \tag{35}$$

where

$$\mu_1 = \frac{1}{2}\left(p_{14} + \frac{1}{2}\sqrt{P^2 + (p_{14} + p_{24})^2}\right) ,$$

$$\mu_2 = \frac{1}{2}\left(p_{24} + \frac{1}{2}\sqrt{P^2 + (p_{14} + p_{24})^2}\right) .$$

In these relations with the symbol (+) we denote the operation of translation on the surface (12). This operation belongs to the motion group SO(2,3). Explicitly:

$$\begin{aligned} (a)_\mu &= (b (+) c)_\mu = b_\mu + c_\mu \left(b_4 - \frac{bc}{1+b_4}\right) \\ (a)_4 &= (b (+) c)_4 = b_4 c_4 - bc . \end{aligned} \tag{36}$$

In the flat limit, obviously $\text{bac} \to b+c$.

Coming back to (34) and comparing these formulae with (33) we conclude that together with P_μ ,"independent" variable is the 4-momentum q_μ which belongs, in contrast to P_μ to the De-Sitter space (12).

Now it is clear that in the theory we developed also the quantities, which are in the flat limit <u>coordinate differences</u>,will be <u>essentially modified</u>. We shall denote these "relative" coordinates by ξ (compare with (9)). Evidently the ξ-space is canonically conjugated to the curved p-space (12)in the spirit of the correspondent Fourier transformation. Later on we shall denote the kernel of this transformation by $\langle\xi|p,p_4\rangle$.

<u>§ 6.</u> Quantities $\langle\xi|p,p_4\rangle$ are eigenfunctions of the Casimir's operators of the group SO(2,3);

$$-\frac{1}{\sqrt{g}}\frac{\partial}{\partial p_\mu}\left(g^{-1}_{\mu\nu}\sqrt{|g|}\,\frac{\partial}{\partial p_\nu}\right)\langle\xi|p,p_4\rangle = \lambda\langle\xi|p,p_4\rangle \qquad (37)$$

($g_{\mu\nu}$ is the metric tensor of the curved 4-space (12), $g = \det\|g_{\mu\nu}\|$ and ξ is a complete set of observables in the new configuration representation). Without going into details let us only notice that the λ -spectrum in (37) corresponds to the maximally degenerate series of unitary representations of the group SO(2,3)[4] and consists of two branches-discrete and continuous:

$$\lambda = \begin{cases} L(L+3) & , \; L = -1, 0, 1, \ldots \qquad (38a) \\ -\left(\frac{3}{2}\right)^2 - \Lambda^2 & , \; 0 \leq \Lambda \leq \infty \qquad (38b) \end{cases}$$

In the flat limit (37) becomes the eigenvalue problem for the operator of the pseudoeuclidean interval $\left(i\frac{\partial}{\partial p}\right)^2$. The L -region goes into the timelike region and the Λ -region into the spacelike region. Let us emphasize that there is no analogue of the light-cone in the spectrum (38). This surface appears <u>only</u> in the flat limit.

The basis functions $\langle \xi | p, p_4 \rangle$ corresponding to the spectrum (38) may be written in relativistic invariant way

$$\langle \xi_L | p, p_4 \rangle = (p_4 - i p N)^{-L-3}, \quad N_\mu = (N_0, \vec{N}), \; N^2 = 1, \tag{39}$$

$$\langle \xi_\Lambda | p, p_4 \rangle = (p_4 + p N)_+^{-3/2 + i\Lambda}, \quad N_\mu = (N_0, \vec{N}), \; N^2 = -1. \tag{40}$$

The functions (39) in accordance with the discrete character of the spectrum (38a) are square integrable in the metrics $d\Omega_p$ and (40) have to be considered as generalized function of the type(23).

In the flat limit any of these quantities transforms in usual exponent:

$$\langle \xi_L | p, p_4 \rangle = (p_4 - i p N)^{-L-3} \to e^{i L(Np)} = e^{i \xi p}, \quad \xi_\mu = L N_\mu \tag{41}$$

$$\langle \xi_\Lambda | p, p_4 \rangle = (p_4 + p N)_+^{-3/2 + i\Lambda} \to e^{i \Lambda (Np)} = e^{i \xi p}, \quad \xi_\mu = \Lambda N_\mu. \tag{42}$$

In order to perform Fourier transformation in De-Sitter p-space one may also use basis functions in which the complete set of variables ξ differs from the set (39)-(40). Let us consider in this connection the generators of the 5-rotations in the ($\mu 4$)-planes:

$$M^{\mu 4} = -i \, p_4 \frac{\partial}{\partial p_\mu}. \tag{43}$$

The zero component of this 4-vector in terms of $(\omega, \vec{p})$-coordinates (see (21)) is equal to $M^{04} = i\frac{\partial}{\partial \omega}$. From here, imposing periodicity in ω , we obtain that in any Lorentz reference frame the eigenvalues of the operator M^{04} are integers $n = 0, \pm 1, \pm 2, \ldots$ and the correspondent eigenfunctions have the form:

$$\langle n|\omega\rangle = e^{in\omega}. \tag{44}$$

Since M^{04} commutes with the Casimir's operator of the group $SO(2,3)$, n may be included in the complete set of observables together with λ. In such a way we get one more set of basis functions. Their use is particularly attractive because of the simple ω-dependence:

$$\langle \lambda, n, \ldots | p, p_4\rangle = \langle n|\omega\rangle\langle\lambda, n, \ldots|\vec{p}\rangle. \tag{45}$$

The dots correspond to the other variables in the complete set. As an illustration we give the full expression for a function of the considered type in the case of the discrete spectrum:

$$\langle \lambda, n, \theta_n, \varphi_n|\omega, p, \theta_p, \varphi_p\rangle = \frac{1}{\sqrt{A}} e^{in\omega} C^{L+3}_{|n|-(L+3)}\left[\frac{p}{\sqrt{1+p^2}}(\cos\theta_n\cos\theta_p - \sin\theta_n\sin\theta_p\cos(\varphi_n-\varphi_p))\right]. \tag{46}$$

Here C^{α}_{k} is Gegenbauer polynomial and A is a normalization constant. The function (46) is different from zero if:

$$|n| \geqslant L+3 = 2,3,4,\ldots \tag{47}$$

The discrete parameter n we shall call "time", because in the flat limit the quantity M^{04} coincides with the time operator $-i\frac{\partial}{\partial p_0}$ of the usual theory.

A remarkable property of the discrete time n is the invariance of its sign in the representations of $SO(2,3)$ corresponding to the discrete "timelike" region(38):

$$\frac{n}{|n|} = \text{invar}\ , \quad \text{if}\ \lambda = L(L+3). \tag{48}$$

The inequality (47) in this case plays the role of "timelikeness" condition.

Because of (48) the operators in the L-region can be ordered in invariant way in the parameter n. The correspondent "step" function has the form:

$$\theta(n) = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \frac{e^{in\omega}}{\operatorname{tg}\frac{\omega}{2} - i\varepsilon}\, d\omega = \begin{cases} 1, & n > 0 \\ 0, & n < 0 \end{cases} \tag{49}$$

Now we are able to make the following conclusion: the new ξ -space consists of two regions, L and Λ , which are analogous to the timelike and spacelike regions of the pseudoeuclidean space. Moreover - in the L-region one can order in an invariant way in terms of the discrete time. Therefore in our disposal we have all necessary machinary in order to formulate the causality condition in the developed theory. This condition, as we already said, can be written only off the mass shell. Therefore it would be sensitive to the accepted by us way of extension off the mass shell.

§ 7. In order to be able to attack the problem of causality some preliminary work is needed.

Let us consider the operator:

$$\varphi(\xi) = \frac{1}{(2\pi)^{3/2}} \int \langle \xi | p, p_4 \rangle\, \varphi(p, p_4)\, d\Omega_p . \tag{50}$$

Let us now apply to it the translation transformation (14):

$$\frac{1}{(2\pi)^{3/2}} \int \langle \xi | p, p_4 \rangle\, \varphi(p, p_4)\, e^{ipa}\, d\Omega_p \equiv \varphi_a(\xi). \tag{51}$$

In the flat limit we have, of course:

$$\varphi_a(\xi) = \varphi(\xi + a). \tag{52}$$

In the present case the quantities a and ξ have completely different mathematical nature and this is the reason why $\varphi_a(\xi) \neq \varphi(\xi+a)$.

Let us further put by definition:

$$\frac{1}{(2\pi)^{3/2}} \int \varphi(p, p_4)\, e^{ipa}\, d\Omega_p \equiv \varphi_a(0). \tag{53}$$

and let us consider the commutator $[\varphi_a(\xi), \varphi_a(0)]$. With the help of (29) it is easy to demonstrate that it does not depend on a, i.e., is translation invariant:

$$[\varphi_a(\xi),\varphi_a(0)] \equiv \frac{1}{i}\mathcal{D}(\xi,0) = \frac{-1}{(2\pi)^3}\int \langle \xi | p, p_4\rangle \varepsilon(p_4)\delta(2p_4-2m_4)\, d\Omega_p . \tag{54}$$

From (54) it follows that:

$$[\varphi_a(\xi),\varphi_a(0)] = 0 , \tag{55}$$

if ξ is in the continuous spacelike series (38b).

Completely in the same manner one can prove that:

$$[\varphi_a(\xi),\varphi_a(-\xi)] = 0 , \qquad a\text{ - arbitrary}, \xi \in \Lambda\text{-series} \tag{56}$$

Let us mention, by the way, that the variable ξ plays the role of a "relative" coordinate in the considered commutation relations. The equalities (55) and (56) may be taken as pattern in the formulation of the locality condition in a theory with interaction.

Now let us consider the chronological product of free φ - fields:

$$T\varphi_a(\xi)\varphi_a(0) = \theta(n)\varphi_a(\xi)\varphi_a(0) + \theta(-n)\varphi_a(0)\varphi_a(-\xi) . \tag{57}$$

This product, because of (48a) and (55) is invariant. After putting (57) in normal form and taking into account (49) we obtain:

$$T\varphi_a(\xi)\varphi_a(0) = :\varphi_a(\xi)\varphi_a(0): + \langle 0 | T\varphi_a(\xi)\varphi_a(0)|0\rangle , \tag{58}$$

$$\langle 0|T\{\varphi_a(\xi)\varphi_a(0)\}|0\rangle \equiv \frac{1}{i}\mathcal{D}^c(\xi) = \frac{1}{i(2\pi)^4}\int \langle \xi | p, p_4\rangle \frac{d\Omega_p}{2(p_4-m_4-i\varepsilon)} \tag{59}$$

Therefore we have the right to interprete the quantity

$$\mathcal{D}^c(p) = \frac{1}{2(p_4-m_4-i\varepsilon)} \tag{60}$$

as the propagator of a free particle. Let us also notice that

$\mathcal{D}^c(p)$ is Green's function of the equation (18).

§ 8. Let us define the current operator in De-Sitter space by:

$$j(p,p_\nu) = i\,\frac{\delta S}{\delta\Phi(-p,p_\nu)}\,S^+ . \tag{61}$$

Evidently under transformations (14) and because of the invariance of the S-matrix we shall have:

$$j(p,p_\nu) \to e^{ipa} j(p,p_\nu) . \tag{62}$$

Applying simultaneously the transformations (50) and (62) to the operator (61) we obtain in complete analogy with (51):

$$j_a(\xi) = \frac{1}{(2\pi)^{5/2}} \int e^{ipa} \langle\xi|p,p_\nu\rangle\, j(p,p_\nu)\, d\Omega_p . \tag{63}$$

Let us now introduce the "bilocal" variation derivative:

$$\left(\frac{\delta}{\delta\varphi(\xi)}\right)_a = \frac{1}{(2\pi)^{5/2}} \int d\Omega_p\, e^{-ipa} \langle -\xi|p,p_\nu\rangle \frac{\delta}{\delta\varphi(p,p_\nu)} \tag{64}$$

We postulate that our current operator satisfies the condition:

$$\left(\frac{\delta}{\delta\varphi(\xi)}\right)_a j_a(\xi) = 0 \qquad \text{for } \xi \gtrsim 0 \;,\quad a\text{- arbitrary}, \tag{65}$$

where the symbol $\xi \gtrsim 0$ we understand in the following way:

1) either $\xi \in L$ - series and $n > 0$

2) or $\xi \in \Lambda$ -series.

Evidently the flat limit of (65) is the Bogolubov's causality condition

$$\frac{\delta j(\xi+a)}{\delta\varphi(-\xi+a)} = 0 \qquad \text{for } \xi \gtrsim 0,\ a\text{- arbitrary} \tag{66}$$

Here the symbol $\xi \gtrsim 0$ has already the usual sense:

1) either $\xi^2 > 0$, $\xi_0 > 0$, 2) or $\xi^2 < 0$.

Using the well known "solvability condition", following from the commutativity of the variation derivatives $\frac{\delta}{\delta\varphi}$, it is easy to write relation (65) in the following form:

$$\left(\frac{\delta}{\delta\Psi(-\xi)}\right)_{a} j_a(\xi) = i\,\theta(-n)\,[\,j_a(\xi), j_a(-\xi)] \tag{67}$$

$$[\,j_a(\xi), j_a(-\xi)] = 0 \ \text{, if } \xi \in \Lambda\text{-series, } a \text{ is arbitrary.} \tag{68}$$

Equality (68) has to be considered as the locality condition for the current in the new scheme (c.f.[56]). It ensures the relativistic invariance of the equation (67).

Relations (67) and (68) may be used in order to obtain the S-matrix c.f., for example in perturbation theory.

The product of the step θ-function and the current commutator appearing in (67) have to be considered with closest attention since in the usual theory such kind of products can not be uniquely defined (see above). In other words a reasonable question arises: what happens with the products of generalized functions with coinciding singularities in the present scheme?

<u>§ 9.</u> In order to clarify this question let us consider in the framework of the usual theory the product of the functions $\delta(\xi_0)$ and $\theta(\xi_0)$. Since $\theta(\xi_0) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE\, \frac{e^{iE\xi_0}}{E - i0}$, then formally

$$\theta(\xi_0)\,\delta(\xi_0) = \left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{dE}{E - i0}\right\}\delta(\xi_0) = \infty\,\delta(\xi_0)\,.$$

A more rigorous approach based on the theory of the generalized functions gives:

$\theta(\xi_0)\,\delta(\xi_0) = C\,\delta(\xi_0)$, where C is an arbitrary constant.

The analogue of $\theta(\xi_0)\,\delta(\xi_0)$ in the new scheme is the expression $\theta(n)\,\delta_{n,0}$, where $\theta(n)$ is the step function (49), and $\delta_{n,m}$ is the Kronecker symbol. Therefore

$$\theta(n)\delta_{n,0} = \left\{\frac{1}{4\pi i}\int_{-\pi}^{\pi}\frac{d\omega}{\operatorname{tg}\frac{\omega}{2} - i\varepsilon}\right\}\delta_{n,0} = \left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dE}{E - i\varepsilon}\,\frac{1}{1+E^2}\right\}\delta_{n,0} = \frac{1}{2}\,\delta_{n,0} \tag{69}$$

($\hbar \frac{\omega}{2} = E$ - is the new integration variable).

The conclusion which can be drawn by the considered example is that the functions $\theta(n)$ and $\delta_{n,0}$ are, contrary to their continuous analogues, ordinary (not generalized) functions and their product is constructed in an unique way.

It turns out that a similar situation holds in the more general case. For instance, the commutator (54) for zero mass particles is given by the expression:

$$\mathcal{D}(\xi,0)\Big|_{m=0} = \frac{1}{2\pi}\,\varepsilon(n)\frac{1}{L+2}\,\delta_{L,-1}\,; \quad \varepsilon(n) = \theta(n)-\theta(-n)\,, \qquad |n| \geqslant L+3\,,\ L=-1,0,1,\dots \tag{70}$$

In the "classical" case we should have correspondingly:

$$\mathcal{D}(\xi)\Big|_{m=0} = \frac{1}{2\pi}\,\varepsilon(\xi^0)\,\delta(\xi^2)\,. \tag{71}$$

A comparison of formulae (70) and (71) demonstrates that the first one has a completely well defined mathematical sense and can be interpreted as an ordinary product of ordinary functions*) and at the same time the second formula is a typical for the orthodox field theory example of multiplying of singular generalized functions with coinciding singularities.

It should be clearly understood that the appearance in our formalism of discrete (quantized) variables L and n is directly connected with the boundedness of the new p -space in timelike direction in the sense of De-Sitter metrics. Owing to the same

*) A similar statement is true for the function $\mathcal{D}(\xi,0)$ for $m \neq 0$, for all commutation functions and propagators and also for arbitrary powers and products of these quantities.

reason the "plane waves" (39) and (46), corresponding to the timelike L-series are square integrable functions. The last circumstance will play an important role in the example, which we consider below.

Let

$$J_a(\xi) = :(\varphi_a(\xi))^n: \tag{72}$$

is a "bilocal" operator, constructed of the fields (51). From (56) it is obvious that:

$$[J_a(\xi), J_a(-\xi)] = 0 \quad , \qquad \text{if } \xi \in \Lambda\text{-series} \tag{73}$$

and a is arbitrary

It is clear also that

$$\langle 0|[J_a(\xi), J_a(-\xi)]|0\rangle = \langle 0|[J(\xi), J(-\xi)]|0\rangle ,$$

where

$$J(\xi) = :(\varphi(\xi))^n:$$

Now let us consider the integral:

$$g(p) = i\int \langle \xi | p, p_4\rangle \theta(n_0)\langle 0|[J(\xi), J(-\xi)]|0\rangle \langle \xi | p, p_4\rangle \, d\Omega_\xi , \tag{74}$$

where $d\Omega_\xi = \begin{cases} 2(L+1)(L+2)(L+\frac{3}{2})\,\delta(N^2-1)\,d^4N \\ 2\Lambda(\Lambda^2+\frac{1}{4})\,\mathrm{th}\,\pi\Lambda\, d\Lambda\, \delta(N^2-1)\, d^4N \end{cases}$ is the volume element of the configuration space. In the flat limit this quantity coincides up to a constant factor with the real part of the one particle propagator, calculated in second order of the perturbation theory, in a model with interaction of the type: $:\varphi^{n+1}(x):(n\geq 2)$. As it is well known in this case the correspondent integral is divergent. The reason is that the product of the generalized functions $\theta(\xi_0)$ and $\langle 0|[J(\xi), J(-\xi)]|0\rangle$ is not integrable because of coincidence of their singularities at $\xi = 0$. In the considered case, because of the locality condition (73) only the

L -region contributes to (74). Taking into account that the functions $\langle \xi | p, p_4 \rangle$ are square-integrable in this region it is easy to prove that the integral (74) is absolutely convergent.

The considered examples testify, apparently, that the extension of the S-matrix off the m a s s shell, based on De-Sitter p-space is less singular and mathematically more correct than the extension based on Minkowski momentum space.

It is interesting to note here that in our scheme the generalized functions are removed, in certain way, from the dynamical part of the theory to the kinematical (plane waves, volume element) and it happens that on their new place they allow unique regularization.

§ 10. Our exposition is inevitably fragmentary. We did not say about numerous applications of our theory:systems of integral equations for the Green's functions, spectral representations,three-dimensional formulation of the two-body problem, different consequences for the phenomenological approaches, etc. We only wanted the main idea of this work to be correctly understood: there exists a possibility of extension the S-matrix off the mass shell in a four-dimensional momentum space with De-Sitter geometry and this possibility is internally consistent.

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