ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2 - 7909

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A NONLINEAR IRREDUCIBLE FORM OF THE SCHRÖDINGER EQUATION FOR THE GROUND STATE OF THE SECOND-QUANTIZED HAMILTONIAN

1974

ЛАБОРАТОРИЯ
ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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Submitted to *Communications in Mathematical Physics*

1. In quantum field theory and many-body problem the Schrödinger eq.

$$H\Omega = E\Omega \tag{1}$$

is of great importance. Here Hamiltonian H is some rather simple function of the creation and annihilation operators.

1.1. We will conduct our consideration by the example of the simplest Hamiltonian (2-4), however, this consideration is valid for any Hamiltonian, e.g., for Yukawa and $g\phi^4$ models of *quantum field theory.*

1.2. Thus, we take

$$H = h + h_1 \tag{2}$$

$$h = \int dp E(p) \left[a_+^*(p) a_+(p) + a_-^*(p) a_-(p) \right] + \\ + \int dk \omega(k) b^*(k) b(k) + \\ + g \int dp dq dk A(p,q,k) a_+(p) a_-(q) b(k) \delta(p+q+k) \tag{3}$$

$$h_1 = + g \int dp dq dk A^*(p,q,k) a_+^*(p) a_-^*(q) b^*(k) \delta(p+q+k). \tag{4}$$

Here $a_\pm$, $a^*_\pm$ are the annihilation and creation operators of two sorts of fermions

$$[a_\alpha(p), a^*_\beta(q)]_+ = \delta_{\alpha\beta}\,\delta(p-q)$$
$$[a_\alpha(p), a_\beta(q)]_+ = 0, \tag{5}$$

b, b^* boson annihilation and creation operators

$$[b(k), b^*(s)]_- = \delta(k-s)$$
$$[b(k), b(s)]_- = 0, \tag{6}$$

$E(p)$, $\omega(k)$ bare particle energies, A some function.

2. We state the ground state Ω_0 of the Hamiltonian (2-4) to have the form

$$\Omega_0 = e^{-\kappa}\,|0> \tag{7}$$

$$\kappa = C_{01}b(0) + \int C_{20}(p_1, p_2)\, a^*_+(p_1)\, a^*_-(p_2) \times$$
$$\times\, \delta(p_1 + p_2)\, dp_1\, dp_2 + \int C_{02}(k_1, k_2) \times$$
$$\times\, b^*(k_1)\, b^*(k_2)\, \delta(k_1 + k_2)\, dk_1\, dk_2 + \dots \tag{8}$$
$$= \sum_{n+r>0} \int C_{2n,r}(p_1, p_2, \dots p_n; q_1, q_2, \dots q_n; k_1, k_2, \dots k_r)$$
$$\prod_1^n (a^*_+(p_i)\, a^*_-(q_i)\, dp_i\, dq_i) \prod_1^r (b^*(k_i)\, dk_i) \times$$
$$\times\, \delta\Big(\sum_1^n (p_i + q_i) + \sum_1^r k_i\Big) = \sum P_{2n,r}(a^*_+, a^*_-, b^*),$$

where $|0>$ is the no particle state
$$a_\pm|0> = b|0> = 0.$$

2.1. Note, that all the terms of expansion (8) commute with each other.

2.2. In order to explain why the solution of the Schrödinger eq. in the form (7), (8) is just the ground state, we point out, that eq. (10) implies $C_{2n,r} \to 0$ for $g \to 0$; but at $g = 0$ the ground state is $|0>$; so there exists some interval $0 \lessgtr g < g_0$, where the solution of eq. (1) of the form (7), (8) is the state with the lowest energy (we suppose both functions $E(p)$, $\omega(k)$ to be positive).

3. Now we substitute (7) into the Schrödinger eq. (1). Using formulae

$$b e^{-\kappa} = e^{-\kappa}(-[b,\kappa]_- + b),$$

$$a e^{-\kappa} = e^{-\kappa}(-[a,\kappa]_- + a),$$

$$a_+(p)\, a_-(q)\, b(k)\, e^{-\kappa} = e^{-\kappa}(-[a_+(p),\kappa]_- + a_+(p))$$

$$(-[a_-(q),\kappa]_- + a_-(q))\,(-[b(k),\kappa]_- + b(k)) \qquad (9)$$

we get rid of the factor $e^{-\kappa}$ and rewrite (1) in the form

$$(E_0 + h\kappa - h_1 - \delta)\,|0> = 0. \qquad (10)$$

Here E_0 is the ground state energy,

$$\delta = g \int dp\, dq\, dk\, A(p,q,k)\,(-[a_+(p),\kappa]_-[a_-(q),\kappa]_-[b(k),\kappa]_- +$$

$$+ a_+(p)[a_-(q),\kappa]_-[b(k),\kappa]_- + [a_+(p),\kappa]_- \times \qquad (11)$$

$$\times\, a_-(q)\,[b(k),\kappa]_-)$$

3.1. Getting rid of the annihilation operators in eq. (10) with the help of (5), (6), the left-hand side of eq. (10)

is represented as a series in powers of creation operators. Equating to zero the coefficients of this series gives the system of equations defining the coefficient functions $C_{2n,r}$; this system will be denoted as (10a). Because of (11) this system, unlike eq. (1), is nonlinear.

3.2. The main advantage gained by the transition from linear form (1) of the Schrödinger eq. to nonlinear form (10), (10a) is that the coefficient functions of expansion (8) are nonsingular as follows from (10a). Therefore, none of the homogeneous translationally-invariant polynomials $p_{2n,r}$ of the series (8) contains the part equal to the product of two homogeneous translationally-invariant polynomials (such a product necessary gives the singularity for the function $C_{2n,r}$). In this sense the coefficient functions of the expansion (8) will be called irreducible. Let us consider for comparison another representation of the state (7):

$$\Omega_0 = A|0>,$$

$$A = 1 + \gamma_{01}\, b^*(0) + \int \gamma_{20}(p,-p)\, a_+^*(p)\, a_-^*(-p)\, dp + \dots \tag{12}$$

$$= \sum_{n+r>0} \int \gamma_{2n,r}(p_1, p_2, \dots p_n; q_1, q_2 \dots q_{1n}; k_1, \dots k_r)$$

$$\prod_1^n (a_+^*(p_i)\, a_-^*(q_i)\, dp_i\, dq_i) \prod_1^r (b^*(k_i)\, dk_i) \times$$
$$\times \delta\left(\sum_1^n (p_i + q_i) + \sum_1^r k_i\right) = \Sigma\, q_{2n,r}. \tag{13}$$

Evidently, we have

$$q_{01} = -p_{01}, \quad q_{20} = -p_{20},$$
$$q_{40} = -p_{40} + \frac{1}{2}(p_{20})^2, \dots \tag{14}$$

Eq. (14) and the irreducibility of coefficient functions $C_{2n,r}$ imply the coefficient functions of expansion (13) (e.g., γ_{40}) not to be irreducible.

3.3. Thus, the ground state representation (7), (8) is, in some sense, the simplest possible one.

3.3.1. Note, that the representation (12), (13) in quantum field theory gives the Tamm-Dankoff equations for the coefficient functions $\gamma_{2n,r}$.

3.4. The ground state energy E_0 is separated from the essential part of eqs. (10a); it enters only the first of these equations. So, one has to determine functions $C_{2n,r}$ from the second and subsequent equations of system (10a); then the first eq. (10a) gives the value of E_0.

3.5. The nonlinear form (10) of the Schrödinger equation defines irreducible coefficient functions of expansion (8) for ground state (7).

In this sense eq. (10) itself in the title of the article is called irreducible.

4. One has to search for excited state of Hamiltonian (2) in the form of product

$$\Omega = U\Omega_0 \tag{15}$$

of the ground state (7) and the operator U, which is a series in powers of operators $a^*_\pm$, b^*.

5. The exponential transformations of the type (7) are at present known well enough.

Thus, e.g., to get irreducible Green functions one must take derivatives with respect to the source $j(x)$ not of the generating functional

$$\int \exp[iS + \int j(x)\phi(x)\,dx]\,\delta\phi,$$

but of its logarithm (S is the action).

However, we are the first obtaining the exponential transform of the ground state of the second quantized Schrödinger equation.

5.1. We have used the method described for investigating $g\phi^4$ and Yukawa models in two /1,3/ and threedimensional space-time [2,4] and for studying the scalar electrodynamics in two-dimensional space-time [5]. In the first papers [1,2] we were able to deal with boson quantum field theory only; now we have succeeded in generalizing our method to the case of Fermi particles [3,4].

5.2. In these models it appears to be possible to simplify. equation of the type (10) and to reduce the order of nonlinearity in this equation to the second one. The way is as follows: one should use the representation where the boson field operator $\phi(k)$ is diagonal, then substitute, in expansion (8) $\phi(k)$ for $b^*(k)$, the state of fermion vacuum for $|0>$ and in the Hamiltonian $-i\delta/\delta\phi(k)$ for the momentum $\pi(k)$, canonically conjugated to $\phi(k)$.

5.3. By the present method we have succeeded in investigating strong coupling limit in $g[\phi^4]_2$ model [6,7]. We have also shown [5] that there is no necessity to introduce the term $-g(\phi^*\phi)^2$ into Lagrangian, in order to get the vacuum degeneration in scalar electrodynamics. This result contradicts the basic works [8,9].

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Received by Publishing Department
on April 30, 1974.